Homework #1

1.1.10: Because

$$\begin{aligned} x \in \overline{\lim_{n\to\infty}} A_n &\iff \forall m \geq 1\ \exists n \geq m\ x \in A_n \\ &\iff \exists n_1 < \cdots < n_m < \cdots\ \forall m \geq 1\ x \in A_{n_m} \\ &\iff x \in A_n \text{ for infinitely many } n \geq 1, \end{aligned}$$

$x \in \overline{\lim}_{n\to\infty} A_n$ if and only if $x \in A_n$ for infinitely many $n \geq 1$. Because

$$\underline{\lim}_{n\to\infty} A_n = \left(\overline{\lim_{n\to\infty}} A_n\complement\right)\complement,$$

$$\begin{aligned} x \in \underline{\lim}_{n\to\infty} A_n &\iff x \notin A_n\complement \text{ for infinitely many } n \geq 1 \\ &\iff x \in A_n \text{ for all but finitely many } n \geq 1, \end{aligned}$$

1.1.11: Because $\bigcup_{n\geq m} A_n \searrow \overline{\lim}_{n\to\infty} A_n$,

$$\mathbb{P}\left(\overline{\lim_{n\to\infty}} A_n\right) = \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{n\geq m} A_n\right) \geq \overline{\lim_{m\to\infty}}\, \mathbb{P}(A_m).$$

Because $\bigcap_{n\geq m} A_n \nearrow \underline{\lim}_{n\to\infty} A_n$,

$$\mathbb{P}\left(\underline{\lim}_{n\to\infty} A_n\right) = \lim_{m\to\infty} \mathbb{P}\left(\bigcap_{n\geq m} A_n\right) \leq \underline{\lim}_{m\to\infty}\, \mathbb{P}(A_m).$$

1.2.34: For each k element subset of vertices, there are $2^{-\binom{k}{2}}$ ways in which their edges can be colored, only 2 of which are monotone. Hence the probability that a given k element subset will have a monotone coloring is $2^{1-\binom{m}{2}}$. Since there are $\binom{N}{k}$ k element subsets, the probability that at least one of them has a monotone coloring is no more than $\binom{N}{k}2^{1-\binom{m}{2}}$. Hence, if $\binom{N}{k} < 2^{\binom{m}{2}-1}$, then, with positive probabiliy, no k element subset will be monotone.

1.2.36: Let H_n and T_n be the events "a head on nth toss" and "a tail on nth toss". Then $\mathbb{P}(H_{n+1}\,|\,H_n) = p_1$ and $\mathbb{P}(H_{n+1}\,|\,T_n) = p_2$. Hence, by Bayes's formula, $P_{n+1} = p_1 P_n + p_2(1 - P_n) = -\Delta P_n + p_2$. Since $P_1 = p_1$, one can use induction to show that

$$P_n = (-\Delta)^{n-1} + p_2 \sum_{m=0}^{n-2} (-\Delta)^m = p_1(-\Delta)^{n-1} + p_2 \frac{1 - (-\Delta)^{n-1}}{1 + \Delta} = \frac{(1 - p_1)(-\Delta)^n + p_2}{1 + \Delta}.$$

1.2.37: Let A be the event that a signal passes, C the event that switch 5 is closed, and, for $0 \leq k \leq 4$, B_k the event that precisely k of the switches 1 through 4 are closed. Then

$$\mathbb{P}(A) = \mathbb{P}(B_4) + P(B_3) + \mathbb{P}(A \cap B_2 \cap C) + \mathbb{P}(A \cap B_2 \cap C\complement).$$

Clearly $\mathbb{P}(B_4) = p^4$ and $\mathbb{P}(B_3) = 4p^3(1 - p)$. Further, there are 4 configurations in $A \cap B_2 \cap C$, each of which occurs with probability $p^3(1 - p)^2$, and there are 2 configurations in $A \cap B_2 \cap C\complement$, each of which occurs with probability $p^2(1 - p)^3$. Hence,

$$\mathbb{P}(A) = p^4 + 4p^3(1 - p) + 4p^3(1 - p)^2 + 2p^2(1 - p)^3$$

and

$\mathbb{P}(A\cap C\complement) = \mathbb{P}(B_4\cap C\complement)+\mathbb{P}(B_3\cap\complement)+\mathbb{P}(A\cap B_2\cap C\complement) = (1-p)\big(p^4+4p^3(1-p)+2p^2(1-p)^2 2\big).$

Finally,

$$\mathbb{P}(C\complement\,|\,A) = \frac{(1-p)\big(p^4+4p^3(1-p)+2p^2(1-p)^2 2\big)}{p^4+4p^3(1-p)+4p^3(1-p)^2+2p^2(1-p)^3}.$$

1.2.40: First observe that $\mathbb{P}(\zeta_N^{\{N\}} = N) = \mathbb{P}(W_N = N)$. Now assume that $k < N$. Then

$$\begin{aligned}
&\mathbb{P}(\zeta_N^{\{k\}} = N) = \mathbb{P}\big(\zeta_N^{\{k\}} > N-1 \;\&\; W_{N-1} = k-1 \;\&\; W_N - W_{N-1} = 1\big)\\
&= \tfrac{1}{2}\mathbb{P}\big(\zeta_N^{\{k\}} > N-1 \;\&\; W_{N-1} = k-1\big)\\
&\tfrac{1}{2}\mathbb{P}(W_{N-1} = k-1) - \tfrac{1}{2}\mathbb{P}(\zeta^{\{k\}} > N-1 \;\&\; W_{N-1} = k-1)\\
&= \tfrac{1}{2}\mathbb{P}(W_{N-1} = k-1) - \tfrac{1}{2}\mathbb{P}(W_{N-1} = k+1),
\end{aligned}$$

where, at the last step, one uses (1.2.12). Now use

$$\mathbb{P}(W_{N-1} = \ell) = 2^{1-N}\binom{N-1}{\frac{N+\ell-1}{2}}.$$

1.2.41: Observe that, since the events $A_m\complement$ are mutually independent,

$$\mathbb{P}\left(\bigcup_{n=m}^{N} A_n\right) = 1-\mathbb{P}\left(\left(\bigcup_{n=m}^{N} A_n\right)\complement\right) = 1-\mathbb{P}\left(\bigcap_{n=m}^{N} A_n\complement\right) = 1-\prod_{n=m}^{N}\big(1-\mathbb{P}(A_n)\big).$$

Now follow the outline.