## A Little Fourier Analysis

Given an $\varphi \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, define the **Fourier transform** $\hat{\varphi}$ of $\varphi$ by

$$\hat{\varphi}(\xi) = \int e^{i(\xi,y)_{\mathbb{R}^N}}\varphi(y)\,\lambda_{\mathbb{R}^N}(dy) \quad \text{for } \xi \in \mathbb{R}^N.$$

Clearly, $\hat{\varphi}$ is continuous and $\|\varphi\|_{\mathrm{u}} \le \|\varphi\|_{L^2(\lambda_{\mathbb{R}^N};\mathbb{C})}$. Next, use Fubini's theorem to see that if $\psi$ is a second element of $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, then

$$\int \varphi(y)\overline{\check{\psi}(y)}\,\lambda_{\mathbb{R}^N}(dy) = \int \hat{\varphi}(\xi)\overline{\psi(\xi)}\,\lambda_{\mathbb{R}^N}(d\xi), \tag{1}$$

where $\check{\varphi}(y) = \hat{\varphi}(-y)$ for $y \in \mathbb{R}^N$.

Before carrying out the next step, I need to compute $\hat{g}$ when $g(y) = (2\pi)^{-\frac{N}{2}}e^{-\frac{|y|^2}{2}}$. To this end, consider the analytic function

$$f(z) = \int e^{zy}e^{-\frac{y^2}{2}}\,\lambda_{\mathbb{R}}(dy) \quad \text{for } z \in \mathbb{C},$$

and use integration by parts to see that

$$f'(z) = \int y e^{zy}e^{-\frac{y^2}{2}}\,\lambda_{\mathbb{R}}(dy) = -\int e^{zy}\frac{d}{dy}e^{-\frac{y^2}{2}}\,\lambda_{\mathbb{R}}(dy) = zf(z).$$

Hence, $\frac{d}{dz}\big(f(z)e^{-\frac{z^2}{2}}\big) = 0$, and so $f(z) = f(0)e^{\frac{z^2}{2}} = (2\pi)^{\frac{1}{2}}e^{\frac{z^2}{2}}$. Thus $\hat{g}(\xi) = e^{-\frac{\xi^2}{2}}$ when $N = 1$. Further, by Fubini's theorem, for $N \ge 2$,

$$\hat{g}(\xi) = (2\pi)^{-\frac{N}{2}}\prod_{j=!}^{N}\int e^{i\xi_j y}e^{-\frac{y^2}{2}}\,\lambda_{\mathbb{R}}(dy) = e^{-\frac{|\xi|^2}{2}}.$$

Next, set $g_\epsilon(y) = \epsilon^{-\frac{N}{2}}g(\epsilon^{-\frac{1}{2}}y)$, and use a change of variables to see that

$$\widehat{g_\epsilon}(\xi) = e^{-\epsilon\frac{|\xi|^2}{2}} \quad \text{for } \epsilon > 0 \text{ and } \xi \in \mathbb{R}^N.$$

In particular, if $\psi_\epsilon = \widehat{g_\epsilon}$, then $\check{\psi}_\epsilon = (2\pi)^N g_\epsilon$, and so, by (1),

$$\int \varphi(y)g_\epsilon(y)\,\lambda_{\mathbb{R}^N}(dy) = (2\pi)^{-N}\int \hat{\varphi}(\xi)e^{-\epsilon\frac{|\xi|^2}{2}}\,\lambda_{\mathbb{R}^N}(d\xi).$$

Hence, if $\varphi \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \cap C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ and $\hat{\varphi} \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, then

$$\begin{aligned}\varphi(0) &= \lim_{\epsilon\searrow 0}\int \varphi(y)g_\epsilon(y)\,\lambda_{\mathbb{R}^N}(dy)\\ &= \lim_{\epsilon\searrow 0}(2\pi)^{-N}\int \hat{\varphi}(\xi)e^{-\epsilon\frac{|\xi|^2}{2}}\,\lambda_{\mathbb{R}^N}(d\xi) = (2\pi)^{-N}\int \hat{\varphi}(\xi)\,\lambda_{\mathbb{R}^N}(d\xi).\end{aligned}$$

More generally, if $\varphi \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$ and $\varphi_x(y) = \varphi(x+y)$, then $\widehat{\varphi_x}(\xi) = e^{-i(\xi,x)_{\mathbb{R}^N}}\hat{\varphi}(\xi)$, and so if $\varphi \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \cap C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ and $\hat{\varphi} \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, then

$$\varphi(x) = (2\pi)^{-N}\int e^{-i(\xi,x)_{\mathbb{R}^N}}\hat{\varphi}(\xi)\,d\xi = (2\pi)^{-N}\big(\hat{\varphi}\big)^{\vee}(x). \tag{2}$$

Now suppose that $\varphi$ and $\psi$ are both elements of $L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \cap C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ for which $\hat{\varphi}$ and $\hat{\psi}$ are in $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$. Then, by (1) and (2),

$$\begin{aligned}\int \varphi(y)\overline{\psi(y)}\,\lambda_{\mathbb{R}^N}(dy) &= (2\pi)^{-N}\int \varphi(y)\overline{(\hat{\psi})^{\vee}}(y)\,\lambda_{\mathbb{R}^N}(dy)\\ &= (2\pi)^{-N}\int \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}\,\lambda_{\mathbb{R}^N}(d\xi),\end{aligned}$$

and so we have proved the following version of what is called **Parseval's identity**

$$\int \varphi(y)\overline{\psi(y)}\,\lambda_{\mathbb{R}^N}(dy) = (2\pi)^{-N}\int \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}\,\lambda_{\mathbb{R}^N}(d\xi). \tag{3}$$

Given a Borel probability measure $\mu$ on $\mathbb{R}^N$, define its **characteristic function** $\hat{\mu} : \mathbb{R}^N \longrightarrow \mathbb{C}$ by

$$\hat{\mu}(\xi) = \int e^{i(\xi,y)_{\mathbb{R}^N}}\,\mu(dy).$$

Clearly $\hat{\mu}$ is a continuous function and $\|\hat{\mu}\|_{\mathrm{u}} \le 1$. We will now use (3) to prove that

$$\int \varphi d\mu = (2\pi)^{-N}\lim_{\epsilon\searrow 0}\int e^{-\epsilon\frac{|\xi|^2}{2}}\hat{\varphi}(\xi)\overline{\hat{\mu}(\xi)}\,\lambda_{\mathbb{R}^N}(d\xi) \tag{4}$$

for $\varphi \in L^1(\lambda_{\mathbb{R}^N};\mathbb{C}) \cap C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$. To this end, define

$$\psi_\epsilon(x) = \int g_\epsilon(x-y)\,\mu(dy) \text{ and } \varphi_\epsilon(x) = \int g_\epsilon(x-y)\varphi(y)\,\lambda_{\mathbb{R}^N}(dy),$$

and use Fubini's theorm to see that

$$\int \varphi\psi_\epsilon\,d\lambda_{\mathbb{R}^N} = \int \varphi_\epsilon\,d\mu \text{ and } \widehat{\psi_\epsilon}(\xi) = e^{-\epsilon\frac{|\xi|^2}{2}}\hat{\mu}(\xi).$$

Hence, by (3),

$$\int \varphi_\epsilon\,d\mu = (2\pi)^{-N}\int e^{-\epsilon\frac{|\xi|^2}{2}}\hat{\varphi}(\xi)\overline{\hat{\mu}(\xi)}\,\lambda_{\mathbb{R}^N}(d\xi),$$

and so (4) follows after one lets $\epsilon \searrow 0$.

An important consequence of (4) is that a Borel probability measure on $\mathbb{R}^N$ is determined by its characteristic function. That is, if $\mu$ and $\nu$ are such measures, then

$$\hat{\mu} = \hat{\nu} \implies \mu = \nu.$$

In fact more is true.

THEOREM 5. *Suppose that $\{\mu_n : n \geq 1\}$ is a sequence of Borel probability measures on $\mathbb{R}^N$ and that $\mu$ is Borel probability measure on $\mathbb{R}^N$ for which*

$$\lim_{n\to\infty} \widehat{\mu_n}(\xi) = \hat{\mu}(\xi) \text{ for all } \xi \in \mathbb{R}^N.$$

*Then*

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu \text{ for all } \varphi \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C}).$$

PROOF: Let $\varphi : \mathbb{R}^N \longrightarrow \mathbb{C}$ be a bounded, uniformly continuous element of $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$, and define $\varphi_\epsilon$ as above. Then $\|\varphi_\epsilon - \varphi\|_{\mathrm{u}} \longrightarrow 0$ as $\epsilon \searrow 0$, and so

$$\lim_{\epsilon\searrow 0} \left| \int \varphi_\epsilon \, d\mu - \int \varphi \, d\mu \right| \vee \sup_{n\geq 1} \left| \int \varphi_\epsilon \, d\mu_n - \int \varphi \, d\mu_n \right| = 0.$$

Thus, by (4),

$$\begin{aligned}&(2\pi)^N \varlimsup_{n\to\infty} \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu \right| \\ &= \lim_{\epsilon\searrow 0} \varlimsup_{n\to\infty} \left| \int e^{-\epsilon\frac{|\xi|^2}{2}} \hat{\varphi}(\xi) \big(\overline{\hat{\mu}_n(\xi)} - \overline{\hat{\mu}(\xi)}\big) \, \lambda_{\mathbb{R}^N}(d\xi) \right| = 0.\end{aligned}$$

To remove the uniform continuity and integrability requirements, for $k \in \mathbb{Z}^+$, set

$$\eta_k(x) = 1 \wedge \big((1 + k - |x|) \vee 0\big).$$

Then, for every $k \in \mathbb{Z}^+$ and $\varphi \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$, $\eta_k\varphi$ is a uniformly continuous, $[0,1]$-valued element of $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$. In addition, $\eta_k \nearrow \mathbf{1}$ as $k \to \infty$, and so

$$\lim_{n\to\infty} \int (1 - \eta_k) \, d\mu_n = 1 - \int \eta_k \, d\mu \longrightarrow 0$$

as $k \to \infty$. Hence, for any $\epsilon > 0$, there is a $k_\epsilon$ such that

$$\sup_{n\geq 1} \int (1 - \eta_{k_\epsilon}) \, d\mu_n \vee \int (1 - \eta_{k_\epsilon}) \, d\mu \leq \epsilon.$$

Finally, let $\varphi \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ be given. Since $\eta_{k_\epsilon}\varphi$ is a uniformly continuous element of $L^1(\lambda_{\mathbb{R}^N};\mathbb{C})$,

$$\varlimsup_{n\to\infty} \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu \right| \leq 2\|\varphi\|_{\mathrm{u}}\epsilon + \varlimsup_{n\to\infty} \left| \int \eta_{k_\epsilon}\varphi \, d\mu_n - \int \eta_{k_\epsilon}\varphi \, d\mu \right| = 2\|\varphi\|_{\mathrm{u}}\epsilon.$$

$\square$

**Exercise** Set $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. The measure $P$ with density $f$ is called the Cauchy distribution, and you are to compute its Fourier transform $\hat{P}$.

(**i**) Observe that
$$f(x) = \frac{1}{\pi} \int_0^\infty e^{-t(1+x^2)}\, dt,$$
and use this to show that
$$\hat{f}(\xi) = \sqrt{\pi} \int_0^\infty e^{-t-\frac{\xi^2}{4t}}\, dt.$$

(**ii**) For $a, b \in \mathbb{R} \setminus 0$, show that
$$\int_0^\infty e^{-\frac{a^2}{2t}-\frac{b^2}{2t}}\, dt = \frac{\sqrt{2\pi}}{|a|} e^{-|ab|}.$$
To this end, assume that $a$, $b > 0$ and use the change of variables $\tau = at^{\frac{1}{2}} - bt^{-\frac{1}{2}}$.

(**iii**) Combine (**i**) and (**ii**) to arrive at $\hat{P}(\xi) = e^{-|\xi|}$.

(**iv**) More generally, if $P_y(dx) = \frac{y}{\pi(y^2+x^2)}\, dx$ for $y > 0$, show that $\widehat{P_y}(\xi) = e^{-y|\xi|}$.