A Little Fourier Analysis

Given an \( \varphi \in L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \), define the Fourier transform \( \hat{\varphi} \) of \( \varphi \) by

\[
\hat{\varphi}(\xi) = \int e^{i(\xi, y)} \varphi(y) \lambda_{\mathbb{R}^N}(dy) \quad \text{for } \xi \in \mathbb{R}^N.
\]

Clearly, \( \hat{\varphi} \) is continuous and \( \|\varphi\|_u \leq \|\varphi\|_{L^2(\lambda_{\mathbb{R}^N}; \mathbb{C})} \). Next, use Fubini’s theorem to see that if \( \psi \) is a second element of \( L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \), then

\[
(1) \quad \int \varphi(y) \overline{\psi(y)} \lambda_{\mathbb{R}^N}(dy) = \int \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} \lambda_{\mathbb{R}^N}(d\xi),
\]

where \( \hat{\varphi}(y) = \hat{\varphi}(-y) \) for \( y \in \mathbb{R}^N \).

Before carrying out the next step, I need to compute \( \hat{g} \) when \( g(y) = (2\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}} \). To this end, consider the analytic function

\[
f(z) = \int e^{zy} e^{-\frac{y^2}{4}} \lambda_{\mathbb{R}}(dy) \quad \text{for } z \in \mathbb{C},
\]

and use integration by parts to see that

\[
f'(z) = \int ye^{zy} e^{-\frac{y^2}{4}} \lambda_{\mathbb{R}}(dy) = - \int e^{zy} \frac{d}{dy} e^{-\frac{y^2}{4}} \lambda_{\mathbb{R}}(dy) = zf(z).
\]

Hence, \( \frac{d}{dz} (f(z) e^{-\frac{z^2}{4}}) = 0 \), and so \( f(z) = f(0) e^\frac{z^2}{2} \). Thus \( \hat{g}(\xi) = e^{-\frac{\xi^2}{4}} \) when \( N = 1 \). Further, by Fubini’s theorem, for \( N \geq 2 \),

\[
\hat{g}(\xi) = (2\pi)^{-\frac{N}{2}} \prod_{j=1}^{N} \int e^{\xi_j y} e^{-\frac{y^2}{4}} \lambda_{\mathbb{R}}(dy) = e^{-\frac{|\xi|^2}{4}}.
\]

Next, set \( g_\epsilon(y) = e^{-\frac{\epsilon y^2}{4}} g(e^{-\frac{\epsilon}{2}} y) \), and use a change of variables to see that

\[
\hat{g}_\epsilon(\xi) = e^{-\epsilon \frac{|\xi|^2}{4}} \quad \text{for } \epsilon > 0 \text{ and } \xi \in \mathbb{R}^N.
\]

In particular, if \( \psi_\epsilon = \hat{g}_\epsilon \), then \( \hat{\psi}_\epsilon = (2\pi)^N g_\epsilon \), and so, by \(1\),

\[
\int \varphi(y) g_\epsilon(y) \lambda_{\mathbb{R}^N}(dy) = (2\pi)^{-N} \int \hat{\varphi}(\xi) e^{-\epsilon \frac{|\xi|^2}{4}} \lambda_{\mathbb{R}^N}(d\xi).
\]

Hence, if \( \varphi \in L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \cap C_b(\mathbb{R}^N; \mathbb{C}) \) and \( \hat{\varphi} \in L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \), then

\[
\varphi(0) = \lim_{\epsilon \to 0} \int \varphi(y) g_\epsilon(y) \lambda_{\mathbb{R}^N}(dy)
= \lim_{\epsilon \to 0} (2\pi)^{-N} \int \hat{\varphi}(\xi) e^{-\epsilon \frac{|\xi|^2}{4}} \lambda_{\mathbb{R}^N}(d\xi) = (2\pi)^{-N} \int \hat{\varphi}(\xi) \lambda_{\mathbb{R}^N}(d\xi).
\]
More generally, if $\varphi \in L^1(\lambda_{RN}; \mathbb{C})$ and $\varphi_x(y) = \varphi(x + y)$, then $\hat{\varphi}_x(\xi) = e^{-i(\xi, x)\mu_N}\hat{\varphi}(\xi)$, and so if $\varphi \in L^1(\lambda_{RN}; \mathbb{C}) \cap C_b(\mathbb{R}^N; \mathbb{C})$ and $\hat{\varphi} \in L^2(\lambda_{RN}; \mathbb{C})$, then

$$
\varphi(x) = (2\pi)^{-N} \int e^{-i(\xi, x)\mu_N}\hat{\varphi}(\xi) \, d\xi = (2\pi)^{-N} (\hat{\varphi})^\vee(x).
$$

(2)

Now suppose that $\varphi$ and $\psi$ are both elements of $L^1(\lambda_{RN}; \mathbb{C}) \cap C_b(\mathbb{R}^N; \mathbb{C})$ for which $\hat{\varphi}$ and $\hat{\psi}$ are in $L^1(\lambda_{RN}; \mathbb{C})$. Then, by (1) and (2),

$$
\int \varphi(y)\overline{\psi(y)} \lambda_{RN}(dy) = (2\pi)^{-N} \int \varphi(x)\overline{(\hat{\psi})^\vee(y)} \lambda_{RN}(dy)
$$

$$
= (2\pi)^{-N} \int \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)} \lambda_{RN}(d\xi),
$$

and so we have proved the following version of what is called Parseval’s identity

$$
\int \varphi(y)\overline{\psi(y)} \lambda_{RN}(dy) = (2\pi)^{-N} \int \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)} \lambda_{RN}(d\xi).
$$

(3)

Given a Borel probability measure $\mu$ on $\mathbb{R}^N$, define its characteristic function $\tilde{\mu} : \mathbb{R}^N \rightarrow \mathbb{C}$ by

$$
\tilde{\mu}(\xi) = \int e^{i(\xi, y)\mu_N} \mu(dy).
$$

Clearly $\tilde{\mu}$ is a continuous function and $\|\tilde{\mu}\|_u \leq 1$. We will now use (3) to prove that

$$
\int \varphi \, d\mu = (2\pi)^{-N} \lim_{\epsilon \downarrow 0} \int e^{-\epsilon \frac{|\xi|^2}{\mu_N}} \hat{\varphi}(\xi) \overline{\tilde{\mu}(\xi)} \lambda_{RN}(d\xi)
$$

(4)

for $\varphi \in L^1(\lambda_{RN}; \mathbb{C}) \cap C_b(\mathbb{R}^N; \mathbb{C})$. To this end, define

$$
\psi_\epsilon(x) = \int g_\epsilon(x - y) \mu(dy) \quad \text{and} \quad \varphi_\epsilon(x) = \int g_\epsilon(x - y) \varphi(y) \lambda_{RN}(dy),
$$

and use Fubini’s theorem to see that

$$
\int \varphi_\epsilon \, d\mu = \int \varphi_\epsilon \, d\mu \quad \text{and} \quad \tilde{\psi}(\xi) = e^{-\epsilon \frac{|\xi|^2}{\mu_N}} \tilde{\mu}(\xi).
$$

Hence, by (3),

$$
\int \varphi_\epsilon \, d\mu = (2\pi)^{-N} \int e^{-\epsilon \frac{|\xi|^2}{\mu_N}} \hat{\varphi}(\xi) \overline{\tilde{\mu}(\xi)} \lambda_{RN}(d\xi),
$$

and so (4) follows after one lets $\epsilon \downarrow 0$.

An important consequence of (4) is that a Borel probability measure on $\mathbb{R}^N$ is determined by its characteristic function. That is, if $\mu$ and $\nu$ are such measures, then

$$
\tilde{\mu} = \tilde{\nu} \implies \mu = \nu.
$$

In fact more is true.
Theorem 5. Suppose that \( \{\mu_n : n \geq 1\} \) is a sequence of Borel probability measures on \( \mathbb{R}^N \) and that \( \mu \) is Borel probability measure on \( \mathbb{R}^N \) for which

\[
\lim_{n \to \infty} \hat{\mu}_n(\xi) = \hat{\mu}(\xi) \text{ for all } \xi \in \mathbb{R}^N.
\]

Then

\[
\lim_{n \to \infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu \quad \text{for all } \varphi \in C_b(\mathbb{R}^N; \mathbb{C}).
\]

Proof: Let \( \varphi : \mathbb{R}^N \to \mathbb{C} \) be a bounded, uniformly continuous element of \( L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \), and define \( \varphi_\varepsilon \) as above. Then \( \|\varphi_\varepsilon - \varphi\|_u \to 0 \) as \( \varepsilon \downarrow 0 \), and so

\[
\lim_{\varepsilon \downarrow 0} \left| \int \varphi_\varepsilon \, d\mu - \int \varphi \, d\mu \right| \leq \sup_{n \geq 1} \left| \int \varphi_\varepsilon \, d\mu_n - \int \varphi \, d\mu_n \right| = 0.
\]

Thus, by (4),

\[
(2\pi)^N \lim_{n \to \infty} \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu \right| = \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \left| \int e^{-\varepsilon |\xi|^2} \varphi(\xi) (\hat{\mu}_n(\xi) - \hat{\mu}(\xi)) \lambda_{\mathbb{R}^N}(d\xi) \right| = 0.
\]

To remove the uniform continuity and integrability requirements, for \( k \in \mathbb{Z}^+ \), set

\[
\eta_k(x) = 1 \wedge (1 + k - |x|) \vee 0.
\]

Then, for every \( k \in \mathbb{Z}^+ \) and \( \varphi \in C_b(\mathbb{R}^N; \mathbb{C}) \), \( \eta_k \varphi \) is a uniformly continuous, \([0,1]\)-valued element of \( L^1(\lambda_{\mathbb{R}^N}; \mathbb{C}) \). In addition, \( \eta_k \not\to 1 \) as \( k \to \infty \), and so

\[
\lim_{n \to \infty} \int (1 - \eta_k) \, d\mu_n = 1 - \int \eta_k \, d\mu \to 0
\]

as \( k \to \infty \). Hence, for any \( \varepsilon > 0 \), there is a \( k_\varepsilon \) such that

\[
\sup_{n \geq 1} \left| \int (1 - \eta_k) \, d\mu_n \vee \int (1 - \eta_k) \, d\mu \leq \varepsilon.
\]

Finally, let \( \varphi \in C_b(\mathbb{R}^N; \mathbb{C}) \) be given. Then \( \eta_k \varphi \) is uniformly continuous, and therefore

\[
\lim_{n \to \infty} \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu \right| \leq 2\|\varphi\|_u \varepsilon + \lim_{n \to \infty} \left| \int \eta_k \varphi \, d\mu_n - \int \eta_k \varphi \, d\mu \right| = 2\|\varphi\|_u \varepsilon.
\]

\( \square \)
Exercise Set \( f(x) = \frac{1}{\pi(1+x^2)} \) for \( x \in \mathbb{R} \). The measure \( P \) with density \( f \) is called the Cauchy distribution, and you are to compute its Fourier transform \( \hat{P} \).

(i) Observe that
\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} e^{-t(1+x^2)} \, dt,
\]
and use this to show that
\[
\hat{f}(\xi) = \sqrt{\pi} \int_{0}^{\infty} e^{-t - \frac{\xi^2}{4t}} \, dt.
\]

(ii) For \( a, b \in \mathbb{R} \setminus \{0\} \), show that
\[
\int_{0}^{\infty} e^{-a^2 t^2 - \frac{b^2}{2t}} \, dt = \frac{\sqrt{2\pi}}{|a|} e^{-|ab|}.
\]
To this end, assume that \( a, b > 0 \) and use the change of variables \( \tau = at^2 - \frac{b}{2t} \).

(iii) Combine (i) and (ii) to arrive at \( \hat{P}(\xi) = e^{-|\xi|} \).

(iv) More generally, if \( P_y(dx) = \frac{y}{\pi(y^2+x^2)} \, dx \) for \( y > 0 \), show that \( \hat{P}_y(\xi) = e^{-y|\xi|} \).