Chapter VIII
Gaussian Measures on a Banach Space

As I said at the end of § 4.3.2, the distribution of Brownian motion is called Wiener measure because Wiener was the first to construct it. Wiener's own thinking about his measure had little or nothing in common with the Lévy–Khinchine program. Instead, he looked upon his measure as a Gaussian measure on an infinite dimensional space, and most of what he did with his measure is best understood from that perspective. Thus, in this chapter, we will look at Wiener measure from a strictly Gaussian point of view. More generally, we will be dealing here with measures on a real Banach space E which are centered Gaussian in the sense that, for each x^* in the dual space E^*, $x \in E \longmapsto \langle x, x^* \rangle \in \mathbb{R}$ is a centered Gaussian random variable. Not surprisingly, such a measure will be said to be a **centered Gaussian measure** on E .

Although the ideas which I will use are implicit in Wiener's work, it was I. Segal and his school, especially L. Gross,* who gave them the form presented here.

§ 8.1 The Classical Wiener Space

In order to motivate what follows, it is helpful to first understand Wiener measure from the point of view which I will be adopting here.

§ 8.1.1. Classical Wiener Measure. Up until now I have been rather casual about the space from which Brownian paths come. Namely, because Brownian paths are continuous, I have thought of their distribution as being a probability on the space $C(\mathbb{R}^N) = C\big([0,\infty);\mathbb{R}^N\big)$. In general, there is no harm done by choosing $C(\mathbb{R}^N)$ as the sample space for Brownian paths. However, for my purposes here, I need our sample spaces to be separable Banach spaces, and, although it is a complete, separable metric space, $C(\mathbb{R}^N)$ is not a Banach space. With this in mind, define $\Theta(\mathbb{R}^N)$ to be the space of continuous paths $\boldsymbol{\theta} : [0,\infty) \longrightarrow \mathbb{R}^N$ with the properties that $\boldsymbol{\theta}(0) = 0$ and $\lim_{t\to\infty} t^{-1}|\boldsymbol{\theta}(t)| = 0$.

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* See I.E. Segal's "Distributions in Hilbert space and canonical systems of operators," *T.A.M.S.* **88** (1958) and L. Gross's "Abstract Wiener spaces," *Proc. 5th Berkeley Symp. on Prob. & Stat., 2* (1965). A good exposition of this topic can be found in H.-H. Kuo's *Gaussian Measures in Banach Spaces*, publ. by Springer–Verlag Math. Lec. Notes., no. **463**.

LEMMA 8.1.1. *The map*

$$\boldsymbol{\psi} \in C(\mathbb{R}^N) \longmapsto \|\boldsymbol{\psi}\|_{\Theta(\mathbb{R}^N)} \equiv \sup_{t\geq 0} \frac{|\boldsymbol{\psi}(t)|}{1+t} \in [0,\infty]$$

is lower semicontinuous, and the pair $\big(\Theta(\mathbb{R}^N), \|\cdot\|_{\Theta(\mathbb{R}^N)}\big)$ *is a separable Banach space which is continuously embedded as a Borel measurable subset of* $C(\mathbb{R}^N)$. *In particular,* $\mathcal{B}_{\Theta(\mathbb{R}^N)}$ *coincides with* $\mathcal{B}_{C(\mathbb{R}^N)}[\Theta(\mathbb{R}^N)] = \big\{A \cap \Theta(\mathbb{R}^N) : A \in \mathcal{B}_{C(\mathbb{R}^N)}\big\}$. *Moreover, the dual space* $\Theta(\mathbb{R}^N)^*$ *of* $\Theta(\mathbb{R}^N)$ *can be identified with the space of* $\mathbb{R}^N$*-valued, Borel measures* $\boldsymbol{\lambda}$ *on* $[0,\infty)$ *with the properties that* $\boldsymbol{\lambda}(\{\mathbf{0}\}) = 0$ *and*[†]

$$\|\boldsymbol{\lambda}\|_{\Theta(\mathbb{R}^N)^*} \equiv \int_{[0,\infty)} (1+t)\,|\boldsymbol{\lambda}|(dt) < \infty,$$

when the duality relation is given by

$$\langle \boldsymbol{\theta}, \boldsymbol{\lambda}\rangle = \int_{[0,\infty)} \boldsymbol{\theta}(t)\cdot\boldsymbol{\lambda}(dt).$$

Finally, if $(\mathbf{B}(t), \mathcal{F}_t, \mathbb{P})$ *is an* $\mathbb{R}^N$*-valued Brownian motion, then* $\mathbf{B} \in \Theta(\mathbb{R}^N)$ $\mathbb{P}$*-almost surely and*

$$\mathbb{E}\big[\|\mathbf{B}\|^2_{\Theta(\mathbb{R}^N)}\big] \leq 32N.$$

PROOF: It is obvious that the inclusion map taking $\Theta(\mathbb{R}^N)$ into $C(\mathbb{R}^N)$ is continuous. To see that $\|\cdot\|_{\Theta(\mathbb{R}^N)}$ is lower semicontinuous on $C(\mathbb{R}^N)$ and that $\Theta(\mathbb{R}^N) \in \mathcal{B}_{C(\mathbb{R}^N)}$, note that, for any $s \in [0,\infty)$ and $R \in (0,\infty)$,

$$A(s,R) \equiv \Big\{\boldsymbol{\psi} \in C(\mathbb{R}^N) : \big|\boldsymbol{\psi}(t)\big| \leq R(1+t) \text{ for } t \geq s\Big\}$$

is closed in $C(\mathbb{R}^N)$. Hence, since $\|\boldsymbol{\psi}\|_{\Theta(\mathbb{R}^N)} \leq R \iff \boldsymbol{\psi} \in A(0,R)$, $\|\cdot\|_{\Theta(\mathbb{R}^N)}$ is lower semicontinuous. In addition, since $\{\boldsymbol{\psi} \in C(\mathbb{R}^N) : \boldsymbol{\psi}(0) = \mathbf{0}\}$ is also closed,

$$\Theta(\mathbb{R}^N) = \bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\Big\{\boldsymbol{\psi} \in A\left(m, \tfrac{1}{n}\right) : \boldsymbol{\psi}(0) = \mathbf{0}\Big\} \in \mathcal{B}_{C(\mathbb{R}^N)}.$$

In order to analyze the space $\big(\Theta(\mathbb{R}^N), \|\cdot\|_{\Theta(\mathbb{R}^N)}\big)$, define

$$F : \Theta(\mathbb{R}^N) \longrightarrow C_0\big(\mathbb{R};\mathbb{R}^N\big) \equiv \Big\{\boldsymbol{\psi} \in C\big(\mathbb{R};\mathbb{R}^N\big) : \lim_{|s|\to\infty} |\boldsymbol{\psi}(s)| = 0\Big\}$$

by

$$\big[F(\boldsymbol{\theta})\big](s) = \frac{\boldsymbol{\theta}\left(e^s\right)}{1+e^s}, \quad s \in \mathbb{R}.$$

[†] I use $|\boldsymbol{\lambda}|$ to denote the variation measure determined by $\boldsymbol{\lambda}$.

As is well-known, $C_0\big(\mathbb{R};\mathbb{R}^N\big)$ with the uniform norm is a separable Banach space; and it is obvious that F is an isometry from $\Theta(\mathbb{R}^N)$ onto $C_0\big(\mathbb{R};\mathbb{R}^N\big)$. Moreover, by the Riesz Representation Theorem for $C_0\big(\mathbb{R};\mathbb{R}^N\big)$, one knows that the dual of $C_0\big(\mathbb{R};\mathbb{R}^N\big)$ is isometric to the space of totally finite, $\mathbb{R}^N$-valued measures on $\big(\mathbb{R};\mathcal{B}_{\mathbb{R}}\big)$ with the norm given by total variation. Hence, the identification of $\Theta(\mathbb{R}^N)^*$ reduces to the obvious interpretation of the adjoint map F^* as a mapping from totally finite $\mathbb{R}^N$-valued measures onto the space of $\mathbb{R}^N$-valued measures which do not charge $\mathbf{0}$ and whose variation measure integrate $(1+t)$.

Because of the Strong Law in part (**ii**) of Exercise 4.3.11, it is clear that almost every Brownian path is in $\Theta(\mathbb{R}^N)$. In addition, by the Brownian scaling property and Doob's Inequality (cf. Theorem 7.1.9),

$$\begin{aligned}\mathbb{E}\big[\|\mathbf{B}\|^2_{\Theta(\mathbb{R}^N)}\big] &\le \sum_{n=0}^{\infty} 4^{-n+1}\mathbb{E}\left[\sup_{0\le t\le 2^n} |\mathbf{B}(t)|^2\right] \\ &= \sum_{n=0}^{\infty} 2^{-n+2}\mathbb{E}\left[\sup_{0\le t\le 1} |\mathbf{B}(t)|^2\right] \le 32\mathbb{E}\big[|\mathbf{B}(1)|^2\big] = 32N. \quad \square\end{aligned}$$

In view of Lemma 8.1.1, we now know that the distribution of $\mathbb{R}^N$-valued Brownian motion induces a Borel measure $\mathcal{W}^{(N)}$ on the separable Banach space $\Theta(\mathbb{R}^N)$, and throughout this chapter I will refer to this measure as the **classical Wiener measure**.

My next goal is to characterize, in terms of $\Theta(\mathbb{R}^N)$, exactly which measure on $\Theta(\mathbb{R}^N)$ Wiener's is, and for this purpose I will use that following simple fact about Borel probability measures on a separable Banach space.

LEMMA 8.1.2. *Let E with norm $\|\cdot\|_E$ be a separable, real Banach space, and use*

$$(x,x^*)\in E\times E^* \longmapsto \langle x,x^*\rangle\in\mathbb{R}$$

to denote the duality relation between E and its dual space E^*. Then the Borel field $\mathcal{B}_E$ coincides with the σ-algebra generated by the maps $x\in E\longmapsto\langle x,x^*\rangle$ as x^* runs over E^*. In particular, if, for $\mu\in\mathbf{M}_1(E)$, one defines its **Fourier transform** *$\hat\mu: E^*\longrightarrow\mathbb{C}$ by*

$$\hat\mu(x^*) = \int_E \exp\Big[\sqrt{-1}\,\langle x,x^*\rangle\Big]\,\mu(dx), \quad x^*\in E^*,$$

then $\hat\mu$ is a continuous function of weak convergence on Θ^*, and $\hat\mu$ uniquely determines μ in the sense that if ν is a second element of $\mathbf{M}_1(\Theta)$ and $\hat\mu=\hat\nu$ then $\mu=\nu$.*

PROOF: Since it is clear that each of the maps $x\in E\longmapsto\langle x,x^*\rangle\in\mathbb{R}$ is continuous and therefore $\mathcal{B}_E$-measurable, the first assertion will follow as soon as we show that the norm $x \rightsquigarrow \|x\|_E$ can be expressed as a measurable function of these maps. But, because E is separable, we know (cf. Exercise 5.1.19) that the closed unit ball $\overline{B_{E^*}(0,1)}$ in E^* is separable with respect to the weak* topology and therefore that we can find a sequence $\{x_n^* :, n \ge 1\} \subseteq \overline{B_{E^*}(0,1)}$ so that

$$\|x\|_\Theta = \sup_{n\in\mathbb{Z}^+} \langle x, x_n^*\rangle, \quad x \in E.$$

Turning to the properties of $\hat{\mu}$, note that its continuity with respect to weak* convergence is an immediate consequence of Lebesgue's Dominated Convergence Theorem. Furthermore, in view of the preceding, we will know that $\hat{\mu}$ completely determines μ as soon as we show that, for each $n \in \mathbb{Z}^+$ and $X^* = \bigl(x_1^*, \dots, x_n^*\bigr) \in \bigl(E^*\bigr)^n$, $\hat{\mu}$ determines the marginal distribution $\mu_{X^*} \in \mathbf{M}_1(\mathbb{R}^N)$ of

$$x \in E \longmapsto \Bigl(\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle\Bigr) \in \mathbb{R}^n$$

under μ. But this is clear (cf. Lemma 2.3.3), since

$$\widehat{\mu_{X^*}}(\boldsymbol{\xi}) = \hat{\mu}\left(\sum_{m=1}^n \xi_m x_m^*\right) \quad \text{for } \boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \quad \square$$

I will now compute the Fourier transform of $\mathcal{W}^{(N)}$. To this end, first recall that, for an $\mathbb{R}^N$-valued Brownian motion, $\bigl\{\bigl(\boldsymbol{\xi}, \mathbf{B}(t)\bigr)_{\mathbb{R}^N} : t \ge 0 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N\bigr\}$ spans a Gaussian family $\mathfrak{G}(\mathbf{B})$ in $L^2(\mathbb{P};\mathbb{R})$. Hence, $\operatorname{span}\bigl(\bigl\{\bigl(\boldsymbol{\xi}, \boldsymbol{\theta}(t)\bigr) : t \ge 0 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N\bigr\}\bigr)$ is a Gaussian family in $L^2(\mathcal{W}^{(N)};\mathbb{R})$. From this, combined with an easy limit argument using Riemann sum approximations, one sees that, for any $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, $\boldsymbol{\theta} \rightsquigarrow \langle \boldsymbol{\theta}, \boldsymbol{\lambda}\rangle$ is a centered Gaussian random variable under $\mathcal{W}^{(N)}$. Furthermore, because, for $0 \le s \le t$,

$$\mathbb{E}^{\mathcal{W}^{(N)}}\bigl[\bigl(\boldsymbol{\xi}, \boldsymbol{\theta}(s)\bigr)_{\mathbb{R}^N}\bigl(\boldsymbol{\eta}, \boldsymbol{\theta}(t)\bigr)_{\mathbb{R}^N}\bigr] = \mathbb{E}^{\mathcal{W}^{(N)}}\bigl[\bigl(\boldsymbol{\xi}, \boldsymbol{\theta}(s)\bigr)_{\mathbb{R}^N}\bigl(\boldsymbol{\eta}, \boldsymbol{\theta}(s)\bigr)_{\mathbb{R}^N}\bigr] = s\bigl(\boldsymbol{\xi}, \boldsymbol{\eta}\bigr)_{\mathbb{R}^N},$$

we can apply Fubini's Theorem to see that

$$\mathbb{E}^{\mathcal{W}^{(N)}}\bigl[\langle \boldsymbol{\theta}, \boldsymbol{\lambda}\rangle^2\bigr] = \iint_{[0,\infty)^2} s \wedge t \, \boldsymbol{\lambda}(ds) \cdot \boldsymbol{\lambda}(dt).$$

Therefore, we now know that $\mathcal{W}^{(N)}$ is characterized by its Fourier transform

$$\widehat{\mathcal{W}^{(N)}}(\boldsymbol{\lambda}) = \exp\left[-\frac{1}{2}\iint_{[0,\infty)^2} s \wedge t \, \boldsymbol{\lambda}(ds) \cdot \boldsymbol{\lambda}(dt)\right], \quad \boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*. \tag{8.1.3}$$

Equivalently, we have shown that $\mathcal{W}^{(N)}$ is the centered Gaussian measure on $\Theta(\mathbb{R}^N)$ with the property that, for each $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, $\boldsymbol{\theta} \rightsquigarrow \langle \boldsymbol{\theta}, \boldsymbol{\lambda}\rangle$ is a centered Gaussian random variable with variance equal to $\iint_{[0,\infty)^2} s \wedge t \, \boldsymbol{\lambda}(ds) \cdot \boldsymbol{\lambda}(dt)$.

§ **8.1.2. The Classical Cameron–Martin Space.** From the Gaussian standpoint, it is extremely unfortunate that the natural home for Wiener measure is a Banach space rather than a Hilbert space. Indeed, in finite dimensions, every centered, Gaussian measure with non-degenerate covariance can be thought as the canonical, or standard, Gaussian measure on a Hilbert space. Namely, if $\gamma_{\mathbf{0},\mathbf{C}}$ is the Gaussian measure on $\mathbb{R}^N$ with mean $\mathbf{0}$ and non-degenerate covariance $\mathbf{C}$, consider $\mathbb{R}^N$ as a Hilbert space H with inner product $(\mathbf{g},\mathbf{h})_H = (\mathbf{g},\mathbf{Ch})_{\mathbb{R}^N}$, and take λ_H to be the natural Lebesgue measure there: the one which assigns measure 1 to a unit cube in H or, equivalently, the one obtained by pushing the usual Lebesgue measure $\lambda_{\mathbb{R}^N}$ forward under the linear transformation $\mathbf{C}^{\frac{1}{2}}$. Then we can write

$$\gamma_{\mathbf{0},\mathbf{C}}(d\mathbf{h}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{\|\mathbf{h}\|_H^2}{2}} \lambda_H(d\mathbf{h})$$

and

$$\widehat{\gamma_{\mathbf{0},\mathbf{C}}}(\mathbf{h}) = e^{-\frac{\|\mathbf{h}\|_H^2}{2}}.$$

As was already pointed out in Exercise 3.1.11, in infinite dimensions there is no precise analog of the preceding canonical respresentation (cf. Exercise 8.1.7 for further corroboration of this point). Nonetheless, a good deal of insight can be gained by seeing how close one can come. In order to guess on which Hilbert space it is that $\mathcal{W}^{(N)}$ would like to live, I will give R. Feynman's highly questionable but remarkably powerful way of thinking about such matters. Namely, given $n \in \mathbb{Z}^+$, $0 = t_0 < t_1 < \cdots < t_n$, and a set $A \in \left(\mathcal{B}_{\mathbb{R}^N}\right)^n$, we know that $\mathcal{W}^{(N)}$ assigns $\{\boldsymbol{\theta} : \left(\boldsymbol{\theta}(t_1),\dots,\boldsymbol{\theta}(t_n)\right) \in A\}$ probability

$$\frac{1}{Z(t_1,\dots,t_n)} \int_A \exp\left[-\sum_{m=1}^n \frac{|\mathbf{y}_m - \mathbf{y}_{m-1}|^2}{t_m - t_{m-1}}\right] d\mathbf{y}_1 \cdots d\mathbf{y}_n$$

where $\mathbf{y}_0 \equiv \mathbf{0}$ and $Z(t_1,\dots,t_n) = \prod_{m=1}^n \left(2\pi(t_m - t_{m_1})\right)^{\frac{N}{2}}$. Now rename the variable $\mathbf{y}_m$ as "$\boldsymbol{\theta}(t_m)$," and rewrite the preceding as $Z(t_1,\dots,t_n)^{-1}$ times

$$\int_A \exp\left[-\sum_{m=1}^n \frac{t_m - t_{m-1}}{2}\left(\frac{\left|\boldsymbol{\theta}(t_m) - \boldsymbol{\theta}(t_{m-1})\right|}{t_m - t_{m-1}}\right)^2\right] d\boldsymbol{\theta}(t_1)\cdots d\boldsymbol{\theta}(t_n).$$

Obviously, nothing very significant has happened yet since nothing very exciting has been done yet. However, if we now close our eyes, suspend our disbelief, and *pass to the limit* as n tends to infinity and the t_k's become dense, we arrive at the *Feynman's representation** of Wiener's measure:

$$\mathcal{W}^{(N)}(d\boldsymbol{\theta}) = \frac{1}{Z}\exp\left[-\frac{1}{2}\int_{[0,\infty)} \left|\dot{\boldsymbol{\theta}}(t)\right|^2 dt\right] d\boldsymbol{\theta}, \tag{8.1.4}$$

* In truth, Feynman himself never dabbled in considerations so mundane as the ones which follow. He was interested in the Schödinger equation, and so he had a factor $\sqrt{-1}$ multiplying the exponent.

where $\dot{\boldsymbol{\theta}}$ denotes the velocity (i.e., derivative) of $\boldsymbol{\theta}$. Of course, when we reopen our eyes and take a look at (8.1.4), we see that it is riddled with flaws. Not even one of the ingredients on the right-hand side (8.1.4) makes sense! In the first place, the constant Z must be 0 (or maybe ∞). Secondly, since the image of the "measure $d\boldsymbol{\theta}$" under

$$\boldsymbol{\theta} \in \Theta(\mathbb{R}^N) \longmapsto \big(\boldsymbol{\theta}(t_1)\dots,\boldsymbol{\theta}(t_n)\big) \in \big(\mathbb{R}^N\big)^n$$

is Lebesgue measure for every $n \in \mathbb{Z}^+$ and $0 < t_1 \cdots < t_n$, "$d\boldsymbol{\theta}$" must be the nonexistent *translation invariant measure* on the infinite dimensional space $\Theta(\mathbb{R}^N)$. Finally, the integral in the exponent only makes sense if $\boldsymbol{\theta}$ is differentiable in some sense, but almost no Brownian path is. Nonetheless, ridiculous as it is, (8.1.4) is exactly the expression at which one would arrive if one were to make a sufficiently naïve interpretation of the notion that Wiener measure is the standard Gauss measure on the Hilbert space $\mathbf{H}(\mathbb{R}^N)$ consisting of absolutely continuous $\mathbf{h} : [0,\infty) \longrightarrow \mathbb{R}^N$ with $\mathbf{h}(0) = \mathbf{0}$ and

$$\|\mathbf{h}\|_{\mathbf{H}^1(\mathbb{R}^N)} = \|\dot{\mathbf{h}}\|_{L^2([0,\infty);\mathbb{R}^N)} < \infty.$$

Of course, the preceding discussion is entirely heuristic. However, now we know that $\mathbf{H}^1(\mathbb{R}^N)$ is the Hilbert space at which to look, it is easy to provide a mathematically rigorous statement of the connection between $\Theta(\mathbb{R}^N)$, $\mathcal{W}^{(N)}$, and $\mathbf{H}^1(\mathbb{R}^N)$. To this end, observe that $\mathbf{H}(\mathbb{R}^N)$ is continuously embedded in $\Theta(\mathbb{R}^N)$ as a dense subspace. Indeed, if $\mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N)$, then $|\mathbf{h}(t)| \le t^{\frac{1}{2}}\|\mathbf{h}\|_{\mathbf{H}^1(\mathbb{R}^N)}$, and so not only is $\mathbf{h} \in \Theta(\mathbb{R}^N)$ but also $\|\mathbf{h}\|_{\Theta(\mathbb{R}^N)} \le \frac{1}{2}\|\mathbf{h}\|_{\mathbf{H}^1(\mathbb{R}^N)}$. In addition, since $C_c^\infty\big((0,\infty);\mathbb{R}^N\big)$ is already dense in $\Theta(\mathbb{R}^N)$, the density of $\mathbf{H}^1(\mathbb{R}^N)$ in $\Theta(\mathbb{R}^N)$ is clear. Knowing this, abstract reasoning (cf. Lemma 8.2.3) guarantees that $\Theta(\mathbb{R}^N)^*$ can be identified as a subspace of $\mathbf{H}^1(\mathbb{R}^N)$. That is, for each $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, there is a $\mathbf{h}_{\boldsymbol{\lambda}} \in \mathbf{H}^1(\mathbb{R}^N)$ with the property that $\big(\mathbf{h},\mathbf{h}_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^1(\mathbb{R}^N)} = \langle \mathbf{h},\boldsymbol{\lambda}\rangle$ for all $\mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N)$, and in the present setting it is a easy to give a concrete representation of $\mathbf{h}_{\boldsymbol{\lambda}}$. In fact, if $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, then, for any $\mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N)$,

$$\begin{aligned}\langle \mathbf{h},\boldsymbol{\lambda}\rangle &= \int_{(0,\infty)} \mathbf{h}(t)\cdot\boldsymbol{\lambda}(dt) = \int_{(0,\infty)} \left(\int_{(0,t)} \dot{\mathbf{h}}(\tau)\,d\tau\right)\cdot\boldsymbol{\lambda}(dt)\\ &= \int_{(0,\infty)} \dot{\mathbf{h}}(\tau)\cdot\boldsymbol{\lambda}\big((\tau,\infty)\big)\,d\tau = \big(\mathbf{h},\mathbf{h}_\lambda\big)_{\mathbf{H}^1(\mathbb{R}^N)},\end{aligned}$$

where

$$\mathbf{h}_{\boldsymbol{\lambda}}(t) = \int_{(0,t]} \boldsymbol{\lambda}\big((\tau,\infty)\big)\,d\tau.$$

Moreover,

$$\|\mathbf{h}_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}^1(\mathbb{R}^N)} = \int_{(0,\infty)} \big|\boldsymbol{\lambda}\big((\tau,\infty)\big)\big|^2\,d\tau = \int_{(0,\infty)} \left(\iint\limits_{(\tau,\infty)^2} \boldsymbol{\lambda}(ds)\cdot\boldsymbol{\lambda}(dt)\right)d\tau$$
$$= \iint\limits_{(0,\infty)^2} s\wedge t\,\boldsymbol{\lambda}(ds)\cdot\boldsymbol{\lambda}(dt).$$

Hence, by (8.1.3),

$$\widehat{\mathcal{W}^{(N)}}(\boldsymbol{\lambda}) = \exp\left(-\frac{\|\mathbf{h}_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}(\mathbb{R}^N)}}{2}\right), \quad \boldsymbol{\lambda}\in\Theta(\mathbb{R}^N)^*. \tag{8.1.5}$$

Although (8.1.5) is far less intuitively appealing than (8.1.4), it provides a mathematically rigorous way in which to think of $\mathcal{W}^{(N)}$ as the standard Gaussian measure on $\mathbf{H}^1(\mathbb{R}^N)$. Furthermore, there is a is another way to understand why one should accept (8.1.5) as evidence for this way of thinking about $\mathcal{W}^{(N)}$. Indeed, given $\boldsymbol{\lambda}\in\Theta(\mathbb{R}^N)^*$, write

$$\langle\boldsymbol{\theta},\boldsymbol{\lambda}\rangle = \lim_{T\to\infty}\int_{[0,T]}\boldsymbol{\theta}(t)\cdot\boldsymbol{\lambda}(dt) = -\lim_{T\to\infty}\int_0^T \boldsymbol{\theta}(t)\cdot d\boldsymbol{\lambda}\big((t,\infty)\big),$$

where the integral in the last expression is taken in the sense of Riemann-Stieljes. Next, apply the integration by part formula* to conclude that $t\rightsquigarrow\boldsymbol{\lambda}\big((t,\infty)\big)$ is Riemann–Stieljes integrable with respect to $t\rightsquigarrow\boldsymbol{\theta}(t)$ and that

$$-\int_0^T \boldsymbol{\theta}(t)\cdot d\boldsymbol{\lambda}\big((t,\infty)\big) = -\boldsymbol{\theta}(T)\cdot\boldsymbol{\lambda}\big((T,\infty)\big) + \int_0^T \boldsymbol{\lambda}\big((t,\infty)\big)\cdot d\boldsymbol{\theta}(t).$$

Hence, since

$$\lim_{T\to\infty}|\boldsymbol{\theta}(T)||\boldsymbol{\lambda}|(T,\infty) \le \lim_{T\to\infty}\frac{|\boldsymbol{\theta}(T)|}{1+T}\int_{(0,\infty)}(1+t)\,|\boldsymbol{\lambda}|(dt) = 0,$$

$$\langle\boldsymbol{\theta},\boldsymbol{\lambda}\rangle = \lim_{T\to\infty}\int_0^T \dot{\mathbf{h}}_{\boldsymbol{\lambda}}(t)\cdot d\boldsymbol{\theta}(t), \tag{8.1.6}$$

where again the integral is in the sense of Riemann–Stieljes. Thus, if one somewhat casually writes $d\boldsymbol{\theta}(t) = \dot{\boldsymbol{\theta}}(t)\,dt$, one can believe that $\langle\boldsymbol{\theta},\boldsymbol{\lambda}\rangle$ provides a reasonable interpretation of $\big(\boldsymbol{\theta},\mathbf{h}_{\boldsymbol{\lambda}}\big)_{\mathbf{H}(\mathbb{R}^N)}$ for all $\boldsymbol{\theta}\in\Theta(\mathbb{R}^N)$, not just those which are in $\mathbf{H}^1(\mathbb{R}^N)$.

Because R. Cameron and T. Martin were the first mathematicians to systematically exploit the consequences of this line of reasoning, I will call $\mathbf{H}^1(\mathbb{R}^N)$ the **Cameron–Martin space** for classical Wiener measure.

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* See, for example, Theorem 1.2.7 in my *A Concise Introduction to the Theory of Integration* published by Birkhäuser (3rd edition, 1999).

Exercises for §8.1

EXERCISE 8.1.7. Let H be a separable Hilbert space, and, for each $n \in \mathbb{Z}^+$ and subset $\{g_1, \dots, g_n\} \subseteq H$, let $\mathcal{A}(g_1, \dots, g_n)$ denote the σ-algebra over H generated by the mapping

$$h \in H \longmapsto \big((h, g_1)_H, \dots, (h, g_n)_H\big) \in \mathbb{R}^n,$$

and check that

$$\mathcal{A} = \bigcup \big\{\mathcal{A}(g_1, \dots, g_n) : n \in \mathbb{Z}^+ \text{ and } g_1, \dots, g_n \in H\big\}$$

is an algebra which generates $\mathcal{B}_H$. Show that there *always* exists a *finitely additive* $\mathcal{W}_H$ on $\mathcal{A}$ which is uniquely determined by the properties that it is σ-additive on $\mathcal{A}(g_1, \dots, g_n)$ for every $n \in \mathbb{Z}^+$ and $\{g_1, \dots, g_n\} \subseteq H$ and that

$$\int_H \exp\Big[\sqrt{-1}\,(h, g)_H\Big]\,\mathcal{W}_H(dh) = \exp\left[-\frac{\|g\|_H^2}{2}\right], \quad g \in H.$$

On the other hand, as we already know, this finitely additive measure admits a countably additive extension to $\mathcal{B}_H$ if and only if H is finite dimensional.

§8.2 A Structure Theorem for Gaussian Measures

Say that a centered Gaussian measure $\mathcal{W}$ on a separable Banach space E is **non-degenerate** if $\mathbb{E}^{\mathcal{W}}\big[\langle x, x^*\rangle^2\big] > 0$ unless $x^* = 0$. In this section I will show that any non-degenerate, centered Gaussian measure $\mathcal{W}$ on a separable Banach space E shares the same basic structure as $\mathcal{W}^{(N)}$ has on $\Theta(\mathbb{R}^N)$. In particular, I will show that there is always a Hilbert space $H \subseteq E$ for which $\mathcal{W}$ is the standard Gauss measure in the same sense as $\mathcal{W}^{(N)}$ was shown in § 8.1.2 to be the standard Gauss measure for $\mathbf{H}^1(\mathbb{R}^N)$.

§8.2.1. Fernique's Theorem.

In order to carry out my program, I need a basic integrability result about Banach space valued, Gaussian random variables. The one which I will use is due to X. Fernique, and his is arguably the most singularly beautiful result in the theory of Gaussian measures on a Banach space.

THEOREM 8.2.1 (**Fernique's Theorem**). *Let E be a real, separable Banach space, and suppose that X is an E-valued random variable which a centered and Gaussian in the sense that, for each $x^* \in E^*$, $\langle X, x^*\rangle$ is a centered, $\mathbb{R}$-valued Gaussian random variable. If $R = \inf\{r : \mathbb{P}(\|X\|_E \le e) \ge \frac{3}{4})\}$, then*

$$\mathbb{E}\Big[e^{\frac{\|X\|_E^2}{18R^2}}\Big] \le K \equiv e^{\frac{1}{2}} + \sum_{n=0}^{\infty}\left(\frac{e}{3}\right)^{2^n}. \tag{8.2.2}$$

(See Corollary 8.4.3 below for a sharpened statement.)

PROOF: After enlarging the sample space if necessary, I may and will assume that there is an independent, E-valued random variable X' with the same distribution as X. Set $Y = 2^{-\frac{1}{2}}(X + X')$ and $Y' = 2^{-\frac{1}{2}}(X - X')$. Then the pair (Y, Y') has the same distribution as the pair (X, X'). Indeed, this comes down to showing that the $\mathbb{R}^2$-valued random variable $\big(\langle Y, x^*\rangle, \langle Y', x^*\rangle\big)$ has the same distribution as $\big(\langle X, x^*\rangle, \langle X', x^*\rangle\big)$, and that is an elementary application of the additivity property of independent Gaussians.

Turning to the main assertion, let $0 < s \le t$ be given, and use the preceding to justify

$$
\begin{aligned}
&\mathbb{P}\big(\|X\|_E \le s\big)\mathbb{P}\big(\|X\|_E \ge t\big) = \mathbb{P}\big(\|X\|_E \le s \ \&\ \|X'\|_E \ge t\big)\\
&\quad = \mathbb{P}\big(\|X - X'\|_E \le 2^{\frac{1}{2}}s \ \&\ \|X + X'\|_E \ge 2^{\frac{1}{2}}t\big)\\
&\quad \le \mathbb{P}\big(\big|\|X\|_E - \|X'\|_E\big| \le 2^{\frac{1}{2}}s \ \&\ \|X\|_E + \|X'\|_E \ge 2^{\frac{1}{2}}t\big)\\
&\quad \le \mathbb{P}\big(\|X\|_E \wedge \|X'\|_E \ge 2^{-\frac{1}{2}}(t-s)\big) = \mathbb{P}\big(\|X\|_E \ge 2^{-\frac{1}{2}}(t-s)\big)^2.
\end{aligned}
$$

Now suppose that $\mathbb{P}\big(\|X\| \le R\big) \ge \frac{3}{4}$, and define $\{t_n : n \ge 0\}$ by $t_0 = R$ and $t_n = R + 2^{\frac{1}{2}}t_{n-1}$ for $n \ge 1$. Then

$$
\mathbb{P}\big(\|X\|_E \le R\big)\mathbb{P}\big(\|X\|_E \ge t_n\big) \le \mathbb{P}\big(\|X\|_E \ge t_{n-1}\big)^2
$$

and therefore

$$
\frac{\mathbb{P}\big(\|X\|_E \ge t_n\big)}{\mathbb{P}\big(\|X\|_E \le R\big)} \le \left(\frac{\mathbb{P}\big(\|X\|_E \ge t_{n-1}\big)}{\mathbb{P}\big(\|X\|_E \le R\big)}\right)^2
$$

for $n \ge 1$. Working by induction, one gets from this that

$$
\frac{\mathbb{P}\big(\|X\|_E \ge t_n\big)}{\mathbb{P}\big(\|X\|_E \le R\big)} \le \left(\frac{\mathbb{P}\big(\|X\|_E \ge R\big)}{\mathbb{P}\big(\|X\|_E \le R\big)}\right)^{2^n}
$$

and therefore, since $t_n = R\frac{2^{\frac{n+1}{2}}-1}{2^{\frac{1}{2}}-1} \le 32^{\frac{n+1}{2}}R$, that $\mathbb{P}\big(\|X\|_E \ge 32^{\frac{n+1}{2}}R\big) \le 3^{-2^n}$. Hence,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[e^{\frac{\|X\|_E^2}{18R^2}}\Big] &\le e^{\frac{1}{2}}\mathbb{P}\big(\|X\|_E \le 3R\big) + \sum_{n=0}^{\infty} e^{2^n}\,\mathbb{P}\big(32^{\frac{n}{2}}R \le \|X\|_E \le 32^{\frac{n+1}{2}}R\big)\\
&\le e^{\frac{1}{2}} + \sum_{n=0}^{\infty}\left(\frac{e}{3}\right)^{2^n} = K. \quad \square
\end{aligned}
$$

§**8.2.2. The Basic Structure Theorem.** I will now abstract the relationship, proved in § 8.1.2, between $\Theta(\mathbb{R}^N)$, $\mathbf{H}^1(\mathbb{R}^N)$, and $\mathcal{W}^{(N)}$, and for this purpose I will need the following simple lemma.

LEMMA 8.2.3. *Let E be a separable, real Banach space, and suppose that $H \subseteq E$ is a real Hilbert space which is continuously embedded as a dense subspace of E.*

(i) *For each $x^* \in E^*$ there is a unique $h_{x^*} \in H$ with the property that $\big(h, h_{x^*}\big)_H = \langle h, x^*\rangle$ for all $h \in H$, and the map $x^* \in E^* \longmapsto h_{x^*} \in H$ is linear, continuous, one-to-one, onto a dense subspace of H.*

(ii) *If $x \in E$, then $x \in H$ if and only if there is a $K < \infty$ such that $|\langle x, x^*\rangle| \le K\|h_{x^*}\|_H$ for all $x^* \in E^*$. Moreover, for each $h \in H$, $\|h\|_H = \sup\{\langle h, x^*\rangle : x^* \in E^* \;\&\; \|x^*\|_{E^*} \le 1\}$.*

(iii) *If L^* is a subspace of E^*, then there exists a sequence $\{x_n^* : n \ge 0\} \subseteq L^*$ such that $\{h_{x_n} : n \ge 0\}$ is an orthonormal basis for H. Moreover, if $x \in E$, then $x \in H$ if and only if $\sum_{n=0}^\infty \langle x, x_n^*\rangle^2 < \infty$, and*

$$\big(h, h'\big)_H = \sum_{n=0}^\infty \langle h, x_n^*\rangle\langle h', x_n^*\rangle \quad \text{for all } h, h' \in H.$$

PROOF: Because H is continuous embedded in E, there exists a $C < \infty$ such that $\|h\|_E \le C\|h\|_H$. Thus, if $x^* \in E^*$ and $f(h) = \langle h, x^*\rangle$, then f is linear and $|f(h)| \le \|h\|_E\|x^*\|_{E^*} \le C\|x^*\|_{E^*}\|h\|_H$, and so, by the Riesz Representation Theorem for Hilbert spaces, there exists a unique $h_{x^*} \in H$ such that $f(h) = \big(h, h_{x^*}\big)_H$. In fact, $\|h_{x^*}\|_H \le C\|x^*\|_{E^*}$, and uniqueness can be used to check that $x^* \rightsquigarrow h_{x^*}$ is linear. To see that $x^* \rightsquigarrow h_{x^*}$ is one-to-one, it suffices to show that $x^* = 0$ if $h_{x^*} = 0$. But if $h_{x^*} = 0$, then $\langle h, x^*\rangle = 0$, for all $h \in H$, and therefore, because H is dense in E, $x^* = 0$. Because I will use it below, I will prove slightly more than the density of just $\{h_{x^*} : x^* \in E^*\}$ in H. Namely, for any weak* dense subset S^* of E^*, $\{h_{x^*} : x* \in S^*\}$ is dense in H. Indeed, if this were not the case, then there would exist an $h \in H \setminus \{0\}$ with the property that $\langle h, x^*\rangle = \big(h, h_{x^*}\big)_H = 0$ for all $x^* \in S$. But, since S^* is weak* dense in E^*, this would lead to the contradiction that $h = 0$. Thus, (**i**) is now proved.

Obviously, if $h \in H$, then $|\langle h, x^*\rangle| = |(h, h_{x^*})_H| \le \|h_{x^*}\|_H\|h\|_H$ for $x^* \in E^*$. Conversely, if $x \in E$ and $|\langle x, x^*\rangle| \le K\|h_{x^*}\|_H$ for some $K < \infty$ and all $x^* \in E^*$, set $f(h_{x^*}) = \langle x, x^*\rangle$ for $x^* \in E^*$. Then, because $x^* \rightsquigarrow h_{x^*}$ is one-to-one, f is a well-defined, linear functional on $\{h_{x^*} : x^* \in E^*\}$. Moreover, $|f(x^*)| \le K\|h_{x^*}\|_H$, and therefore, since $\{h_{x^*} : x^* \in E^*\}$ is dense, f admits a unique extension as a continuous, linear functional on H. Hence, by Riesz's theorem, there is an $h \in H$ such that

$$\langle x, x^*\rangle = f(h_{x^*}) = \big(h, h_{x^*}\big)_H = \langle h, x^*\rangle, \quad x^* \in E^*,$$

which means that $x = h \in H$. In addition, if $h \in H$, then $\|h\|_H = \sup\{\langle h, x^*\rangle : \|h_{x^*}\|_H\}$ follows from the density of $\{h_{x^*} : x^* \in E^*\}$, and this completes the proof of (**ii**).

Turning to (**iii**), remember that, by Exercise 5.1.19, the weak* topology on E^* is second countable. Hence, the weak* topology on L^* is also second countable and therefore separable. Thus, we can find a sequence in L^* which is weak* dense in E^*, and then, proceeding as in the hint given for that exercise, extract a subsequence of linearly independent elements whose span S^* is weak* dense in E^*. Starting with this subsequence, apply the Grahm–Schmidt orthogonalization procedure to produce a sequence $\{x_n^* : n \geq 0\}$ whose span is S^* and for which $\{h_{x_n^*} : n \geq 0\}$ is orthonormal in H. Moreover, because the span of $\{h_{x_n^*} : n \geq 0\}$ equals $\{h_{x^*} : x^* \in S^*\}$, which, by what we proved earlier, is dense in H, $\{h_{x_n^*} : n \geq 0\}$ is an orthonormal basis in H. Knowing this, it is immediate that

$$\big(h, h'\big)_H = \sum_{n=0}^{\infty} \big(h, h_{x_n}\big)_H \big(h', h_{x_n}\big)_H = \sum_{n=0}^{\infty} \langle h, x_n^* \rangle \langle h', x_n^* \rangle.$$

In particular, $\|h\|_H^2 = \sum_{n=0}^{\infty} \langle h, x_n^* \rangle^2$. Finally, if $x \in E$ and $\sum_{n=0}^{\infty} \langle x, x_n^* \rangle^2 < \infty$, set $g = \sum_{m=0} \langle x, x_n^* \rangle h_{x_n^*}$. Then $g \in H$ and $\langle x - g, x^* \rangle = 0$ for all $x^* \in S^*$. Hence, since S^* is weak* dense in E^*, $x = g \in H$. $\square$

Given a separable real Hilbert space H, a separable real Banach space E, and a $\mathcal{W} \in \mathbf{M}_1(E)$, I will say that the triple $(H, E, \mathcal{W})$ is an **abstract Wiener space** if H is continuously embedded as a dense subspace of E and $\mathcal{W} \in \mathbf{M}_1(E)$ has Fourier transform

$$\widehat{\mathcal{W}}(x^*) = e^{-\frac{\|h_{x^*}\|_H^2}{2}} \quad \text{for all } x^* \in E^*. \tag{8.2.4}$$

The terminology is justified by the fact, demonstrated at the end of § 8.1.2, that $\big(\mathbf{H}^1(\mathbb{R}^N), \Theta(\mathbb{R}^N), \mathcal{W}^{(N)}\big)$ is an abstract Wiener space. The concept of an abstract Wiener space was introduced by Gross, although his description was somewhat different from the one just given (cf. Theorem 8.3.9) for a reconcilation of mine with his definition.)

Theorem 8.2.5. *Suppose that E is a separable, real Banach space and that $\mathcal{W} \in \mathbf{M}_1(E)$ is a centered Gaussian measure which is non-degenerate. Then there exists a unique Hilbert space H such that $(H, E, \mathcal{W})$ is an abstract Wiener space.*

Proof: By Fernique's Theorem, we know that $C \equiv \sqrt{\mathbb{E}^{\mathcal{W}}\big[\|x\|_E^2\big]} < \infty$.

To understand the proof of existence, it is best to start with the proof of uniqueness. Thus, suppose that H is a Hilbert space for which $(E, H, \mathcal{W})$ is an abstract Wiener space. Then, for all x^*, $y^* \in E^*$, $\langle h_{x^*}, y^* \rangle = (h_{x^*}, h_{y^*})_H = \langle h_{y^*}, x^* \rangle$. In addition,

$$\langle h_{x^*}, x^* \rangle = \|h_{x^*}\|_H^2 = \int \langle x, x^* \rangle^2 \, \mathcal{W}(dx),$$

and so, by the symmetry just established,

$$(*)\qquad \langle h_{x^*}, y^*\rangle = \|h_{x^*}\|_H^2 = \int \langle x, x^*\rangle\langle x, y^*\rangle\, \mathcal{W}(dx),$$

for all x^*, $y^* \in E^*$. Next observe that

$$(**)\qquad \int \big\|\langle x, x^*\rangle\, x\big\|_E\, \mathcal{W}(dx) \le C\|h_{x^*}\|_H,$$

and therefore that the integral $\int x\langle x, x^*\rangle\, \mathcal{W}(dx)$ is a well defined element of E. Moreover, by (*)

$$\langle h_{x^*}, y^*\rangle = \left\langle \int x\langle x, x^*\rangle\, \mathcal{W}(dx)\right\rangle \quad \text{for all } y^* \in E^*,$$

and so

$$(***)\qquad h_{x^*} = \int x\langle x, x^*\rangle\, \mathcal{W}(dx).$$

Finally, given $h \in H$, choose $\{x_n^* : n \ge 1\} \subseteq E^*$ so that $h_{x_n^*} \longrightarrow h$ in H. Then

$$\lim_{m\to\infty} \sup_{n>m} \big\|\langle\,\cdot\,, x_n^*\rangle - \langle\,\cdot\,, x_m^*\rangle\big\|_{L^2(\mathcal{W};\mathbb{R})} = \lim_{m\to\infty} \sup_{n>m} \|h_{x_n^*} h_{x_m^*}\|_H = 0,$$

and so, if Ψ denotes the closure of $\{\langle\,\cdot\,, x^*\rangle : x^* \in E^*\}$ in $L^2(\mathcal{W};\mathbb{R})$ and $F : \Psi \longrightarrow E$ is given by

$$F(\psi) = \int x\psi(x)\, \mathcal{W}(dx), \quad \psi \in \Psi,$$

then $h = F(\psi)$. Conversely, if $\psi \in \Psi$ and $\{x_n^* : n \ge 1\}$ is chosen so that $\langle\,\cdot\,, x_n^*\rangle \longrightarrow \psi$ in $L^2(\mathcal{W};\mathbb{R})$, then $\{h_{x_n^*} : n \ge 1\}$ converges in H to some $h \in H$ and it converges in E to $F(\psi)$. Hence, $F(\psi) = h \in H$. In other words, $H = F(\Psi)$.

The proof of existence is now a matter of checking that if Ψ and F are defined as above and if $H = F(\Psi)$ with $\|F(\psi)\|_H = \|\psi\|_{L^2(\mathcal{W};\mathbb{R})}$, then $(H, E, \mathcal{W})$ is an abstract Wiener space. To this end,

$$\langle F(\psi), x*\rangle = \int \langle x, x^*\rangle\psi(x)\, \mathcal{W}(dx) = \big(F(\psi), h_{x^*}\big)_H,$$

and therefore both (*) and (***) hold for this choice of H. Further, given (*), it is clear that $\|h_{x^*}\|_H^2$ is the variance of $\langle\,\cdot\,, x^*\rangle$ and therefore that (8.2.4) holds. At the same time, just as in the derivation of (**), $\|F(\psi)\|_E \le C\|\psi\|_{L^2(\mathcal{W};\mathbb{R})} = C\|F(\psi)\|_H$, and so H is continuously embedded inside E. Finally, by the Hahn–Banach Theorem, to show that H is dense in E it suffices to check the only $x^* \in E^*$ such that $\langle F(\psi), x^*\rangle = 0$ for all $\psi \in \Psi$ is $x^* = 0$. But when $\psi = \langle\,\cdot\,, x^*\rangle$, $\langle F(\psi), x^*\rangle = \int \langle x, x^*\rangle^2\, W(dx)$, and therefore, because $\mathcal{W}$ is non-degenate, such an x^* would have to be 0. $\square$

§**8.2.3. The Cameron–Marin Space.** Given a centered, non-degenerate Gaussian measure $\mathcal{W}$ on E, the Hilbert space H for which $(H, E, \mathcal{W})$ is an abstract Wiener space is called its **Cameron–Martin space**. Here are a couple of important properties of the Cameron–Martin subspace.

Theorem 8.2.6. *If $(H, E, \mathcal{W})$ is an abstract Wiener space, then the map $x^* \in E^* \longmapsto h_{x^*} \in H$ is continuous from the weak* topology on E^* into the strong topology on H. In particular, for each $R > 0$, $\{h_{x^*} : x^* \in \overline{B_{E^*}(0,R)}\}$ is a compact subset of H, $\overline{B_H(0,R)}$ is a compact subset of E, and so $H \in \mathcal{B}_E$. Moreover, when E is infinite dimensional $\mathcal{W}(H) = 0$. Finally, there is a unique linear, isometric map $\mathcal{I} : H \longrightarrow L^2(\mathcal{W};\mathbb{R})$ such that $\mathcal{I}(h_{x^*}) = \langle\,\cdot\,, x^*\rangle$ for all $x^* \in E^*$, and $\{\mathcal{I}(h) : h \in H\}$ is a Gaussian family in $L^2(\mathcal{W};\mathbb{R})$.*

Proof: To prove the intitial assertion, remember that $x^* \rightsquigarrow \widehat{\mathcal{W}}(x^*)$ is continuous with respect to the weak* topology. Hence, if $x_k^* \longrightarrow x^*$ in the weak* topology, then

$$\exp\left(-\frac{\|h_{x_k^*} - h_{x^*}\|_H^2}{2}\right) = \widehat{\mathcal{W}}(x_k^* - x^*) \longrightarrow 1,$$

and so $h_{x_k^*} \longrightarrow h_{x^*}$ in H.

Given the first assertion, the compactness of $\{h_{x^*} : x^* \in \overline{B_{E^*}(0,R)}\}$ in H follows from the compactness (cf. Exercise 5.1.19) of $\overline{B_{E^*}(0,R)}$ in the weak* topology. To see that $\overline{B_H(0,R)}$ is compact in E, again apply Exercise 5.1.19 to check that $\overline{B_H(0,R)}$ is compact in the weak topology on H. Therefore, all that we have to show is that the embedding map $h \in H \longmapsto h \in E$ is continuous from the weak topology on H into the strong topology on E. Thus, suppose that $h_k \longrightarrow h$ weakly in H. Because $\{h_{x^*} : x^* \in \overline{B_{E^*}(0,1)}\}$ is compact in H, for each $\epsilon > 0$ there exist an $n \in \mathbb{Z}^+$ and $\{x_1^*, \dots, x_n^*\} \subseteq \overline{B_{E^*}(0,1)}$ such that

$$\{h_{x^*} : x^* \in \overline{B_{E^*}(0,1)}\} \subseteq \bigcup_1^n B_H(h_{x_m^*}, \epsilon).$$

Now choose ℓ so that $\max_{1\le m\le n} |\langle h_k - h, x_m^*\rangle| < \epsilon$ for all $k \ge \ell$. Then, for any $x^* \in \overline{B_{E^*}(0,1)}$ and all $k \ge \ell$,

$$|\langle h_k - h, x^*\rangle| \le \epsilon + \min_{1\le m\le n}\left|\big(h_k - h, h_{x^*} - h_{x_m^*}\big)_H\right| \le \epsilon + 2\epsilon \sup_{k\ge 1} \|h_k\|_H.$$

Since, by the uniform boundedness principle, $\sup_{k\ge1} \|h_k\|_H < \infty$, this proves that $\|h_k - h\|_E = \sup\{\langle h_k - h, x^*\rangle : x^* \in \overline{B_{E^*}(0,1)}\} \longrightarrow 0$ as $k \to \infty$.

Because $H = \bigcup_1^\infty \overline{B_H(0,n)}$ and $\overline{B_H(0,n)}$ is a compact subset of E for each $n \in \mathbb{Z}^+$, it is clear that $H \in \mathcal{B}_E$. To see that $\mathcal{W}(H) = 0$ when E is infinite dimensional, choose $\{x_n^* : n \ge 0\}$ as in the final part of Lemma 8.2.3, and set $X_n(x) = \langle x, x_n^*\rangle$. Then the X_n's are an infinite sequence of independent, centered, Gaussians with mean value 1, and so, $\sum_{n=0}^\infty X_n^2 = \infty$ $\mathcal{W}$-almost surely. Hence, by Lemma 8.2.3, $\mathcal{W}$-almost no x is in H.

Turning to the map $\mathcal{I}$, define $\mathcal{I}(h_{x^*}) = \langle \,\cdot\,, x^* \rangle$. Then, for each x^*, $\mathcal{I}(h_{x^*})$ a centered Gaussian with variance $\|h_{x^*}\|_H^2$, and so $\mathcal{I}$ is a linear isometry from $\{h_{x^*} : x^* \in E^*\}$ into $L^2(\mathcal{W};\mathbb{R})$. Hence, since $\{h_{x^*} : x^* \in E^*\}$ is dense in H, $\mathcal{I}$ admits a unique extension as a linear isometry from H into $L^2(\mathcal{W};\mathbb{R})$. Moreover, as the $L^2(\mathcal{W};\mathbb{R})$-limit of centered Gaussians, $\mathcal{I}(h)$ is a centered Gaussian for each $h \in H$. $\square$

The map $\mathcal{I}$ in Theorem 8.2.6 was introduced for the classical Wiener space by Paley and Wiener, and so I will call it the **Payley–Wiener map**. To appreciate its importance here, recall the suggestion in § 8.1.2 that $\boldsymbol{\theta} \rightsquigarrow \langle \boldsymbol{\theta}, \boldsymbol{\lambda} \rangle$ be thought of as an extention of $\mathbf{h} \rightsquigarrow \big(\mathbf{h}, \mathbf{h}_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^1(\mathbb{R}^N)}$ to $\Theta(\mathbb{R}^N)$, and, in the same sense, think of $x \rightsquigarrow \langle x, x^* \rangle$ as an extention of the inner product $h \rightsquigarrow \big(h, h_{x^*}\big)_H$ to E. When one adopts this point of view, then $x \rightsquigarrow [\mathcal{I}(h)](x)$ can be interpreted as an extension of $(\,\cdot\,, h)_H$ to E, this time not just for $h \in \{h_{x^*} : x^* \in E^*\}$ but for any $h \in H$. Of course, when E is infinite dimensional, one has to be careful when using this interpretation, since for general $h \in H$, $\mathcal{I}(h)$ is defined only up to a $\mathcal{W}$-null set. Nonetheless, by adopting it, one gets further evidence for the idea that $\mathcal{W}$ wants to be the standard Gauss measure on H. Namely, because

$$\mathbb{E}^{\mathcal{W}}\big[e^{\sqrt{-1}\,\mathcal{I}(h)}\big] = e^{-\frac{\|h\|_H^2}{2}}, \quad h \in H, \tag{8.2.7}$$

if $\mathcal{W}$ lived on H, then it would certainly be the standard Gauss measure there.

Perhaps the most important application of the Paley–Wiener map is the following theorem about the behavior of Gaussian measures under translation. That is, if $y \in E$ and $\tau_y : E \longrightarrow E$ is given by $\tau_y(x) = x + y$, we will be looking at the measure $(\tau_y)_*\mathcal{W}$ and its relationship to $\mathcal{W}$. Using the reasoning suggested above, the result is easy to guess. Namely, if $\mathcal{W}$ really lived on H and were given by a Feynman-type representation

$$\mathcal{W}(dh) = \frac{1}{Z} e^{-\frac{\|h\|_H^2}{2}}\, \lambda_H(dh),$$

then $(\tau_g)_*\mathcal{W}$ should have the Feynman representation

$$\frac{1}{Z} e^{-\frac{\|h-g\|_H^2}{2}}\, \lambda_H(dh),$$

which could be re-written as

$$\big[(\tau_g)_*\mathcal{W}\big](dh) = \exp\big[\big(h, g\big)_H - \tfrac{1}{2}\|g\|_H^2\big]\, \mathcal{W}(dh).$$

Hence, if we assume that $\mathcal{I}(g)$ gives us the correct interpretation of $(\,\cdot\,, g)_H$, we are led to guess that, at least for $g \in H$,

$$\big[(\tau_g)_*\mathcal{W}(dx)\big](dh) = R_g(x)\, W(dx) \quad \text{where } R_g = \exp\big[\mathcal{I}(g) - \tfrac{1}{2}\|g\|_H^2\big]. \tag{8.2.8}$$

That (8.2.8) is correct was proved for the classical Wiener space by Cameron and Martin, and for this reason it is called the **Cameron–Martin formula**. In fact, one has the following result, the second half of which is due to Segal.

THEOREM 8.2.9. *If $(H,E,\mathcal{W})$ is an abstract Wiener space, then, for each $g \in H$, $(\tau_g)_*\mathcal{W} \ll \mathcal{W}$ and the R_g in (8.2.8) is the corresponding Radon–Nikodym derivative. Conversely, if $(\tau_y)_*\mathcal{W}$ is not singular to $\mathcal{W}$, then $y \in H$.*

PROOF: Let $g \in H$, and set $\mu = (\tau_g)_*\mathcal{W}$. Then

$$(*)\qquad \hat{\mu}(x^*) = \mathbb{E}^{\mathcal{W}}\big[e^{\sqrt{-1}\langle x+g,x^*\rangle}\big] = \exp\big[\sqrt{-1}\langle g,x^*\rangle - \tfrac{1}{2}\|h_{x^*}\|_H^2\big].$$

Now define ν by the right hand side of (8.2.8). Clearly $\nu \in \mathbf{M}_1(E)$. Thus, we will have proved the first part once we show that $\hat{\nu}$ is given by the right hand side of (*). To this end, observe that, for any $h_1,h_2 \in H$,

$$\mathbb{E}^{\mathcal{W}}\big[e^{\xi_1\mathcal{I}(h_1)+\xi_2\mathcal{I}(h_2)}\big] = \exp\left[\frac{\xi_1^2}{2}\|h_1\|_H^2 + \xi_1\xi_2\big(h_1,h_2\big)_H + \frac{\xi_2^2}{2}\|h_2\|_H^2\right]$$

for all $\xi_1,\xi_2 \in \mathbb{C}$. Indeed, this is obvious when ξ_1 and ξ_2 are pure imaginary, and, since both sides are entire functions of $(\xi_1,\xi_2) \in \mathbb{C}^2$, it follows in general by analytic continuation. In particular, by taking $h_1 = g$, $\xi_1 = 1$, $h_2 = h_{x^*}$, and $\xi_2 = \sqrt{-1}$, it is easy to check that the right hand side of (*) is equal to $\hat{\nu}(x^*)$.

To prove the second assertion, begin by recalling from Lemma 8.2.3 that if $y \in E$, then $y \in H$ if and only if there is a $K < \infty$ with the property that $|\langle y,x^*\rangle| \le K$ for all $x^* \in E^*$ with $\|h_{x^*}\|_H = 1$. Now suppose that $(\tau_{x^*})_*\mathcal{W} \not\perp \mathcal{W}$, and let R be the Radon–Nikodym derivative of its absolutely continuous part. Given $x^* \in E^*$ with $\|h_{x^*}\|_H = 1$, let $\mathcal{F}_{x^*}$ be the σ-algebra generated by $x \rightsquigarrow \langle x,x^*\rangle$, and check that $(\tau_y)_*\mathcal{W} \restriction \mathcal{F}_{x^*} \ll \mathcal{W} \restriction \mathcal{F}_{x^*}$ with Radon–Nikodym derivative

$$Y(x) = \exp\left(\langle y,x^*\rangle\langle x,x^*\rangle - \frac{\langle y,x^*\rangle^2}{2}\right).$$

Hence,

$$Y \ge \mathbb{E}^{\mathcal{W}}\big[R\,\big|\,\mathcal{F}_{x^*}\big] \ge \mathbb{E}^{\mathcal{W}}\big[R^{\frac{1}{2}}\,\big|\,\mathcal{F}_{x^*}\big]^2,$$

and so (cf. Exercise 8.2.14)

$$\exp\left(-\frac{\langle y,x^*\rangle^2}{8}\right) = \mathbb{E}^{\mathcal{W}}\big[Y^{\frac{1}{2}}\big] \ge \alpha \equiv \mathbb{E}^{\mathcal{W}}\big[R^{\frac{1}{2}}\big] \in (0,1].$$

Since this means that $\langle y,x^*\rangle^2 \le 8\log\frac{1}{\alpha}$, the proof is complete. □

Exercises for § 8.2

EXERCISE 8.2.10. Let $\mathbf{C} \in \mathbb{R}^N \otimes \mathbb{R}^N$ be a positive definite, symmetric matrix, take $E = \mathbb{R}^N$ is the standard Euclidean metric, and $H = \mathbb{R}^N$ with the Hilbert inner product $(\mathbf{x},\mathbf{y})_H = (\mathbf{x},\mathbf{C}^{-1}\mathbf{y})_{\mathbb{R}^N}$. Show that $\big(H,E,\gamma_{\mathbf{0},\mathbf{C}}\big)$ is an abstract Wiener space.

EXERCISE 8.2.11. Referring to the setting in Lemma 8.2.3, show that there is a sequence $\{\|\cdot\|_E^{(n)} : n \geq 0\}$ of norms on E each of which is commensurate with $\|\cdot\|_E$ (i.e., $C_n^{-1}\|\cdot\| \leq \|\cdot\|_E^{(N)} \leq C_n\|\cdot\|$ for some $C_n \in [1,\infty)$) such that, for each $R > 0$,

$$\overline{B_H(0,R)} = \{x \in E : \|x\|_E^{(n)} \leq R \text{ for all } n \geq 0\}.$$

Hint: Choose $\{x_m^* : m \geq 0\} \subseteq E^*$ so that $\{h_{x_m^*} : m \geq 0\}$ is an orthonormal basis for H, define $P_n : E \longrightarrow H$ by $P_n x = \sum_{m=0}^{n} \langle x, x_m^* \rangle h_{x_m^*}$, and

$$\|x\|_E^{(n)} = \sqrt{\|P_n x\|_H^2 + \|x - P_n x\|_E^2}.$$

EXERCISE 8.2.12. Referring to the setting in Fernique's Theorem, observe that all powers of $\|X\|_E$ are integrable, and set $\sigma^2 = \mathbb{E}\big[\|X\|_E^2\big]$. Show that

$$\mathbb{E}\Big[e^{\frac{\|X\|_E^2}{72\sigma^2}}\Big] \leq K.$$

In particular, for any $n \geq 1$, conclude that

$$\mathbb{E}\big[\|X\|_E^{2n}\big] \leq (72)^n n! K \sigma^{2n},$$

which is remarkably close to the equality which holds when $E = \mathbb{R}$.

EXERCISE 8.2.13. Given $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$, I pointed out at the end of § 8.1.2 that the Paley–Wiener integral $[\mathcal{I}(\mathbf{h}_{\boldsymbol{\lambda}})](\boldsymbol{\theta})$ can be interpreted as the Riemann–Stieltjes integral of $\boldsymbol{\lambda}\big((s,\infty)\big)$ with respect to $\boldsymbol{\theta}(s)$. In this exercise, I will use this observation as the starting point for what is called **stochastic integration**.

(**i**) Given $\boldsymbol{\lambda} \in \Theta(\mathbb{R}^N)^*$ and $t > 0$, set $\boldsymbol{\lambda}^t(d\tau) = \mathbf{1}_{[0,t)}(\tau)\boldsymbol{\lambda}(d\tau) + \delta_t \boldsymbol{\lambda}\big([t,\infty)\big)$, and show that for all $\boldsymbol{\theta} \in \Theta(\mathbb{R}^N)$

$$\langle \boldsymbol{\theta}, \boldsymbol{\lambda}^t \rangle = \int_0^t \boldsymbol{\lambda}\big((\tau,\infty)\big) \cdot d\boldsymbol{\theta}(\tau),$$

where the integral on the right is taken in the sense of Riemann–Stieltjes. In particular, conclude that $t \rightsquigarrow \langle \boldsymbol{\theta}, \boldsymbol{\lambda}^t \rangle$ is continuous for each $\boldsymbol{\theta}$.

(**ii**) Given $\mathbf{f} \in C_c^1\big([0,\infty);\mathbb{R}^N\big)$, set $\boldsymbol{\lambda}_{\mathbf{f}}(d\tau) = -\dot{\mathbf{f}}(\tau)\,d\tau$, and show that

$$\langle \boldsymbol{\theta}, \boldsymbol{\lambda}_{\mathbf{f}}^t \rangle = \int_0^t \mathbf{f}(\tau) \cdot d\boldsymbol{\theta}(\tau),$$

where again the integral on the right is Riemann–Stieltjes. Use this to see that the process

$$\left\{\int_0^t \mathbf{f}(\tau) \cdot d\boldsymbol{\theta}(\tau) : t \geq 0\right\}$$

has the same distribution under $\mathcal{W}^{(N)}$ as

$$(*) \qquad \left\{B\left(\int_0^t |\mathbf{f}(\tau)|^2\,d\tau\right) : t \geq 0\right\},$$

where $\{B(t) : t \geq 0\}$ is an $\mathbb{R}$-valued Brownian motion.

(**iii**) Given $\mathbf{f} \in L^2_{\mathrm{loc}}\big([0,\infty);\mathbb{R}^N\big)$ and $t > 0$, set $\mathbf{h}^t_{\mathbf{f}}(\tau) = \int_0^{t\wedge\tau} \mathbf{f}(s)\,ds$. Show that the $\mathcal{W}^{(N)}$-distribution of the process $\big\{\mathcal{I}(\mathbf{h}^t_{\mathbf{f}}) : t \geq 0\big\}$ is the same as that the process in (*). In particular, conclude (cf. part (**ii**) of Exercise 4.3.16) that there is a continuous modification of the process $\{\mathcal{I}(\mathbf{h}^t_{\mathbf{f}}) : t \geq 0\}$. For reasons made clear in (**ii**), such a continuous modification is denoted by

$$\left\{\int_0^t \mathbf{f}(\tau)\cdot d\boldsymbol{\theta}(\tau) : t \geq 0\right\}.$$

Of course, unless $\mathbf{f}$ has bounded variation, the integrals in the preceding are no longer interpretable as Riemann-Stieltjes integrals. In fact, they not even defined $\boldsymbol{\theta}$ by $\boldsymbol{\theta}$ but only as a stochastic process. For this reason, they are called **stochastic integrals**.

Exercise 8.2.14. Define R_g as in (8.2.8), and show that

$$\mathbb{E}^{\mathcal{W}}\big[R_g^p\big]^{\frac{1}{p}} = \exp\left[\frac{(p-1)\|g\|_H^2}{2}\right] \quad \text{for all } p \in (0,\infty).$$

Exercise 8.2.15. Here is another way to think about Segal's half of Theorem 8.2.9. Using Lemma 8.2.3, choose $\{x_n^* : n \geq 0\} \subseteq E^*$ so that $\{h_{x_n^*} : n \geq 0\}$ is an orthonormal basis for H. Next, define $F : E \longrightarrow \mathbb{R}^{\mathbb{N}}$ so that $F(x)_n = \langle x, x_n^*\rangle$ for each $n \in \mathbb{N}$, and show that $F_*\mathcal{W} = \gamma_{0,1}^{\mathbb{N}}$ and $(F\circ\tau_y)_*\mathcal{W} = \prod_0^\infty \gamma_{a_n,1}$, where $a_n = \langle y, x_n^*\rangle$. Conclude from this that $(\tau_y)_*\mathcal{W} \perp \mathcal{W}$ if $\gamma_{0,1}^{\mathbb{N}} \perp \prod_0^\infty \gamma_{a_n,1}$. Finally, use this together with Exercise 5.2.34 to see that $(\tau_y)_*\mathcal{W} \perp \mathcal{W}$ if $\sum_0^\infty a_m^2 = \infty$, which, by Lemma 8.2.3, will be the case if $y \notin H$.

§ 8.3 From Hilbert to Abstract Wiener Space

Up to this point I have been assuming that we already have at hand a non-degenerate, centered Gaussian measure $\mathcal{W}$ on a Banach space E, and, on the basis of this assumption, we produced the associated Cameron–Martin space H. In this section, I will show how one can go in the opposite direction. That is, I will start with a separable, real Hilbert space H and show how to go about finding a separable, real Banach space E for which there exists a $\mathcal{W} \in \mathbf{M}_1(E)$ such that $(H, E, \mathcal{W})$ is an abstract Wiener space. Although I will not adopt his approach, the idea of carrying out such a program is Gross's.

Warning: From now on, unless the contrary is explicitly stated, I will be assuming that the spaces with which I am dealing are all infinite dimensional, separable, and real.

§ 8.3.1. An Isomorphism Theorem. Because, at an abstract level, all infinite dimensional, separable Hilbert spaces are the same, one should expect that, in a related sense, the set of all abstract Wiener spaces for which one Hilbert space is the Cameron–Martin space is the same as the same as the set of all abstract Wiener spaces for which any other Hilbert space is the Cameron–Martin space. The following simple result verifies this conjecture.

Theorem 8.3.1. *Let H and H' be a pair of Hilbert spaces, and suppose that F is a linear isometry from H onto H'. Further, suppose that $(H, E, \mathcal{W})$ is an abstract Wiener space. Then there exists a Banach space $E' \supseteq H'$ and a linear isometry $\tilde{F}$ from E onto E' such that $\tilde{F} \restriction H = F$ and $\big(H', E', \tilde{F}_* W\big)$ is an abstract Wiener space.*

Proof: Define $\|h'\|_{E'} = \|F^{-1}h'\|_E$ for $h' \in H'$, and let E' be the Banach space obtained by completing H' with respect to $\|\cdot\|_{E'}$. Trivially, H' is continuously embedded in E' as a dense subspace, and F admits a unique extension $\tilde{F}$ as an isometry from E onto E'. Moreover, if $(x')^* \in (E')^*$ and $\tilde{F}^\top$ is the adjoint map from $(E')^*$ onto E^*, then

$$\begin{aligned}\big(h', h'_{(x')^*}\big)_{H'} &= \langle h', (x')^*\rangle = \langle F^{-1}h', \tilde{F}^\top (x')^*\rangle \\ &= \big(F^{-1}h', h_{\tilde{F}^\top (x')^*}\big)_H = \big(h', F h_{\tilde{F}^\top (x')^*}\big)_{H'},\end{aligned}$$

and so $h'_{(x')^*} = F h_{\tilde{F}^\top (x')^*}$. Hence,

$$\begin{aligned}\mathbb{E}^{\tilde{F}_*\mathcal{W}}\Big[e^{\sqrt{-1}\,\langle x', (x')^*\rangle}\Big] &= \mathbb{E}^{\mathcal{W}}\Big[e^{\sqrt{-1}\,\langle \tilde{F}x, (x')^*\rangle}\Big] = \mathbb{E}^{\mathcal{W}}\Big[e^{\sqrt{-1}\,\langle x, \tilde{F}^\top (x')^*\rangle}\Big] \\ &= e^{-\frac{1}{2}\|h_{\tilde{F}^\top (x')^*}\|_H^2} = e^{-\frac{1}{2}\|F^{-1}h'_{(x')^*}\|_H^2} = e^{-\frac{1}{2}\|h'_{(x')^*}\|_{H'}^2},\end{aligned}$$

which completes the proof that $\big(H', E', \tilde{F}_*\mathcal{W}\big)$ is an abstract Wiener space. □

Theorem 8.3.1 says that there is a one-to-one correspondence between the abstract Wiener spaces associated with one Hilbert space and the abstract Wiener spaces associated with any other. In particular, it allows us to prove the theorem of Gross which states that every Hilbert space is the Cameron–Martin space for some abstract Wiener space.

Corollary 8.3.2. *Given a separable, real Hilbert space H, there exists a separable Banach space E and a $\mathcal{W} \in \mathbf{M}_1(E)$ such that $(H, E, \mathcal{W})$ is an abstract Wiener space.*

Proof: Let $F : H^1(\mathbb{R}) \longrightarrow H$ be an isometric isomorphism, and use Theorem 8.3.1 to construct a separable Banach space E and an isometric, isomorphism $\tilde{F} : \Theta(\mathbb{R}) \longrightarrow E$ so that $(H, E, \mathcal{W})$ is an abstract Wiener space when $\mathcal{W} = \tilde{F}_*\mathcal{W}^{(1)}$. □

It is important to recognize that although a non-degenerate, centered Gaussian measure on a Banach space E determines a unique Cameron–Martin space H, a given H will be the Cameron–Martin space for an uncountable number of abstract Wiener spaces. For example, in the classical case when $H = \mathbf{H}^1(\mathbb{R}^N)$, we could have replaced $\Theta(\mathbb{R}^N)$ by a subspace which reflected the fact that almost every Brownian path is locally Hölder continuous of any order less than a half. We see a definitive, general formulation of this point in Corollary 8.3.10 below.

§ **8.3.2. Wiener Series.** The proof which I gave of Corollary 8.3.2 is too non-constructive to reveal much about the relationship between H and the abstract Wiener spaces for which it is the Cameron–Martin space. Thus, in this subsection I will develop another, entirely different, way of constructing abstract Wiener spaces for a Hilbert space.

The approach here has its origins in one of Wiener's own constructions of Brownian motion and is based on the following line of reasoning. Given H, choose an orthonormal basis $\{h_n : n \geq 0\}$. If there were a standard Gauss measure $\mathcal{W}$ on H, then the random variables $\{X_n : n \geq 0\}$ given by $X_n(h) = \big(h, h_n\big)_H$ would be independent, standard normal $\mathbb{R}$-valued random variables, and, for each $h \in H$, $\sum_0^\infty X_n(h)h_n$ would converge in H to h. Even though $\mathcal{W}$ cannot live on H, this line of reasoning suggests that a way to construct an abstract Wiener space is to start with a sequence $\{X_n : n \geq 0\}$ of $\mathbb{R}$-valued, independent standard normal random variables on some probability space, find a Banach space E in which $\sum_0^\infty X_n h_n$ converges with probability 1, and take $\mathcal{W}$ on E to the distribution of this series.

To convince oneself that this line of reasoning has a chance of leading somewhere, one should observe that Lévy's construction corresponds to a particular choice of the orthonormal basis $\{\mathbf{h}_m : m \geq 0\}$.* To see this, determine $\{\dot{h}_{k,n} : (k,n) \in \mathbb{N}^2\}$ by

$$\dot{h}_{k,0} = \mathbf{1}_{[k,k+1)} \text{ and } \dot{h}_{k,n} = 2^{\frac{n-1}{2}} \begin{cases} 1 & \text{on } \big[k2^{1-n}, (2k+1)2^{-n}\big) \\ -1 & \text{on } \big[(2k+1)2^{-n}, (k+1)2^{1-n}\big) \\ 0 & \text{elsewhere} \end{cases}$$

for $n \geq 1$. Clearly, the $\dot{h}_{k,n}$'s are orthonormal in $L^2\big([0,\infty);\mathbb{R}\big)$. In addition, for each $n \in \mathbb{N}$, the span of $\{\dot{h}_{k,n} : k \in \mathbb{N}\}$ equals that of $\{\mathbf{1}_{[k2^{-n},(k+1)2^{-n})} : k \in \mathbb{N}\}$. Perhaps the easiest way to check this is to do so by dimension counting. That is, for a given $(\ell, n) \in \mathbb{N}^2$, note that

$$\{\dot{h}_{\ell,0}\} \cup \{\dot{h}_{k,m} : \ell 2^{m-1} \leq k < (\ell+1)2^{m-1} \text{ and } 1 \leq m \leq n\}$$

has the same number of elements as $\{\mathbf{1}_{[k2^{-n},(k+1)2^{-n})} : \ell 2^n \leq k < (\ell+1)2^n\}$ and that the first is contained in the span of the second. As a consequence, we know that $\{\dot{h}_{k,n} : (k,n) \in \mathbb{N}^2\}$ is an orthonormal basis in $L^2\big([0,\infty);\mathbb{R}\big)$, and so, if $h_{k,n}(t) = \int_0^t \dot{h}_{k,n}(\tau)\,d\tau$ and $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is an orthonormal basis in $\mathbb{R}^N$, then

$$\big\{\mathbf{h}_{k,n,i} \equiv h_{k,n}\mathbf{e}_i : (k,n,i) \in \mathbb{N}^2 \times \{1, \dots, N\}\big\}$$

* The observation that Lévy's construction (cf. § 4.3.2) can be interpreted in terms of Wiener series is due to Z. Ciesielski. To be more precise, initially Ciesielski himself was thinking entirely in terms of orthogonal series and did not realize that he was giving a re-interpretation of Lévy's construction. Only later did the connection become clear.

is an orthonormal basis, known as the **Haar basis**, in $\mathbf{H}^1(\mathbb{R}^N)$. Finally, if $\big\{X_{k,n,i} : (k,n,i) \in \mathbb{N}^2 \times \{1,\dots,N\}\big\}$ is a family of independent, $N(0,1)$-random variables and $\mathbf{X}_{k,n} = \sum_{i=1}^N X_{k,n,i}\mathbf{e}_i$, then

$$\sum_{m=0}^{n}\sum_{k=0}^{\infty}\sum_{i=1}^{N} X_{k,m,i}\mathbf{h}_{k,m,i}(t) = \sum_{m=0}^{n}\sum_{k=0}^{\infty} h_{k,m}(t)\mathbf{X}_{k,m}$$

is precisely the polygonalization which I denoted by $\mathbf{B}_n(t)$ in Lévy's construction (cf. § 4.3.2).

The construction by Wiener, alluded to above, was essentially the same, only he chose a different basis for $\mathbf{H}^1(\mathbb{R}^N)$. Wiener took $\dot{h}_{k,0}(t) = \mathbf{1}_{[k,k+1)}(t)$ for $k \in \mathbb{N}$ and $\dot{h}_{k,n}(t) = 2^{\frac{1}{2}}\mathbf{1}_{[k,k+1)}(t)\cos\big(\pi n(t-k)\big)$ for $(k,n) \in \mathbb{N}\times\mathbb{Z}^+$, which means that he was looking at the series

$$\sum_{k=0}^{\infty}(t-k)\mathbf{1}_{[k,k+1)}(t)\mathbf{X}_{k,0} + \sum_{(k,n)\in\mathbb{N}\times\mathbb{Z}^+}\mathbf{1}_{[k,k+1)}(t)\frac{2^{\frac{1}{2}}\sin\big(\pi n(t-k)\big)}{\pi n}\mathbf{X}_{k,n},$$

where again $\{\mathbf{X}_{k,n} : (k,n) \in \mathbb{N}^2\}$ is a family of independent, $\mathbb{R}^N$-valued, $N(\mathbf{0},\mathbf{I})$-random variables. The reason why Lévy's choice is easier to handle than Wiener's is that, in Lévy's case, for each $n \in \mathbb{Z}^+$ and $t \in [0,\infty)$, $h_{k,n}(t) \neq 0$ for precisely one $k \in \mathbb{N}$. Wiener's choice has no such property.

With these preliminaries, the following theorem should come as no surprise.

Theorem 8.3.3. *Let H be an infinite dimensional, separable, real Hilbert space and E a Banach space into which H is continuously embedded as a dense subspace. If for some orthonormal basis $\{h_m : m \geq 0\}$ in H the series*

$$\sum_{m=0}^{\infty}\xi_m h_m \text{ converges in } E \quad \text{for } \gamma_{0,1}^{\mathbb{N}}\text{-almost every } \boldsymbol{\xi} = (\xi_0,\dots,\xi_m,\dots) \in \mathbb{R}^{\mathbb{N}} \tag{8.3.4}$$

and if $S : \mathbb{R}^{\mathbb{N}} \longrightarrow E$ is given by

$$S(\boldsymbol{\xi}) = \begin{cases} \sum_{m=0}^{\infty}\xi_m h_m & \text{when the series converges in } E \\ 0 & \text{otherwise,}\end{cases}$$

then $\big(H,E,\mathcal{W}\big)$ with $\mathcal{W} = S_*\gamma_{0,1}^{\mathbb{N}}$ is an abstract Wiener space. Conversely, if $(H,E,\mathcal{W})$ is an abstract Wiener space and $\{h_m : m \geq 0\}$ is an orthogonal sequence in H such that, for each $m \in \mathbb{N}$, either $h_m = 0$ or $\|h_m\|_H = 1$, then

$$\mathbb{E}^{\mathcal{W}}\left[\sup_{n\geq 0}\left\|\sum_{m=0}^{n}\mathcal{I}(h_m)h_m\right\|_E^p\right] < \infty \quad \text{for all } p \in [1,\infty), \tag{8.3.5}$$

and, for $\mathcal{W}$-almost every $x \in E$, $\sum_{m=0}^{\infty}[\mathcal{I}(h_m)](x)h_m$ converges in E to the $\mathcal{W}$-conditional expectation value of x given $\sigma\big(\{\mathcal{I}(h_m) : m \geq 0\}\big)$. Moreover,

$$\sum_{m=0}^{\infty}[\mathcal{I}(h_m)](x)h_m \text{ is } \mathcal{W}\text{-independent of } x - \sum_{m=0}^{\infty}[\mathcal{I}(h_m)](x)h_m.$$

Finally, if $\{h_m : m \geq 0\}$ is an orthonormal basis in H, then, for $\mathcal{W}$-almost every $x \in E$, $\sum_{m=0}^{\infty}[\mathcal{I}(h_m)](x)h_m$ converges in E to x, and the convergence is also in $L^p(\mathcal{W}; E)$ for every $p \in [1, \infty)$.

PROOF: First assume that (8.3.4) holds for some orthonormal basis, and set $S_n(\boldsymbol{\xi}) = \sum_{m=0}^{n} \xi_m h_m$ and $\mathcal{W} = S_* \gamma_{0,1}^{\mathbb{N}}$. Then, because $S_n(\boldsymbol{\xi}) \longrightarrow S(\boldsymbol{\xi})$ in E for $\gamma_{0,1}^{\mathbb{N}}$-almost every $\boldsymbol{\xi} \in \mathbb{R}^{\mathbb{N}}$,

$$\widehat{\mathcal{W}}(x^*) = \lim_{n\to\infty} \mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[e^{\sqrt{-1}\langle S_n, \lambda\rangle}\right] = \lim_{n\to\infty} \prod_{m=0}^{n} e^{-\frac{1}{2}(h_{x^*}, h_m)_H^2} = e^{-\frac{1}{2}\|h_{x^*}\|_H^2},$$

which proves that $(H, E, \mathcal{W})$ is an abstract Wiener space.

Next suppose that $(H, E, \mathcal{W})$ is an abstract Wiener space and that $\{h_m : m \geq 0\}$ is an orthogonal sequence with $\|h_m\|_H \in \{0, 1\}$ for each $m \geq 0$. By Theorem 8.2.1, $x \in L^p(\mathcal{W}; E)$ for every $p \in [1, \infty)$. Next, for each $n \in \mathbb{N}$, set $\mathcal{F}_n = \sigma\big(\{\mathcal{I}(h_m) : 0 \leq m \leq n\}\big)$. Clearly, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\mathcal{F} \equiv \bigvee_{n=0}^{\infty} \mathcal{F}_n$ is the σ-algebra generated by $\{\mathcal{I}(h_m) : m \geq 0\}$. Moreover, if $S_n = \sum_{m=0}^{n} \mathcal{I}(h_m)h_m$, then, since $\{\mathcal{I}(h_m) : m \geq 0\}$ is a Gaussian family and $\langle x - S_n(x), x^*\rangle$ is perpendicular in $L^2(\mathcal{W}; \mathbb{R})$ to $\mathcal{I}(h_m)$ for all $x^* \in E^*$ and $0 \leq m \leq n$, $x - S_n(x)$ is $\mathcal{W}$-independent of $\mathcal{F}_n$, . Thus $S_n = \mathbb{E}^{\mathcal{W}}[x \,|\, \mathcal{F}_n]$, and so, by Theorem 6.1.12, we know both that (8.3.5) holds and that $S_n \longrightarrow \mathbb{E}^{\mathcal{W}}[x \,|\, \mathcal{F}]$ $\mathcal{W}$-almost surely. In addition, the $\mathcal{W}$-independence of $S_n(x)$ from $x - S_n(x)$ implies that the limit quantities possess the same independence property.

In order to complete the proof at this point, all that I have to do is show that $x = \mathbb{E}^{\mathcal{W}}[x \,|\, \mathcal{F}]$ $\mathcal{W}$-almost surely when $\{h_m : m \geq 0\}$ is an orthonormal basis. Equivalently, I must check that $\mathcal{B}_E$ is contained in the $\mathcal{W}$-completion $\overline{\mathcal{F}}^{\mathcal{W}}$ of $\mathcal{F}$. To this end, note that for each $h \in H$, because $\sum_{m=0}^{n}(h, h_m)_H h_m$ converges in H to h,

$$\sum_{m=0}^{n} \big(h, h_m\big)_H \mathcal{I}(h_m) = \mathcal{I}\left(\sum_{m=0}^{n} \big(h, h_m\big)_H h_m\right) \longrightarrow \mathcal{I}(h) \quad \text{in } L^2(\mathcal{W}; \mathbb{R}).$$

Hence, $\mathcal{I}(h)$ is $\overline{\mathcal{F}}^{\mathcal{W}}$-measurable for every $h \in H$. In particular, this means that $x \rightsquigarrow \langle x, x^*\rangle$ is $\overline{\mathcal{F}}^{\mathcal{W}}$-measurable for every $x^* \in E^*$, and so, since $\mathcal{B}_E$ is generated by $\{\langle\,\cdot\,, x^*\rangle : x^* \in E^*\}$, $\mathcal{B}_E \subseteq \overline{\mathcal{F}}^{\mathcal{W}}$. $\square$

It is important to acknowledge that the preceding theorem does not give another proof of Wiener's theorem that Brownian motion exists. Instead, it simply says that, knowing it exists, there are lots of ways in which to construct it. See Exercise 8.3.20 for a more satisfactory proof of the same conclusion in the classical case, one that does not require the *a priori* existence of $\mathcal{W}^{(N)}$.

The following result shows that, in some sense, a non-degenerate, centered, Gaussian measure $\mathcal{W}$ on a Banach space does not fit on a smaller space.

COROLLARY 8.3.6. *If $\mathcal{W}$ is a non-degenerate, centered Gaussian measure on a separable Banach space E, then E is the support of $\mathcal{W}$ in the sense that $\mathcal{W}$ assigns positive probability to every, non-empty open subset of E.*

PROOF: Let H be the Cameron–Martin space for $\mathcal{W}$. Since H is dense in E, it suffices to show that $\mathcal{W}\big(B_E(g,r)\big) > 0$ for every $g \in H$ and $r > 0$. Moreover, since, by the Cameron–Martin formula (8.2.8) (cf. Exercise 8.2.14)

$$\begin{aligned}\mathcal{W}\big(B_E(0,r)\big) &= (\tau_{-g})_*\mathcal{W}\big(B_E(g,r)\big) = \mathbb{E}^{\mathcal{W}}\big[R_{-g},\, B_E(g,r)\big]\\ &\le e^{\frac{\|g\|_H^2}{2}}\sqrt{\mathcal{W}\big(B_E(g,r)\big)},\end{aligned}$$

I need only show that $\mathcal{W}\big(B_E(0,r)\big) > 0$ for all $r > 0$. To this end, choose an orthonormal basis $\{h_m : m \ge 0\}$ in H, and set $S_n = \sum_{m=0}^n \mathcal{I}(h_m)h_m$. Then, by Theorem 8.3.3, $x \rightsquigarrow S_n(x)$ is $\mathcal{W}$-independent of $x \rightsquigarrow x - S_n(x)$ and $S_n(x) \longrightarrow x$ in E for $\mathcal{W}$-almost every $x \in E$. Hence, $\mathcal{W}\big(\{\|x - S_n(x)\|_E < \frac{r}{2}\}\big) \ge \frac{1}{2}$ for some $n \in \mathbb{N}$, and therefore

$$\mathcal{W}\big(B_E(0,r)\big) \ge \tfrac{1}{2}\mathcal{W}\big(\|S_n\|_E < \tfrac{r}{2}\big).$$

But $\|S_n\|_E^2 \le C\|S_n\|_H^2 = \sum_{m=0}^n \mathcal{I}(h_m)^2$ for some $C < \infty$, and so

$$\mathcal{W}\big(\|S_n\|_E < \tfrac{r}{2}\big) \ge \gamma_{0,1}^{n+1}\Big(B_{\mathbb{R}^{n+1}}\big(\mathbf{0},\, \tfrac{r}{2C}\big)\Big) > 0$$

for any $r > 0$. $\square$

§ 8.3.3. Orthogonal Projections. Associated with any closed, linear subspace L of a Hilbert space H, there is a an orthogonal projection map $\Lambda_L : H \longrightarrow L$ determined by the property that, for each $h \in H$, $h - \Pi_L h \perp L$. Equivalently, $\Pi_L h$ is the element of L which is closest to h. In this subsection I will show that if $(H, E, \mathcal{W})$ is an abstract Wiener space and L is a finite dimensional subspace of H, then Π_L admits a $\mathcal{W}$-almost surely unique extension P_L to E. In addition, I will show that $P_L x \longrightarrow x$ in $L^2(\mathcal{W}; E)$ as $L \nearrow H$.

LEMMA 8.3.7. *Let $(H, E, \mathcal{W})$ be an abstract Wiener space and $\{h_m : m \ge 0\}$ an orthonormal basis in H. Then, for each $h \in H$, $\sum_{m=0}^\infty (h, h_m)_H \mathcal{I}(h_m)$ converges to $\mathcal{I}(h)$ $\mathcal{W}$-almost surely and in $L^p(\mathcal{W}; \mathbb{R})$ for every $p \in [1, \infty)$.*

PROOF: Define the σ-algebras $\mathcal{F}_n$ and $\mathcal{F}$ as in the proof of Theorem 8.3.3. Then, by the same argument as I used there, one can identify $\sum_{m=0}^{n}(h,h_m)_H\mathcal{I}(h_m)$ as $\mathbb{E}^{\mathcal{W}}[\mathcal{I}(h)\,|\,\mathcal{F}_n]$. Thus, since $\overline{\mathcal{F}}^{\mathcal{W}} \supseteq \mathcal{B}_E$, the required convergence statement is an immediate consequence of Corollary 5.2.4. $\square$

THEOREM 8.3.8. *Let $(H,E,\mathcal{W})$ be an abstract Wiener space. For each finite dimensional subspace L of H there is a $\mathcal{W}$-almost surely unique map $P_L : E \longrightarrow H$ such that, for every $h \in H$ and $\mathcal{W}$-almost every $x \in E$, $\big(h, P_Lx\big)_H = \mathcal{I}(\Pi_L h)(x)$, where Π_L denotes orthogonal projection from H onto L. In fact, if $\{g_1,\dots,g_{\dim(L)}\}$ is an orthonormal basis for L, then $P_Lx = \sum_1^{\dim(L)}[\mathcal{I}(g_i)](x)g_i$, and so $P_Lx \in L$ for $\mathcal{W}$-almost every $x \in E$. In particular, the distribution of $x \in E \longmapsto P_Lx \in L$ under $\mathcal{W}$ is the same as that of $(\xi_1,\dots,\xi_{\dim(L)}) \in \mathbb{R}^{\dim(L)} \longmapsto \sum_1^{\dim(L)}\xi_i g_i \in L$ under $\gamma_{0,1}^{\dim(L)}$. Finally, $x \rightsquigarrow P_Lx$ is $\mathcal{W}$-independent of $x \rightsquigarrow x - P_Lx$.*

PROOF: It suffices to note that

$$\mathcal{I}(\Pi_L h) = \mathcal{I}\left(\sum_{k=1}^{\ell}(h,g_k)_H g_k\right) = \sum_{k=1}^{\ell}(h,g_k)_H\mathcal{I}(g_k) = \left(\sum_{k=1}^{\ell}\mathcal{I}(g_k)g_k, h\right)_H$$

for all $h \in H$ $\square$

We now have the preparations needed to prove a result which shows that my definition of abstract Wiener space is the same as Gross's. Specifically, Gross's own definition was based on the property proved in the following.

THEOREM 8.3.9. *Let $(H,E,\mathcal{W})$ be an abstract Wiener space, $\{h_n : n \ge 0\}$ an orthonormal basis for H, and set $L_n = \operatorname{span}\big(\{h_0,\dots,h_n\}\big)$. Then, for all $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\mathbb{E}^{\mathcal{W}}\big[\|P_Lx\|_E^2\big] \le \epsilon^2$ whenever L is a finite dimensional subspace which is perpendicular to L_n.*

PROOF: Without loss in generality, I will assume that $\|\cdot\|_E \le \|\cdot\|_H$.

Arguing by contradiction, I will show that if the asserted property does not hold then there would exist an orthonormal basis $\{f_n : n \ge 0\}$ for H such that $\sum_0^\infty \mathcal{I}(f_n)f_n$ fails to converge in $L^2(\mathcal{W};E)$. Thus, suppose that there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exists a finite dimensional $L \perp L_n$ with $\mathbb{E}^{\mathcal{W}}\big[\|P_Lx\|_E^2\big] \ge \epsilon^2$. Under this assumption, define $\{n_m : m \ge 0\} \subseteq \mathbb{N}$, $\{\ell_m : m \ge 0\} \subseteq \mathbb{N}$, and $\big\{\{f_0,\dots,f_{n_m}\} : m \ge 0\big\} \subseteq H$ inductively by the following prescription. First, take $n_0 = 0$ and $f_0 = h_0$. Next, knowing n_m and $\{f_0,\dots,f_{n_m}\}$, choose a finite dimensional subspace $L \perp L_{n_m}$ so that $\mathbb{E}^{\mathcal{W}}\big[\|P_Lx\|_E^2\big] \ge \epsilon^2$, set $\ell_m = \dim(L)$, and let $\{g_{m,1},\dots,g_{m,\ell_m}\}$ be an orthonormal basis for L. For any $\delta > 0$ there exists an $n \ge n_m + \ell_m$ such that

$$\sum_{i,j=1}^{\ell_m}\big|\big(\Pi_{L_n}g_{m,i}, \Pi_{L_n}g_{m,j}\big) - \delta_{i,j}\big| \le \delta.$$

In particular (cf. Exercise 8.3.16), if $\delta \in (0,1)$, then the elements of $\{\Pi_{L_n} g_{m,i} : 1 \le i \le \ell_m\}$ are linearly independent and the orthonormal set $\{\tilde{g}_{m,i} : 1 \le i \le \ell_m\}$ obtained from them via the Gram–Schmidt orthogonalization procedure satisfies

$$\sum_{i=1}^{\ell_m} \|\tilde{g}_{m,i} - g_{m,i}\|_H \le K_{\ell_m} \sum_{i,j=1}^{\ell_m} \bigl|\bigl(\Pi_{L_n} g_{m,i}, \Pi_{L_n} g_{m,j}\bigr) - \delta_{i,j}\bigr|$$

for some $K_m < \infty$ which depends only on ℓ_m. Moreover, and because $L \perp L_{n_m}$, $\tilde{g}_{m,i} \perp L_{n_m}$ for all $1 \le i \le \ell_m$. Hence, we can find an $n_{m+1} \ge n_m + \ell_m$ so that $\operatorname{span}\bigl(\{h_n : n_m < n \le n_{m+1}\}\bigr)$ admits an orthonormal basis $\{f_{n_m+1}, \dots, f_{n_{m+1}}\}$ with the property that $\sum_1^{\ell_m} \|g_{m,i} - f_{n_m+i}\|_H \le \frac{\epsilon}{4}$.

Clearly $\{f_n : n \ge 0\}$ is an orthonormal basis for H. On the other hand,

$$\begin{aligned}
\mathbb{E}^{\mathcal{W}}\left[\left\|\sum_{n=n_m+1}^{n_m+\ell_m} \mathcal{I}(f_n) f_n\right\|_E^2\right]^{\frac{1}{2}} &\ge \epsilon - \mathbb{E}^{\mathcal{W}}\left[\left\|\sum_1^{\ell_m} \bigl(\mathcal{I}(g_{m,i}) g_{m,i} - \mathcal{I}(f_{n_m+i}) f_{n_m+i}\bigr)\right\|_E^2\right]^{\frac{1}{2}} \\
&\ge \epsilon - \sum_1^{\ell_m} \mathbb{E}^{\mathcal{W}}\bigl[\bigl\|\mathcal{I}(g_{m,i}) g_{m,i} - \mathcal{I}(f_{n_m+i}) f_{n_m+i}\bigr\|_H^2\bigr]^{\frac{1}{2}},
\end{aligned}$$

and so, since $\mathbb{E}^{\mathcal{W}}\bigl[\bigl\|\mathcal{I}(g_{i,m}) g_{m,i} - \mathcal{I}(f_{n_m+i}) f_{n_m+i}\bigr\|_H^2\bigr]^{\frac{1}{2}}$ is dominated by

$$\begin{aligned}
&\mathbb{E}^{\mathcal{W}}\bigl[\bigl\|\bigl(\mathcal{I}(g_{m,i}) - \mathcal{I}(f_{n_m+i})\bigr) g_{m,i}\bigr\|_H^2\bigr]^{\frac{1}{2}} + \mathbb{E}^{\mathcal{W}}\bigl[\mathcal{I}(f_{n_m+i})^2\bigr]^{\frac{1}{2}} \|g_{m,i} - f_{n_m+i}\|_H \\
&\quad \le 2\|g_{m,i} - f_{n_m+i}\|_H,
\end{aligned}$$

we have that

$$\mathbb{E}^{\mathcal{W}}\left[\left\|\sum_{n_m+1}^{n_m+\ell_m} \mathcal{I}(f_n) f_n\right\|_E^2\right]^{\frac{1}{2}} \ge \frac{\epsilon}{2} \quad \text{for all } m \ge 0,$$

and this means that $\sum_0^\infty \mathcal{I}(f_n) f_n$ cannot be converging in $L^2(\mathcal{W}; E)$. □

Besides showing that my definition of an abstract Wiener space is the same as Gross's, Theorem 8.3.9 allows us to prove a very convincing statement, again due to Gross, of just how non-unique is the Banach space for which a given Hilbert space is the Cameron–Martin space.

Corollary 8.3.10. *If $(H, E, \mathcal{W})$ is an abstract Wiener space, then there exists a separable Banach space E_0 which is continuously embedded in E as a measurable subset and has the properties that $\mathcal{W}(E_0) = 1$, bounded subsets of E_0 are relatively compact in E, and $\bigl(H, E_0, \mathcal{W} \restriction E_0\bigr)$ is again an abstract Wiener space.*

PROOF: Again I will assume that $\|\cdot\|_E \le \|\cdot\|_H$.

Choose $\{x_n^* : n \ge 0\} \subseteq E^*$ so that $\{h_n : n \ge 0\}$ is an orthonormal basis in H when $h_n = h_{x_n^*}$, and set $L_n = \operatorname{span}\big(\{h_0, \dots, h_n\}\big)$. Next, using Theorem 8.3.9, choose an increasing sequence $\{n_m : m \ge 0\}$ so that $n_0 = 0$ and $\mathbb{E}^{\mathcal{W}}\big[\|P_L x\|_E^2\big]^{\frac{1}{2}} \le 2^{-m}$ for $m \ge 1$ and finite dimensional $L \perp L_{n_m}$, and define Q_ℓ for $\ell \ge 0$ on E into H so that

$$Q_0 x = \langle x, x_0^* \rangle h_0 \quad \text{and} \quad Q_\ell x = \sum_{n=n_{\ell-1}+1}^{n_\ell} \langle x, x_n^* \rangle h_n \quad \text{when } \ell \ge 1.$$

Finally, set $S_m = P_{L_{n_m}} = \sum_{\ell=0}^m Q_\ell$, and define E_0 to be the set of $x \in E$ such that

$$\|x\|_{E_0} \equiv \|Q_0 x\|_E + \sum_{\ell=1}^\infty \ell^2 \big\|Q_\ell x\|_E < \infty \quad \text{and} \quad \|S_m x - x\|_E \longrightarrow 0.$$

To show that $\|\cdot\|_{E_0}$ is a norm on E_0 and that E_0 with norm $\|\cdot\|_{E_0}$ is a Banach space, first note that if $x \in E_0$ then

$$\|x\|_E = \lim_{m\to\infty} \|S_m x\|_E \le \|Q_0 x\|_E + \varliminf_{m\to\infty} \sum_{\ell=1}^m \|Q_\ell x\|_E \le \|x\|_{E_0},$$

and therefore $\|\cdot\|_{E_0}$ is certainly a norm on E_0. Next, suppose that the sequence $\{x_k : k \ge 1\} \subseteq E_0$ is a Cauchy sequence with respect to $\|\cdot\|_{E_0}$. By the preceding, we know that $\{x_k : k \ge 1\}$ is also Cauchy convergent with respect to $\|\cdot\|_E$, and so there exists an $x \in E$ such that $x_k \longrightarrow x$ in E. We need to show that $x \in E_0$ and that $\|x_k - x\|_{E_0} \longrightarrow 0$. Because $\{x_k : k \ge 1\}$ is bounded in E_0, it is clear that $\|x\|_{E_0} < \infty$. In addition, for any $m \ge 0$ and $k \ge 1$,

$$\begin{aligned}
\|x - S_m x\|_E &= \lim_{\ell\to\infty} \|x_\ell - S_m x_\ell\|_E \le \varliminf_{\ell\to\infty} \|x_\ell - S_m x_\ell\|_{E_0} \\
&= \varliminf_{\ell\to\infty} \sum_{n>m} n^2 \|Q_n x_\ell\|_E \le \sum_{n>m} n^2 \|Q_n x_k\| + \sup_{\ell>k} \|x_\ell - x_k\|_{E_0}.
\end{aligned}$$

Thus, by choosing k for a given $\epsilon > 0$ so that $\sup_{\ell>k} \|x_\ell - x_k\|_{E_0} < \epsilon$, we conclude that $\varlimsup_{m\to\infty} \|x - S_m x\|_E < \epsilon$ and therefore that $S_m x \longrightarrow x$ in E. Hence, $x \in E_0$. Finally, to see that $x_k \longrightarrow x$ in E_0, simply note that

$$\begin{aligned}
&\|x - x_k\|_{E_0} = \|Q_0(x - x_k)\|_E + \sum_{m=1}^\infty m^2 \|Q_m(x - x_k)\|_E \\
&\le \varliminf_{\ell\to\infty} \left(\|Q_0(x_\ell - x_k)\|_E + \sum_{m=1}^\infty m^2 \|Q_m(x_\ell - x_k)\|_E \right) \le \sup_{\ell>k} \|x_\ell - x_k\|_{E_0}
\end{aligned}$$

which tends to 0 as $k \to \infty$.

To show that bounded subsets of E_0 are relatively compact in E, it suffices to show that if $\{x_\ell : \ell \geq 1\} \subseteq \overline{B_{E_0}(0,R)}$, then there is an $x \in E$ to which a subsequence converges in E. For this purpose, observe that, for each $m \geq 0$, there is a subsequence $\{x_{\ell_k} : k \geq 1\}$ along which $S_m x_{\ell_k}$ converges in L_{n_m}. Hence, by a diagonalization argument, $\{x_{\ell_k} : k \geq 1\}$ can be chosen so that $\{S_m x_{\ell_k} : k \geq 1\}$ converges in L_{n_m} for all $m \geq 0$. Since, for $1 \leq j < k$,

$$\begin{aligned}\|x_{\ell_k} - x_{\ell_j}\|_E &\leq \|S_m x_{\ell_k} - S_m x_{\ell_j}\|_E + \sum_{n>m} \|Q_n(x_{\ell_k} - x_{\ell_j})\|_E \\ &\leq \|S_m x_{\ell_k} - S_m x_{\ell_j}\|_E + 2R \sum_{n>m} \frac{1}{n^2},\end{aligned}$$

it follows that $\{x_{\ell_k} : k \geq 1\}$ is Cauchy convergent in E and therefore that it converges in E.

I must still show that $E_0 \in \mathcal{B}_E$ and that $(H, E_0, \mathcal{W}_0)$ is an abstract Wiener space when $\mathcal{W}_0 = \mathcal{W} \restriction E_0$. To see the first of these, observe that $x \in E \longmapsto \|x\|_{E_0} \in [0,\infty]$ is lower semi-continuous and that $\{x : \|S_m x - x\|_E \longrightarrow 0\} \in \mathcal{B}_E$. In addition, because, by Theorem 8.3.3, $\|S_m x - x\|_E \longrightarrow 0$ for $\mathcal{W}$-almost every $x \in E$, we will know that $\mathcal{W}(E_0) = 1$ once I show that $\mathcal{W}\big(\|x\|_{E_0} < \infty\big) = 1$, which follows immediately from

$$\begin{aligned}\mathbb{E}^{\mathcal{W}}\big[\|x\|_{E_0}\big] &= \mathbb{E}^{\mathcal{W}}\big[\|Q_0 x\|_E\big] + \sum_1^\infty m^2 \mathbb{E}^{\mathcal{W}}\big[\|Q_m x\|_E\big] \\ &\leq \mathbb{E}^{\mathcal{W}}\big[\|Q_0 x\|_E\big] + \sum_1^\infty m^2 \mathbb{E}^{\mathcal{W}}\big[\|Q_m x\|_E^2\big]^{\frac{1}{2}} < \infty.\end{aligned}$$

The next step is to check that H is continuously embedded in E_0. Certainly $h \in H \implies \|S_m h - h\|_E \leq \|S_m h - h\|_H \longrightarrow 0$. Next suppose that $h \in H \setminus \{0\}$ and that $h \perp L_{n_m}$, and let L be the line spanned by h. Then $P_L x = \|h\|_H^{-2}[\mathcal{I}(h)](x)h$, and so, because $L \perp L_{n_m}$,

$$\frac{1}{2^m} \geq \mathbb{E}^{\mathcal{W}}\big[\mathcal{I}(h)^2\big]^{\frac{1}{2}} \frac{\|h\|_E}{\|h\|_H^2} = \frac{\|h\|_E}{\|h\|_H}.$$

Hence, we now know that $h \perp L_{n_m} \implies \|h\|_E \leq 2^{-m}\|h\|_H$. In particular, $\|Q_{m+1}h\|_E \leq 2^{-m}\|Q_{m+1}h\|_H \leq 2^{-m}\|h\|_H$ for all $m \geq 0$ and $h \in H$, and so

$$\|h\|_{E_0} = \|Q_0 h\|_E + \sum_{m=1}^\infty m^2 \|Q_m h\|_E \leq \left(1 + 2\sum_{m=1}^\infty \frac{m^2}{2^m}\right)\|h\|_H = 25\|h\|_H.$$

To complete the proof, I must show that H is dense in E_0 and that, for each $y^* \in E_0^*$, $\widehat{\mathcal{W}_0}(y^*) = e^{-\frac{1}{2}\|h_{y^*}\|_H^2}$, where $\mathcal{W}_0 = \mathcal{W} \restriction E_0$ and $h_{y^*} \in H$ is determined by $\big(h, h_{y^*}\big)_H = \langle h, y^* \rangle$ for $h \in H$. Both these facts rely on the observation that

$$\|x - S_m x\|_{E_0} = \sum_{n>m} n^2 \|Q_n x\|_E \longrightarrow 0 \quad \text{for all } x \in E_0.$$

Knowing this, the density of H in E_0 is obvious. Finally, if $y^* \in E_0^*$, then, by the preceding and Lemma 8.3.7,

$$\begin{aligned}\langle x, y^* \rangle &= \lim_{m\to\infty} \langle S_m x, y^* \rangle = \lim_{m\to\infty} \sum_{n=0}^{n_m} \langle x, x_n^* \rangle \langle h_n, y^* \rangle \\ &= \lim_{m\to\infty} \sum_{n=0}^{n_m} \big(h_{y^*}, h_n\big)_H \big[\mathcal{I}(h_n)\big](x) = \big[\mathcal{I}(h_{y^*})\big](x)\end{aligned}$$

for $\mathcal{W}_0$-almost every $x \in E_0$. Hence $\langle\,\cdot\,, y^*\rangle$ under $\mathcal{W}_0$ is a centered Gaussian with variance $\|h_{y^*}\|_H^2$. $\square$

§ **8.3.4. Pinned Brownian Motion.** Theorem 8.3.8 has a particularly interesting application to the classical abstract Wiener space $\big(\mathbf{H}^1(\mathbb{R}^N), \Theta(\mathbb{R}^N), \mathcal{W}^{(N)}\big)$. Namely, suppose that $0 = t_0 < t_1 < \cdots < t_n$, and let L be the span of $\big\{h_{t_m}\mathbf{e} : 1 \le m \le n \text{ and } \mathbf{e} \in \mathbb{S}^{N-1}\big\}$, where $h_t(\tau) \equiv t \wedge \tau$. In this case,

$$P_L \boldsymbol{\theta} = \sum_{m=1}^{n} \frac{h_{t_m} - h_{t_{m-1}}}{t_m - t_{m-1}} \big(\boldsymbol{\theta}(t_m) - \boldsymbol{\theta}(t_{m-1})\big),$$

and so

$$\tag{8.3.11} \begin{aligned}&\boldsymbol{\theta}_{(t_1,\dots,t_n)}(t) \equiv [\boldsymbol{\theta} - P_L\boldsymbol{\theta}](t) \\ &\quad = \begin{cases} \boldsymbol{\theta}(t) - \boldsymbol{\theta}(t_{m-1}) - \frac{t - t_{m-1}}{t_m - t_{m-1}}\big(\boldsymbol{\theta}(t_m) - \boldsymbol{\theta}(t_{m-1})\big) & \text{if } t \in [t_{m-1}, t_m] \\ \boldsymbol{\theta}(t) - \boldsymbol{\theta}(t_n) & \text{if } t \in [t_n, \infty). \end{cases}\end{aligned}$$

Thus, if $(\boldsymbol{\theta}, \vec{\mathbf{y}}) \in \Theta(\mathbb{R}^N) \times (\mathbb{R}^N)^n \longmapsto \boldsymbol{\theta}_{(t_1,\dots,t_n),\vec{\mathbf{y}}} \in \Theta(\mathbb{R}^N)$ is given by

$$\boldsymbol{\theta}_{(t_1,\dots,t_n),\vec{\mathbf{y}}} = \boldsymbol{\theta}_{(t_1,\dots,t_n)} + \sum_{m=1}^{n} \frac{h_{t_m} - h_{t_{m-1}}}{t_m - t_{m-1}} (\mathbf{y}_m - \mathbf{y}_{m-1})$$

where $\vec{\mathbf{y}} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$ and $\mathbf{y}_0 \equiv \mathbf{0}$, then, for any Borel measurable $F : \Theta(\mathbb{R}^N) \times (\mathbb{R}^N)^n \longrightarrow [0, \infty)$,

$$\tag{8.3.12} \begin{aligned}&\int_{\Theta(\mathbb{R}^N)} F\Big(\boldsymbol{\theta}, \big(\boldsymbol{\theta}(t_1), \dots, \boldsymbol{\theta}(t_n)\big)\Big)\, \mathcal{W}^{(N)}(d\boldsymbol{\theta}) \\ &\quad = \int_{(\mathbb{R}^N)^n} \left(\int_{\Theta(\mathbb{R}^N)} F\big(\boldsymbol{\theta}_{(t_1,\dots,t_n),\vec{\mathbf{y}}}, \vec{\mathbf{y}}\big)\, \mathcal{W}^{(N)}(d\boldsymbol{\theta}) \right) \gamma_{\mathbf{0},\mathbf{C}(t_1,\dots,t_n)}(d\vec{\mathbf{y}}),\end{aligned}$$

where $\mathbf{C}(t_1,\dots,t_n)_{(m,i),(m'i')} = t_m \wedge t_{m'}\delta_{i,i'}$ for $1 \le m, m' \le n$ and $1 \le i, i' \le N$ is the covariance of $\boldsymbol{\theta} \rightsquigarrow (\boldsymbol{\theta}(t_1),\dots,\boldsymbol{\theta}(t_n))$ under $\mathcal{W}^{(N)}$. Equivalently, if

$$\check{\boldsymbol{\theta}}_{(t_1,\dots,t_n),\vec{\mathbf{y}}} = \boldsymbol{\theta}_{(t_1,\dots,t_n)} + \sum_{m=1}^{n} \frac{h_{t_m} - h_{t_{m-1}}}{t_m - t_{m-1}} \mathbf{y}_m,$$

then

$$\begin{aligned}&\int_{\Theta(\mathbb{R}^N)} F\Big(\boldsymbol{\theta}, \big(\boldsymbol{\theta}(t_1) - \boldsymbol{\theta}(t_0), \dots, \boldsymbol{\theta}(t_n) - \boldsymbol{\theta}(t_{n-1})\big)\Big)\, \mathcal{W}^{(N)}(d\boldsymbol{\theta}) \\ &= \int\limits_{(\mathbb{R}^N)^n} \Bigg(\int\limits_{\Theta(\mathbb{R}^N)} F\big(\check{\boldsymbol{\theta}}_{(t_1,\dots,t_n),\vec{\mathbf{y}}}, \vec{\mathbf{y}}\big)\, \mathcal{W}^{(N)}(d\boldsymbol{\theta}) \Bigg) \gamma_{\mathbf{0},\mathbf{D}(t_1,\dots,t_n)}(d\vec{\mathbf{y}}),\end{aligned} \tag{8.3.13}$$

where $\mathbf{D}(t_1,\dots,t_n)_{(m,i),(m',i')} = (t_m - t_{m-1})\delta_{m,m'}\delta_{i,i'}$ for $1 \le m, m' \le n$ and $1 \le i, i' \le N$ is the covariance matrix for $\big(\boldsymbol{\theta}(t_1) - \boldsymbol{\theta}(t_0), \dots, \boldsymbol{\theta}(t_n) - \boldsymbol{\theta}(t_{n-1})\big)$ under $\mathcal{W}^{(N)}$.

There are several comments which should be made about these conclusions. In the first place, it is clear from (8.3.11) that $t \rightsquigarrow \boldsymbol{\theta}_{(t_1,\dots,t_n)}(t)$ returns to the origin at each of the times $\{t_m : 1 \le m \le n\}$. In addition, the excursions $\boldsymbol{\theta}_{(t_1,\dots,t_n)} \restriction [t_{m-1}, t_m]$, $1 \le m \le n$, are independent of each other and of $\boldsymbol{\theta}_{(t_1,\dots,t_n)} \restriction [t_n, \infty)$. Secondly, if $\mathcal{W}^{(N)}_{(t_1,\dots,t_n),\vec{\mathbf{y}}}$ denotes the $\mathcal{W}^{(N)}$-distribution of $\boldsymbol{\theta} \rightsquigarrow \boldsymbol{\theta}_{(t_1,\dots,t_n),\vec{\mathbf{y}}}$, then (8.3.12) says that

$$\boldsymbol{\theta} \rightsquigarrow \mathcal{W}^{(N)}_{(t_1,\dots,t_n),(\boldsymbol{\theta}(t_1),\dots,\boldsymbol{\theta}(t_n))}$$

is a regular conditional probability distribution (cf. § 9.2) of $\mathcal{W}^{(N)}$ given the σ-algebra generated by $\{\boldsymbol{\theta}(t_1),\dots,\boldsymbol{\theta}(t_n)\}$. Expressed in more colloquial terms, the process $\big\{\boldsymbol{\theta}_{(t_1,\dots,t_n),\vec{\mathbf{y}}}(t) : t \ge 0\big\}$ is **Brownian motion pinned to the points** $\{\mathbf{y}_m : 1 \le m \le n\}$ **at times** $\{t_m : 1 \le m \le n\}$.

§ **8.3.5. Orthogonal Invariance.** Consider the standard Gauss distribution $\gamma_{\mathbf{0},\mathbf{I}}$ on $\mathbb{R}^N$. Obviously, $\gamma_{\mathbf{0},\mathbf{I}}$ is rotation invariant. That is, if $\mathcal{O}$ is an orthogonal transformation on $\mathbb{R}^N$, then $\gamma_{\mathbf{0},\mathbf{I}}$ is invariant under the transformation $T_{\mathcal{O}} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ given by $T_{\mathcal{O}}\mathbf{x} = \mathcal{O}\mathbf{x}$. On the other hand, none of these transformations can be ergodic, since any radial function on $\mathbb{R}^N$ is invariant under $T_{\mathcal{O}}$ for every $\mathcal{O}$.

Now think about the analogous situation when $\mathbb{R}^N$ is replaced by an infinite dimensional Hilbert space H and $(H, E, \mathcal{W})$ is an associated abstract Wiener space. As I am about to show, $\mathcal{W}$ still enjoys rotation invariance with respect to orthogonal transformations on H. On the other hand, because $\|x\|_H = \infty$ for $\mathcal{W}$-almost every $x \in E$, there are no non-trivial radial functions now, a fact which leaves open the possibility that some orthogonal transformation of H give rise to ergodic transformations for $\mathcal{W}$. The purpose of this subsection is to investigate these matters, and I begin with the following formulation of the rotation invariance of $\mathcal{W}$.

THEOREM 8.3.14. *Let $(H, E, \mathcal{W})$ be an abstract Wiener space and $\mathcal{O}$ an orthogonal transformation on H. Then there is $\mathcal{W}$-almost surely unique, Borel measurable map $T_{\mathcal{O}} : E \longrightarrow E$ such that $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^\top h)$ $\mathcal{W}$-almost surely for each $h \in H$. Moreover, $\mathcal{W} = (T_{\mathcal{O}})_* \mathcal{W}$.*

PROOF: To prove uniqueness, note that if T and T' both satisfy the defining property for $T_{\mathcal{O}}$, then, for each $x^* \in E^*$,

$$\langle Tx, x^* \rangle = \mathcal{I}(h_{x^*})(Tx) = \mathcal{I}(\mathcal{O}^\top h_{x^*}) = \mathcal{I}(h_{x^*})(T'x) = \langle T'x, x^* \rangle$$

for $\mathcal{W}$-almost every $x \in E$. Hence, since E^* is separable in the weak* topology, $Tx = T'x$ for $\mathcal{W}$-almost every $x \in E$.

To prove existence, choose an orthonormal basis $\{h_m : m \geq 0\}$ for H, and let C be the set of $x \in E$ for which both $\sum_{m=0}^\infty [\mathcal{I}(h_m)](x) h_m$ and $\sum_{m=0}^\infty [\mathcal{I}(h_m)](x) \mathcal{O} h_m$ converge in E. By Theorem 8.3.3, we know that $\mathcal{W}(C) = 1$ and that

$$x \rightsquigarrow T_{\mathcal{O}} x \equiv \begin{cases} \sum_{m=0}^\infty [\mathcal{I}(h_m)](x) \mathcal{O} h_m & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

has distribution $\mathcal{W}$. Hence, all that remains is to check that $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^\top h)$ $\mathcal{W}$-almost surely for each $h \in H$. To this end, let $x^* \in E^*$, and observe that

$$\begin{aligned} [\mathcal{I}(h_{x^*})](T_{\mathcal{O}} x) = \langle T_{\mathcal{O}} x, x^* \rangle &= \sum_{m=0}^\infty \big(h_{x^*}, \mathcal{O} h_m \big)_H [\mathcal{I}(h_m)](x) \\ &= \sum_{m=0}^\infty \big(\mathcal{O}^\top h_{x^*}, h_m \big)_H [\mathcal{I}(h_m)](x) \end{aligned}$$

for $\mathcal{W}$-almost every $x \in E$. Thus, since, by Lemma 8.3.7, the last of these series convergences $\mathcal{W}$-almost surely to $\mathcal{I}(\mathcal{O}^\top h_{x^*})$, we have that $\mathcal{I}(h_{x^*}) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^\top h_{x^*})$ $\mathcal{W}$-almost surely. To handle general $h \in H$, simply note that both $h \in H \longmapsto \mathcal{I}(h) \circ T_{\mathcal{O}} \in L^2(\mathcal{W}; \mathbb{R})$ and $h \in H \longmapsto \mathcal{I}(\mathcal{O}^\top h) \in L^2(\mathcal{W}; \mathbb{R})$ are isometric, and remember that $\{h_{x^*} : x^* \in E^*\}$ is dense in H. $\square$

I next want to discuss the possibility of $T_{\mathcal{O}}$ being ergodic for some orthogonal transformations $\mathcal{O}$. First notice that $T_{\mathcal{O}}$ cannot be ergodic if $\mathcal{O}$ has a non-trivial, finite dimensional invariant subspace L, since if $\{h_1, \dots, h_n\}$ were an orthonormal basis for L, then $\sum_{m=1}^n \mathcal{I}(h_m)^2$ would be a non-constant, $T_{\mathcal{O}}$-invariant function. Thus, the only candidates for ergodicity are $\mathcal{O}$'s which have no non-trivial, finite dimensional, invariant subspaces. In a more general and highly adstract context, I. Segal* showed that the existence of a non-trivial, finite dimensional subspace for $\mathcal{O}$ is the only obstruction to $T_{\mathcal{O}}$ being ergodic. Here I will show less.

* See I.E. Segal's "Ergodic subsgroups of the orthogonal group on a real Hibert Space," Annals of Math., vol. 66 # 2, 1957. For a treatment in the setting here, see my article ?

THEOREM 8.3.15. *Let $(H, E, \mathcal{W})$ be an abstract Wiener space. If $\mathcal{O}$ is an orthogonal transformation on H with the property that, for every $g, h \in H$, $\varliminf_{n\to\infty}\big(\mathcal{O}^n g, h\big)_H = 0$, then $T_{\mathcal{O}}$ is ergodic.*

PROOF: What I have to show is that any $T_{\mathcal{O}}$-invariant element $\Phi \in L^2(\mathcal{W};\mathbb{R})$ is $\mathcal{W}$-almost surely constant, and for this purpose it suffices to check that

$$(*) \qquad \varliminf_{n\to\infty}\big|\mathbb{E}^{\mathcal{W}}\big[(\Phi\circ T_{\mathcal{O}}^n)\Phi\big]\big| = 0$$

for all $\Phi \in L^2(\mathcal{W};\mathbb{R})$ with mean value 0. In fact, if $\{h_m : m \geq 1\}$ is an orthonormal basis for H, then it suffices to check (*) when

$$\Phi(x) = F\big([\mathcal{I}(h_1)](x), \dots, [\mathcal{I}(h_N)](x)\big)$$

for some $N \in \mathbb{Z}^+$ and bounded, Borel measurable $F : \mathbb{R}^N \longrightarrow \mathbb{R}$. The reason why it is sufficient to check it for such Φ's is that, because $T_{\mathcal{O}}$ is $\mathcal{W}$-measure preserving, the set of Φ's for which (*) holds is closed in $L^2(\mathcal{W};\mathbb{R})$. Hence, if we start with any $\Phi \in L^2(\mathcal{W};\mathbb{R})$ with mean value 0, we can first approximate it in $L^2(\mathcal{W};\mathbb{R})$ by bounded functions with mean value 0 and then condition these bounded approximates with respect to $\sigma\big(\{\mathcal{I}(h_1), \dots, \mathcal{I}(h_N)\}\big)$ to give them the required form.

Now suppose that $\Phi = F\big(\mathcal{I}(h_1), \dots, \mathcal{I}(h_N)\big)$ for some N and bounded, measurable F. Then

$$\mathbb{E}^{\mathcal{W}}\big[\Phi\circ T_{\mathcal{O}}^n\Phi\big] = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} F(\boldsymbol{\xi})F(\boldsymbol{\eta})\,\gamma_{\mathbf{0},\mathbf{C}_n}(d\boldsymbol{\xi}\times d\boldsymbol{\eta}),$$

where

$$\mathbf{C}_n = \begin{pmatrix} \mathbf{I} & \mathbf{B}_n \\ \mathbf{B}_n^\top & \mathbf{I} \end{pmatrix} \quad \text{with } \mathbf{B}_n = \Big(\Big(\big(h_k, \mathcal{O}^n h_\ell\big)_H\Big)\Big)_{1\leq k,\ell\leq N}$$

and the block structure corresponds to $\mathbb{R}^N \times \mathbb{R}^N$. Finally, by our hypothesis about $\mathcal{O}$, we can find a subsequence $\{n_m : m \geq 0\}$ such that $\lim_{m\to\infty}\mathbf{B}_{n_m} = \mathbf{0}$, from which it is clear that $\gamma_{\mathbf{0},\mathbf{C}_{n_m}}$ tends to $\gamma_{\mathbf{0},\mathbf{I}} \times \gamma_{\mathbf{0},\mathbf{I}}$ in variation and therefore

$$\lim_{m\to\infty}\mathbb{E}^{\mathcal{W}}\big[(\Phi\circ T_{\mathcal{O}}^{n_m})\Phi\big] = \mathbb{E}^{\mathcal{W}}[\Phi]^2 = 0. \quad \square$$

Perhaps the best tests for whether an orthogonal transformation satisfies the hypothesis in Theorem 8.3.15 come from spectral theory. To be more precise, if H_c and $\mathcal{O}_c$ are the space and operator obtained by complexifying H and $\mathcal{O}$, the Spectral Theorem for normal operators allows one to write

$$\mathcal{O}_c = \int_0^{2\pi} e^{\sqrt{-1}\alpha}\, dE_\alpha,$$

where $\{E_\alpha : \alpha \in [0,2\pi)\}$ is a resolution of the identity in H_c by orthogonal projection operators. The spectrum of $\mathcal{O}_c$ is said to be **absolutely continuous** if, for each $h \in H_c$, the non-decreasing function $\alpha \rightsquigarrow \big(E_\alpha h, h\big)_{H_c}$ is absolutely continuous, which, by polarization, means that $\alpha \rightsquigarrow \big(E_\alpha h, h'\big)_{H_c}$ is absolutely continuous for all h, $h' \in H_c$. The reason for introducing this concept here is that, by combining the Riemann-Lebesgue Lemma with Theorem 8.3.15, one can prove that $T_{\mathcal{O}}$ *is ergodic if the spectrum of* $\mathcal{O}_c$ *is absolutely continuous.** Indeed, given h, $h' \in H$, let f be the Radon–Nikodym derivative of $\alpha \rightsquigarrow \big(E_\alpha h, h'\big)_{H_c}$, and apply the Riemann–Lebesgue Lemma to see that

$$\big(\mathcal{O}^n h, h'\big)_H = \int_0^{2\pi} e^{\sqrt{-1}n\alpha} f(\alpha)\,d\alpha \longrightarrow 0 \quad \text{as } n \to \infty.$$

See Exercises 8.3.23, 8.3.24, and 8.5.15 below for a more concrete examples.

Exercises for § 8.3

EXERCISE 8.3.16. The purpose of this exercise is to provide the linear algebraic facts which I used in the proof of Theorem 8.3.9. Namely, I want to show that if a set $\{h_1, \dots, h_n\} \subseteq H$ is approximately orthonormal, then the vectors h_i differ by very little from their Gram–Schmidt orthognalizaion.

(**i**) Suppose that $A = \big((a_{ij})\big)_{1 \le i,j \le n} \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a lower triangular matrix whose diagonal entries are non-negative. Show that there is a $C_n < \infty$, depending only on n, such that $\|\mathbf{I}_{\mathbb{R}^n} - A\|_{\text{op}} \le C_n \|\mathbf{I}_{\mathbb{R}^n} - AA^\top\|_{\text{op}}$.

Hint: Show that it suffices to treat the case when $AA^\top \le 2\mathbf{I}_{\mathbb{R}^n}$, and set $\Delta = \mathbf{I}_{\mathbb{R}^n} - AA^\top$. Assuming that $AA^\top \le 2\mathbf{I}_{\mathbb{R}^n}$, work by induction on n, at each step using the lower triangularity of A to see that

$$|a_{\ell\,\ell} a_{n\,\ell}| \le |\Delta_{n\,\ell}| + (AA^\top)^{\frac{1}{2}}_{n\,n} \left(\sum_{j=1}^{\ell} a_{\ell\,j}^2\right)^{\frac{1}{2}} \quad \text{if } 1 \le \ell < n$$

$$\big|1 - a_{n\,n}^2\big| \le |\Delta_{n\,n}| + \sum_{\ell=1}^{n-1} a_{n\,\ell}^2.$$

(**ii**) Let $\{h_1, \dots, h_n\} \subseteq H$, set $B = \big(((h_i, h_j)_H)\big)_{1 \le i,j \le n}$, and assume that $\|\mathbf{I}_{\mathbb{R}^n} - B\|_{\text{op}} < 1$. Show that the h_i's are linearly independent.

(**iii**) Continuing part (**ii**), let $\{f_1, \dots, f_n\}$ be the orthonormal set obtained from the h_i's by the Gram–Schmidt orthogonalization procedure, and let A be matrix whose (i,j)th entry is $(h_i, f_j)_H$. Show that A is lower triangular and that its diagonal entries are non-negative. In addition, show that $AA^\top = B$.

* This conclusion highlights the poverty of the result here in comparison to Segal's result, which says that $T_{\mathcal{O}}$ is ergodic as soon as the spectrum of $\mathcal{O}_c$ is continuous.

(**iv**) By combining (**i**) and (**iii**), show that there is a $K_n < \infty$, depending only on n, such that

$$\sum_{i=1}^{n} \|h_i - f_i\|_H \le K_n \sum_{i,j=1}^{n} \bigl|\delta_{i,j} - (h_i, h_j)_H\bigr|.$$

Hint: Note that $h_i = \sum_{j=1}^{n} a_{ij} f_j$ and therefore that

$$\|h_i - f_i\|_H^2 = \sum_{j=1}^{n} \bigl(\mathbf{I}_{\mathbb{R}^n} - A\bigr)_{ij}^2 \le n \|\mathbf{I}_{\mathbb{R}^n} - A\|_{\mathrm{op}}^2.$$

Exercise 8.3.17. Given a Hilbert space H, the problem of determining for which Banach spaces H arises as the Cameron–Martin space is an extremely delicate one. For example, one might hope that H will be the Cameron–Martin space for E if H is dense in E and its closed unit ball $\overline{B_H(0,1)}$ is compact in E. However, this is not the case. For example, take $H = \ell^2(\mathbb{N};\mathbb{R})$ and let E be the completion of H with respect to $\|\boldsymbol{\xi}\|_E \equiv \sqrt{\sum_{n=0}^{\infty} \frac{\xi_n^2}{n+1}}$. Show that $\overline{B_H(0,1)}$ is compact as a subset of E but that there is no $\mathcal{W} \in \mathbf{M}_1(E)$ for which $(H, E, \mathcal{W})$ is an abstract Wiener space.

Hint: The first part is an easy application of the standard diagonalization argument combined with the obvious fact that $\sum_{n \ge m} \frac{\xi_n^2}{n+1} \le \frac{1}{m+1} \|\boldsymbol{\xi}\|_{\ell^2(\mathbb{N};\mathbb{R})}$. To prove the second part, note that in order for $\mathcal{W}$ to exist it would be necessary for $\sum_{n=0}^{\infty} \frac{\xi_n^2}{n+1}$ to be $\gamma_{0,1}^{\mathbb{N}}$-almost surely convergent.

Exercise 8.3.18. Let $N = 1$. Using Theorem 8.3.3, take Wiener's choice of orthonormal basis and check that there are independent, standard normal random variables $\{X_m : m \ge 1\}$ under $\mathcal{W}^{(1)}$, such that, for $\mathcal{W}^{(1)}$-almost almost every θ,

$$\theta(t) = t X_0(\theta) + 2^{\frac{1}{2}} \sum_{m=1}^{\infty} X_m(\theta) \frac{\sin(\pi m t)}{m\pi}, \quad t \in [0,1],$$

where the convergence is uniform. From this, show that, $\mathcal{W}^{(1)}$-almost surely,

$$\int_0^1 \theta(t)^2\, dt = \frac{X_0(\theta)^2}{3} + \frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{X_m(\theta)^2 + \sqrt{8} X_0(\theta) X_m(\theta)}{m^2},$$

where the convergence of the series is absolute. Using the preceding, conclude that, for any $\alpha \in (0,\infty)$,

$$\mathbb{E}^{\mathcal{W}^{(1)}}\left[-\alpha \int_0^1 \theta(t)^2\, dt\right] = \left[\prod_{m=1}^{\infty} \left(1 + \frac{2\alpha}{m^2\pi^2}\right)\right]^{-\frac{1}{2}} \left[1 + 4\alpha \sum_{m=1}^{\infty} \frac{1}{m^2\pi^2 + 2\alpha}\right]^{-\frac{1}{2}}.$$

Finally, recall Euler's product formula

$$\sinh z = \prod_{m=1}^{\infty}\left(1+\frac{z^2}{m^2\pi^2}\right), \quad z \in \mathbb{C},$$

and arrive first at

$$\mathbb{E}^{\mathcal{W}^{(1)}}\left[\exp\left(-\alpha\int_0^1 \theta(t)^2\,dt\right)\right] = \left[\cosh\sqrt{2\alpha}\right]^{-\frac{1}{2}}$$

and then, after an application of Brownian rescaling, at

$$\mathbb{E}^{\mathcal{W}^{(1)}}\left[\exp\left(-\alpha\int_0^T \theta(t)^2\,dt\right)\right] = \left[\cosh\sqrt{2\alpha}\,T\right]^{-\frac{1}{2}}.$$

This is a famous calculation which can be made using many different methods. We will return to it in § 10.1.3. See, in addition, Exercise 8.4.6.

Hint: Use Euler's product formula to see that

$$\frac{d}{dt}\log\frac{\sinh t}{t} = 2t\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+t^2} \quad \text{for} \quad t \in \mathbb{R}.$$

Exercise 8.3.19. Related to the preceding exercise, but easier, is finding the Laplace transform of the variance

$$V_T(\theta) \equiv \frac{1}{T}\int_0^T \theta(t)^2\,dt - \left(\frac{1}{T}\int_0^T \theta(t)\,dt\right)^2$$

of a Brownian path over the interval $[0,T]$. To do this calculation, first use Brownian scaling to show that

$$\mathbb{E}^{\mathcal{W}^{(1)}}\left[e^{-\alpha V_T}\right] = \mathbb{E}^{\mathcal{W}^{(1)}}\left[e^{-\alpha T V_1}\right].$$

Next, use elementary Fourier series to show that (cf. part (**iii**) of Exercise 8.2.13)

$$V_1(\theta) = 2\sum_{k=1}^{\infty}\left(\int_0^1 \theta(t)\cos(k\pi t)\,dt\right)^2 = \sum_{k=1}^{\infty}\frac{\left(\int_0^1 f_k(t)\,d\theta(t)\right)^2}{k^2\pi^2},$$

where $f_k(t) = 2^{\frac{1}{2}}\sin(k\pi t)$ for $k \geq 1$. Since the f_k's are orthonormal as elements of $L^2\big([0,\infty);\mathbb{R}\big)$, this leads to

$$\mathbb{E}^{\mathcal{W}^{(1)}}\left[e^{-\alpha V_1}\right] = \prod_{k=1}^{\infty}\left(1+\frac{2\alpha}{k^2\pi^2}\right)^{-\frac{1}{2}}.$$

Now apply Euler's formula to arrive at

$$\mathbb{E}^{\mathcal{W}}\left[e^{-\alpha V_T}\right] = \sqrt{\frac{\sqrt{2\alpha T}}{\sinh(\sqrt{2\alpha T})}}.$$

Finally, using Wiener's choice of basis, show that $\theta \rightsquigarrow V_1(\theta)$ has the same distribution as $\theta \rightsquigarrow \int_0^1\big(\theta(t)-t\theta(1)\big)^2\,dt$ under $\mathcal{W}^{(1)}$, a fact for which I have no conceptual explanation.

EXERCISE 8.3.20. The purpose of this exercise is to show that, without knowing ahead of time that $\mathcal{W}^{(N)}$ lives on $\Theta(\mathbb{R}^N)$, for the Hilbert space $\mathbf{H}^1(\mathbb{R}^N)$ one can give a proof that any Wiener series converges $\gamma_{0,1}^{\mathbb{N}}$-almost surely in $\Theta(\mathbb{R}^N)$. Thus, let $\{\mathbf{h}_m : m \geq 0\}$ be an orthonormal basis in $\mathbf{H}(\mathbb{R}^N)$ and, for $n \in \mathbb{N}$ and $\boldsymbol{\omega} = (\omega_0, \dots, \omega_m, \dots) \in \mathbb{R}^{\mathbb{N}}$, set $\mathbf{S}_n(t, \boldsymbol{\omega}) = \sum_{m=0}^n \omega_m \mathbf{h}_m(t)$. The goal is to show that $\{\mathbf{S}_n(\,\cdot\,, \boldsymbol{\omega}) : n \geq 0\}$ converges in $\Theta(\mathbb{R}^N)$ for $\gamma_{0,1}^{\mathbb{N}}$-almost every $\boldsymbol{\omega} \in \mathbb{R}^{\mathbb{N}}$.

(**i**) For $\boldsymbol{\xi} \in \mathbb{R}^N$, set $\mathbf{h}_{t,\boldsymbol{\xi}}(\tau) = t \wedge \tau \boldsymbol{\xi}$, check that $\big(\boldsymbol{\xi}, \mathbf{S}_n(t)\big)_{\mathbb{R}^N} = \big(\mathbf{h}_{t,\boldsymbol{\xi}}, \mathbf{S}_n(t)\big)_{\mathbf{H}^1(\mathbb{R}^N)}$, and apply Theorem 1.4.2 to show that $\lim_{n\to\infty} \big(\boldsymbol{\xi}, \mathbf{S}_n(t)\big)_{\mathbb{R}^N}$ exists both $\gamma_{0,1}^{\mathbb{N}}$-almost surely and in $L^2(\gamma_{0,1}^{\mathbb{N}}; \mathbb{R})$ for each $(t, \boldsymbol{\xi}) \in [0,\infty) \times \mathbb{R}^N$. Conclude from this that, for each $t \in [0, \infty)$, $\lim_{n\to\infty} \mathbf{S}_n(t)$ exists both $\gamma_{0,1}^{\mathbb{N}}$-almost surely and in $L^2(\gamma_{0,1}^{\mathbb{N}}; \mathbb{R}^N)$.

(**ii**) On the basis of part (**i**), show that we will be done once we know that, for $\gamma_{0,1}^{\mathbb{N}}$-almost every $\mathbf{x} \in \mathbb{R}^N$, $\{\mathbf{S}_n(\,\cdot\,, \mathbf{x}) : n \geq 0\}$ is equicontinuous on finite intervals and that $\sup_{n\geq 0} t^{-1} |\mathbf{S}_n(t, \mathbf{x})| \longrightarrow 0$ as $t \to \infty$. Show that both these will follow from the existence of a $C < \infty$ such that

$$(*)\qquad \mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{0\leq s<t\leq T}\sup_{n\geq 0}\frac{\big|\mathbf{S}_n(t) - \mathbf{S}_n(s)\big|}{(t-s)^{\frac{1}{8}}}\right] \leq C T^{\frac{3}{8}} \quad \text{for all } T \in (0,\infty).$$

(**iii**) As an application of Theorem 4.3.2, show that (*) will follow once one checks that

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0} |\mathbf{S}_n(t) - \mathbf{S}_n(s)|^4\right] \leq B(t-s)^2, \quad 0 \leq s < t,$$

for some $B < \infty$. Next, apply (6.1.14) to see that

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0} |\mathbf{S}_n(t) - \mathbf{S}_n(s)|^4\right] \leq \left(\frac{4}{3}\right)^4 \sup_{n\geq 0} \mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|\mathbf{S}_n(t) - \mathbf{S}_n(s)|^4\big].$$

In addition, because $\mathbf{S}_n(t) - \mathbf{S}_n(s)$ is a centered Gaussian, argue that

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|\mathbf{S}_n(t) - \mathbf{S}_n(s)|^4\big] \leq 3\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|\mathbf{S}_n(t) - \mathbf{S}_n(s)|^2\big]^2.$$

Finally, repeat the sort of reasoning used in (**i**) to check that

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|\mathbf{S}_n(t) - \mathbf{S}_n(s)|^2\big] \leq N(t-s) \quad \text{for } 0 \leq s < t.$$

EXERCISE 8.3.21. In this exercise we discuss some properties of pinned Brownian motion. Given $T > 0$, set $\boldsymbol{\theta}_T(t) = \boldsymbol{\theta}(t) - \frac{t\wedge T}{T}\boldsymbol{\theta}(T)$. As I pointed out at the end of § 8.3.2, the $\mathcal{W}^{(N)}$-distribution of $\boldsymbol{\theta}_T$ is that of a Brownian motion conditioned to be back at $\mathbf{0}$ at time T. Next take $\Theta_T(\mathbb{R}^N)$ to be the space of continuous paths $\boldsymbol{\theta} : [0,T] \longrightarrow \mathbb{R}^N$ satisfying $\boldsymbol{\theta}(0) = \mathbf{0} = \boldsymbol{\theta}(T)$, and let $\mathcal{W}_T^{(N)}$ denote the $\mathcal{W}^{(N)}$-distribution of $\boldsymbol{\theta} \in \Theta(\mathbb{R}^N) \longmapsto \boldsymbol{\theta}_T \in \Theta_T(\mathbb{R}^N)$.

(**i**) Show that the $\mathcal{W}^{(N)}$-distribution of $\{\boldsymbol{\theta}_T(t) : t \geq 0\}$ is the same as that of $\{T^{\frac{1}{2}}\boldsymbol{\theta}_1(T^{-1}t) : t \geq 0\}$.

(**ii**) Set $\mathbf{H}^1_T(\mathbb{R}^N) = \{\mathbf{h} \restriction [0,T] : \mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N) \ \& \ \mathbf{h}(T) = 0\}$, and define $\|\mathbf{h}\|_{\mathbf{H}^1_T(\mathbb{R}^N)} = \|\dot{\mathbf{h}}\|_{L^2([0,T];\mathbb{R}^N)}$. Show that the triple $\big(\mathbf{H}^1_T(\mathbb{R}^N), \Theta_T(\mathbb{R}^N), \mathcal{W}^{(N)}_T\big)$ is an abstract Wiener space. In addition, show that $\mathcal{W}^{(N)}_T$ is invariant under **time reversal**. That is, $\{\boldsymbol{\theta}(t) : t \in [0,T]\}$ and $\{\boldsymbol{\theta}(T-t) : t \in [0,T]\}$ have the same distribution under $\mathcal{W}^{(N)}_T$.

Hint: Begin by identifying $\Theta_T(\mathbb{R}^N)^*$ as the space of finite, $\mathbb{R}^N$-valued Borel measures $\boldsymbol{\lambda}$ on $[0,T]$ such that $\boldsymbol{\lambda}(\{\mathbf{0}\}) = \mathbf{0} = \boldsymbol{\lambda}(\{T\})$.

Exercise 8.3.22. Say that $D \subseteq E^*$ is determining if $x = y$ whenever $\langle x, x^*\rangle = \langle y, x^*\rangle$ for all $x^* \in D$. Next, referring to Theorem 8.3.14, suppose that $\mathcal{O}$ is an orthogonal transformation on H and that $F : E \longmapsto E$ has the properties that $F \restriction H = \mathcal{O}$ and that $x \rightsquigarrow \langle F(x), x^*\rangle$ is continuous for all x^*'s from a determining set D. Show that $T_{\mathcal{O}}x = F(x)$ for $\mathcal{W}$-almost every $x \in E$.

Exercise 8.3.23. Consider $\big(\mathbf{H}^1(\mathbb{R}^N), \Theta(\mathbb{R}^N), \mathcal{W}^{(N)}\big)$, the classical Wiener space. Given $\alpha \in (0,\infty)$, define $\mathcal{O}_\alpha : \mathbf{H}^1(\mathbb{R}^N) \longrightarrow \mathbf{H}(\mathbb{R}^N)$ by $[\mathcal{O}_\alpha \mathbf{h}](t) = \alpha^{-\frac{1}{2}}\mathbf{h}(\alpha t)$, show that $\mathcal{O}_\alpha$ is an orthogonal transformation, and apply Exercise 8.3.22 to see that $T_{\mathcal{O}_\alpha}$ is the Brownian scaling map S_α given by $S_\alpha\boldsymbol{\theta}(t) = \alpha^{-\frac{1}{2}}\boldsymbol{\theta}(\alpha t)$ discussed in part (**iii**) of Exercise 4.3.10. The main goal of this exercise is to apply Theorem 8.3.15 to show that $T_{\mathcal{O}_\alpha}$ is ergodic for every $\alpha \in (0,\infty)\setminus\{1\}$.

(**i**) Given an orthogonal transformation $\mathcal{O}$ on $\mathbf{H}^1(\mathbb{R}^N)$, show that $\big(\mathcal{O}^n\mathbf{h}, \mathbf{h}'\big)_{\mathbf{H}^1(\mathbb{R}^N)}$ tends to 0 for all $\mathbf{h}$, $\mathbf{h}' \in \mathbf{H}^1(\mathbb{R}^N)$ if $\lim_{n\to\infty}\big(\mathcal{O}^n\mathbf{h}, \mathbf{h}'\big)_{\mathbf{H}^1(\mathbb{R}^N)} = 0$ for all $\mathbf{h}$, $\mathbf{h}' \in \mathbf{H}(\mathbb{R}^N)$ with $\dot{\mathbf{h}}$, $\dot{\mathbf{h}}' \in C^\infty_{\mathrm{c}}\big((0,\infty);\mathbb{R}^N\big)$.

(**ii**) Complete the program by showing that $\big(\mathcal{O}^n_\alpha\mathbf{h}, \mathbf{h}'\big)_{\mathbf{H}^1(\mathbb{R}^N)}$ tends to 0 for all $\alpha \in (0,\infty)\setminus\{1\}$ and $\mathbf{h}$, $\mathbf{h}' \in \mathbf{H}^1(\mathbb{R}^N)$ with $\dot{\mathbf{h}}$, $\dot{\mathbf{h}}' \in C^\infty_{\mathrm{c}}\big((0,\infty);\mathbb{R}^N\big)$.

(**iii**) There is another way to think about the operator $\mathcal{O}_\alpha$. Namely, define $U : \mathbf{H}(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R};\mathbb{R}^N)$ by $Uh(x) = e^{\frac{x}{2}}\dot{h}(e^x)$, and show that U is an isometry from $\mathbf{H}^1(\mathbb{R}^N)$ onto $L^2(\mathbb{R};\mathbb{R}^N)$. Further, show that $U \circ \mathcal{O}_\alpha = \tau_{\log\alpha} \circ U$, where $\tau_\alpha : L^2(\mathbb{R};\mathbb{R}^N) \longrightarrow L^2(\mathbb{R};\mathbb{R}^N)$ is the translation map $\tau_\alpha f(x) = f(x+\alpha)$. Conclude from this that

$$\big(\mathcal{O}^n_\alpha\mathbf{h}, \mathbf{h}'\big)_{\mathbf{H}^1(\mathbb{R}^N)} = (2\pi)^{-1}\int_{\mathbb{R}} e^{-\sqrt{-1}n\xi\log\alpha}\big(\widehat{U\mathbf{h}}(\xi), \widehat{U\mathbf{h}'}\big)_{\mathbb{C}^N}\,d\xi,$$

and use this, together with the Riemann–Lebesgue Lemma, to give a second proof that if $\alpha \neq 1$ then $\big(\mathcal{O}^n_\alpha\mathbf{h}, \mathbf{h}'\big)_{\mathbf{H}^1(\mathbb{R}^N)}$ tends to 0 as $n \to \infty$.

(**iv**) As a consequence of the above and Theorem 6.2.8, show that for each $\alpha \in (0,\infty)\setminus\{1\}$, $q \in [1,\infty)$, and $F \in L^q(\mathcal{W}^{(N)};\mathbb{C})$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1} F\big(S_{\alpha^n}\boldsymbol{\theta}\big) = \mathbb{E}^{\mathcal{W}^{(N)}}[F]\ \mathcal{W}^{(N)}\text{-almost surely and in } L^q(\mathcal{W}^{(N)};\mathbb{C}).$$

Next, replace Theorem 6.2.8 by Theorem 6.2.13 to show that

$$\lim_{t\to\infty}\frac{1}{\log t}\int_1^t \tau^{-1}F\big(S_\tau\boldsymbol{\theta}\big)\,d\tau = \mathbb{E}^{\mathcal{W}^{(N)}}[F]$$

$\mathcal{W}^{(N)}$-almost surely and in $L^q(\mathcal{W}^{(N)};\mathbb{C})$. In particular, use this to show that, for $n \in \mathbb{N}$,

$$\lim_{t\to\infty}\frac{1}{\log t}\int_1^t \tau^{-\frac{n}{2}-1}\theta(\tau)^n\,d\tau = \begin{cases} \prod_{m=1}^{\frac{n}{2}}(2m-1) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Exercise 8.3.24. Here is a second reasonably explicit example to which Theorem 8.3.15 applies. Again consider the classical case when $H = \mathbf{H}^1(\mathbb{R}^N)$, and assume that $N \in \mathbb{Z}^+$ is even. Choose a skew-symmetric $A \in \mathrm{Hom}(\mathbb{R}^N;\mathbb{R}^N)$ whose kernel is $\{\mathbf{0}\}$. That is, $A^\top = -A$ and $A\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.

(**i**) Define $\mathcal{O}_A$ on $\mathbf{H}^1(\mathbb{R}^N)$ by

$$\mathcal{O}_A\mathbf{h}(t) = \int_0^t e^{\tau A}\dot{\mathbf{h}}(\tau)\,d\tau,$$

and show that $\mathcal{O}_A$ is an orthogonal transformation which satisfies the hypotheses in Theorem 8.3.15.

Hint: Using elementary spectral theory, show that there exist non-zero, real numbers $\alpha_1,\dots,\alpha_{\frac{N}{2}}$ and an orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_N)$ in $\mathbb{R}^N$ such that $A\mathbf{e}_{2m-1} = \alpha_m\mathbf{e}_{2m}$ and $A\mathbf{e}_{2m} = -\alpha_m\mathbf{e}_{2m-1}$ for $1 \le m \le \frac{N}{2}$. Thus, if L_m is the space spanned by $\mathbf{e}_{2m-1}$ and $\mathbf{e}_{2m}$, then L_m is invariant under A and the action of $e^{\tau A}$ on L_m in terms of this basis is given by

$$\begin{pmatrix}\cos(\alpha_m\tau) & -\sin(\alpha_m\tau)\\ \sin(\alpha_m\tau) & \cos(\alpha_m\tau)\end{pmatrix}.$$

Finally, observe that $\mathcal{O}_A^n = \mathcal{O}_{nA}$, and apply the Riemann–Lebesgue Lemma.

(**ii**) With the help of Exercise 8.3.22, show that

$$T_{\mathcal{O}_A}\boldsymbol{\theta}(t) = \int_0^t e^{\tau A}\,d\boldsymbol{\theta}(\tau),$$

where the integral is taken in the sense of Riemann-Stieltjes.

§ 8.4 A Large Deviations Result and Strassen's Theorem

In this section I will prove the analog of Corollary 1.3.13 for non-degenerate, centered Gaussian measures on a Banach space. Once we have that result, I will apply it to prove Strassen's Theorem, which is the law of the iterated logarithm for such measures.

§ 8.4.1. Large Deviations for Abstract Wiener Space. The goal of this subsection is to derive the following result.

THOEREM 8.4.1. *Let $(H, E, \mathcal{W})$ be an abstract Wiener space, and, for $\epsilon > 0$, denote by $\mathcal{W}_\epsilon$ the $\mathcal{W}$-distribution of $x \rightsquigarrow \epsilon^{\frac{1}{2}} x$. Then, for each $\Gamma \in \mathcal{B}_E$,*

$$\begin{aligned} -\inf_{h\in\Gamma^\circ} \frac{\|h\|_H^2}{2} &\le \varliminf_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \\ &\le \varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \le -\inf_{h\in\overline{\Gamma}} \frac{\|h\|_H^2}{2}. \end{aligned} \tag{8.4.2}$$

The original version of Theorem 8.4.1 was proved by M. Schilder for the classical Wiener measure using a method which does not extend easily to the general case. The statement which I have given is due to Donsker and S.R.S. Varadhan, and my proof derives from an approach (which very much resembles the argument's given in § 1.3 to prove Cramér's Theorem) was introduced introduced into this context by Varadhan.

The lower bound is an easy application of the Cameron–Martin formula. Indeed, all that we have to do is show that if $h \in H$ and $r > 0$, then

$$\varliminf_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon\big(B_E(h,r)\big) \ge -\frac{\|h\|_H^2}{2}. \tag{*}$$

To this end, note that, for any $x^* \in E^*$ and $\delta > 0$,

$$\begin{aligned} \mathcal{W}_\epsilon\big(B_E(h_{x^*},\delta)\big) &= \mathcal{W}\big(B_E(\epsilon^{-\frac{1}{2}} h_{x^*}, \epsilon^{-\frac{1}{2}}\delta)\big) \\ &= \mathbb{E}^{\mathcal{W}}\Big[e^{-\epsilon^{-\frac{1}{2}}\langle x,x^*\rangle - \frac{1}{2\epsilon}\|h_{x^*}\|_H^2}, B_E(0,\epsilon^{-\frac{1}{2}}\delta)\Big] \\ &\ge e^{-\delta\epsilon^{-1}\|x^*\|_{E^*} - \frac{1}{2\epsilon}\|h_{x^*}\|_H^2}\mathcal{W}\big(B_E(0,\epsilon^{-\frac{1}{2}}\delta)\big), \end{aligned}$$

which means that

$$B_E(h_{x^*},\delta) \subseteq B_E(h,r) \implies \varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon\big(B_E(h_{x^*},r)\big) \ge -\delta\|x^*\|_{E^*} - \frac{\|h_{x^*}\|_H^2}{2},$$

and therefore, after letting $\delta \searrow 0$ and remembering that $\{h_{x^*} : x \in E^*\}$ is dense in H, that (*) holds.

The proof of the upper bound in (8.4.2) is a little more involved. The first step is to show that it suffices to treat the case when Γ is relatively compact. To this end, refer to Corollary 8.3.10, and set C_R equal to the closure in E of $B_{E_0}(0,R)$. By Fernique's Theorem applied to $\mathcal{W}$ on E_0, we know that $\mathbb{E}^{\mathcal{W}}\bigl[e^{\alpha\|x\|^2_{E_0}}\bigr] \le K < \infty$ for some $\alpha > 0$. Hence

$$\mathcal{W}_\epsilon\bigl(E \setminus C_R\bigr) = \mathcal{W}\bigl(E \setminus C_{\epsilon^{-\frac{1}{2}}R}\bigr) \le Ke^{-\alpha\frac{R^2}{\epsilon}},$$

and so, for any $\Gamma \in \mathcal{B}_E$ and $R > 0$,

$$\mathcal{W}_\epsilon\bigl(\Gamma\bigr) \le 2\mathcal{W}\epsilon(\Gamma \cap C_R) \vee \bigl(Ke^{-\alpha\frac{R^2}{\epsilon}}\bigr).$$

Thus, if we can prove the upper bound for relatively compact Γ's, then, because $\Gamma \cap C_R$ is relatively compact, we will know that, for all $R > 0$,

$$\varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \le -\left[\left(\inf_{h\in\overline{\Gamma}} \frac{\|h\|_H^2}{2}\right) \wedge \bigl(\alpha R^2\bigr)\right],$$

from which the general result is immediate.

To prove the upper bound when Γ is relatively compact, I will show that, for any $y \in E$,

$$(**) \qquad \varlimsup_{r\searrow 0}\varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon\bigl(B_E(y,r)\bigr) \le \begin{cases} -\frac{\|y\|_H^2}{2} & \text{if } y \in H \\ -\infty & \text{if } y \notin H. \end{cases}$$

To see that (**) is enough, assume that it is true and let $\Gamma \in \mathcal{B}_E\setminus\{\emptyset\}$ be relatively compact. Given $\beta \in (0,1)$, for each $y \in \overline{\Gamma}$ choose $r(y) > 0$ and $\epsilon(y) > 0$ so that

$$\mathcal{W}_\epsilon\bigl(B_E(y,r(y))\bigr) \le \begin{cases} e^{-\frac{(1-\beta)}{2\epsilon}\|y\|_H^2} & \text{if } y \in H \\ e^{-\frac{1}{\beta\epsilon}} & \text{if } y \notin H \end{cases}$$

for all $0 < \epsilon \le \epsilon(y)$. Because Γ is relatively compact, we can find $N \in \mathbb{Z}^+$ and $\{y_1,\dots,y_N\} \subseteq \overline{\Gamma}$ such that $\Gamma \subseteq \bigcup_1^N B_E(y_n,r_n)$, where $r_n = r(y_n)$. Then, for sufficiently small $\epsilon > 0$,

$$\mathcal{W}_\epsilon(\Gamma) \le N\exp\left(-\left[\left(\frac{1-\beta}{2\epsilon}\inf_{h\in\overline{\Gamma}}\|h\|_H^2\right) \wedge \frac{1}{\epsilon\beta}\right]\right),$$

and so

$$\varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon(\Gamma) \le -\left[\left(\frac{1-\beta}{2}\inf_{h\in\overline{\Gamma}}\|h\|_H^2\right) \wedge \frac{1}{\beta}\right].$$

Now let $\beta \searrow 0$.

Finally, to prove (**), observe that

$$\mathcal{W}_\epsilon\big(B_E(y,r)\big) = \mathcal{W}\Big(B_E(\tfrac{y}{\sqrt{\epsilon}}, \tfrac{r}{\sqrt{\epsilon}})\Big) = \mathbb{E}^{\mathcal{W}}\Big[e^{-\epsilon^{-\frac12}\langle x,x^*\rangle}e^{\epsilon^{-\frac12}\langle x,x^*\rangle},\, B_E(\tfrac{y}{\sqrt{\epsilon}}, \tfrac{r}{\sqrt{\epsilon}})\Big]$$
$$\le e^{-\epsilon^{-1}(\langle y,x^*\rangle - r\|x^*\|_{E^*})}\mathbb{E}^{\mathcal{W}}\big[e^{\epsilon^{-\frac12}\langle x,x^*\rangle}\big] = e^{-\epsilon^{-1}\left(\langle y,x^*\rangle - \frac{\|h_{x^*}\|_H^2}{2} - r\|x^*\|_{E^*}\right)},$$

for all $x^* \in E$. Hence,

$$\varlimsup_{r\searrow 0}\varlimsup_{\epsilon\searrow 0} \epsilon \log \mathcal{W}_\epsilon\big(B_E(y,r)\big) \le -\sup_{x^*\in E^*}\Big(\langle y,x^*\rangle - \tfrac12\|h_{x^*}\|_H^2\Big).$$

Finally, note that the preceding supremum is the same as half the supremum of $\langle y,x^*\rangle$ over x^* with $\|h_{x^*}\|_H = 1$, which, by Lemma 8.2.3, is equal to $\frac{\|y\|_H^2}{2}$ if $y \in H$ and to ∞ if $y \notin H$.

An interesting corollary of Theorem 8.4.1 is the following sharpening, due to Donsker and Varadhan, of Fernique's Theorem.

Corollary 8.4.3. *Let $\mathcal{W}$ be a non-degenerate, centered, Gaussian measure on the separable Banach space E, let H be the associated Cameron–Martin space, and determine $\Sigma > 0$ by $\Sigma^{-1} = \inf\{\|h\|_H : \|h\|_E = 1\}$. Then*

$$\lim_{R\to\infty} R^{-2}\log\mathcal{W}\big(\|x\|_E \ge R\big) = -\frac{1}{2\Sigma^2}.$$

In particular, $\mathbb{E}^{\mathcal{W}}\big[e^{\frac{\alpha^2}{2}\|x\|_E^2}\big]$ is finite if $\alpha < \Sigma^{-1}$ and infinite if $\alpha \ge \Sigma^{-1}$.

Proof: Set $f(r) = \inf\{\|h\|_H : \|h\|_E \ge r\}$. Clearly $f(r) = rf(1)$ and $f(1) = \Sigma^{-1}$. Thus, by the upper bound in (8.4.2), we know that

$$\varlimsup_{R\to\infty} R^{-2}\log\mathcal{W}\big(\|x\|_E \ge R\big) = \varlimsup_{R\to\infty} R^{-2}\log\mathcal{W}_{R^{-2}}\big(\|x\|_E \ge 1\big) \le -\frac{f(1)^2}{2} = \frac{\Sigma^{-2}}{2}.$$

Similarly, by the lower bound in (8.4.2), for any $\delta \in (0,1)$,

$$\varliminf_{R\to\infty} R^{-2}\log\mathcal{W}\big(\|x\|_E \ge R\big) \ge \varliminf_{R\to\infty} R^{-2}\log\mathcal{W}\big(\|x\|_E > R\big)$$
$$\ge -\inf\left\{\frac{\|h\|_H^2}{2} : \|h\|_E > R\right\} \ge -\frac{f(1+\delta)^2}{2} = -(1+\delta)^2\frac{1}{2\Sigma^2}.$$

and so we have now proved the first assertion.

Given the first assertion, it is obvious that $\mathbb{E}^{\mathcal{W}}\big[e^{\frac{\alpha^2\|x\|_E^2}{2}}\big]$ is finite when $\alpha < \Sigma^{-1}$ and infinite when $\alpha > \Sigma^{-1}$. The case when $\alpha = \Sigma^{-1}$ is more delicate. To handle it, I first show that $\Sigma = \sup\{\|h_{x^*}\|_H : \|x^*\|_{E^*} = 1\}$. Indeed, if $x^* \in E^*$ and $\|x^*\|_{E^*} = 1$, set $g = \frac{h_{x^*}}{\|h_{x^*}\|_E}$, note that $\|g\|_E = 1$, and check that

$1 \geq \langle g, x^* \rangle = \big(g, h_{x^*}\big)_H = \|g\|_H \|h_{x^*}\|_H$. Hence $\|h_{x^*}\|_H \leq \|g\|_H^{-1} \leq \Sigma$. Next, suppose that $h \in H$ with $\|h\|_E = 1$. Then, by the Hahn–Banach Theorem, there exists a $x^* \in E^*$ with $\|x\|_{E^*} = 1$ and $\langle h, x^* \rangle = 1$. In particular, $\|h\|_H \|h_{x^*}\|_H \geq \big(h, h_{x^*}\big)_H = \langle h, x^* \rangle = 1$, and therefore $\|h\|_H^{-1} \leq \|h_{x^*}\|_H$, which, together with the preceding, completes the verification.

The next step is to show that there exists an $x^* \in E^*$ with $\|x^*\|_{E^*} = 1$ such that $\|h_{x^*}\|_H = \Sigma$. To this end, choose $\{x_k^* : k \geq 1\} \subseteq E^*$ with $\|x_k^*\|_{E^*} = 1$ so that $\|h_{x_k^*}\|_H \longrightarrow \Sigma$. Because $\overline{B_{E^*}(0,1)}$ is compact in the weak* topology and, by Theorem 8.2.6, $x^* \in E^* \longmapsto h_{x^*} \in H$ is continuous from the weak* topology into the strong topology, we can assume that $\{x_k^* : k \geq 1\}$ is weak* convergent to some $x^* \in \overline{B_{E^*}(0,1)}$ and that $\|h_{x^*}\|_H = \Sigma$, which is possible only if $\|x^*\|_{E^*} = 1$. Finally, knowing that this x^* exists, note that $\langle \cdot\,, x^* \rangle$ is a centered Gaussian under $\mathcal{W}$ with variance Σ^2. Hence, since $\|x\|_E \geq |\langle x, x^* \rangle|$,

$$\mathbb{E}^{\mathcal{W}}\Big[e^{\frac{\|x\|_E^2}{2\Sigma^2}}\Big] \geq \int_{\mathbb{R}} e^{\frac{\xi^2}{2\Sigma^2}}\, \gamma_{0,\Sigma^2}(d\xi) = \infty. \quad \square$$

§ **8.4.2. Strassen's Law of the Iterated Logarithm.** Just as in § 1.5 we were able to prove a law of the iterated logarithm on the basis of the large deviation estimates in § 1.3, so here the estimates in the preceding subsection will allow us to prove a law of the iterated for centered Gaussian random variables on a Banach space. Specifically, I will prove the following theorem, whose statement is modeled on V. Strassen's famous law of the iterated for Brownian motion (cf. § 8.6.3 below).

Recall from § 1.5 the notation $\Lambda_n = \sqrt{2n \log_{(2)}(n \vee 3)}$ and $\tilde{S}_n = \frac{S_n}{\Lambda_n}$, where $S_n = \sum_1^n X_m$.

Theorem 8.4.4. *Suppose that $\mathcal{W}$ is a non-degenerate, centered, Gaussian measure on the Banach space E, and let H be its Cameron–Martin space. If $\{X_n : n \geq 1\}$ is a sequence of independent, E-valued, $\mathcal{W}$-distributed random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then, $\mathbb{P}$-almost surely, the sequence $\{\tilde{S}_n : n \geq 1\}$ is relatively compact in E and the closed unit ball $\overline{B_H(0,1)}$ in H coincides with its set of limit points. Equivalently, $\mathbb{P}$-almost surely, $\lim_{n\to\infty} \|\tilde{S}_n - \overline{B_H(0,1)}\|_E = 0$ and, for each $h \in \overline{B_H(0,1)}$, $\underline{\lim}_{n\to\infty} \|\tilde{S}_n - h\|_E = 0$.*

Because, by Theorem 8.2.6, $\overline{B_H(0,1)}$ is compact in E, the equivalence of the two formulations is obvious, and so I will concentrate on the second formulation.

I begin by showing that $\lim_{n\to\infty} \|\tilde{S}_n - \overline{B_H(0,1)}\|_E = 0$ $\mathbb{P}$-almost surely, and the fact which underlies my proof is the estimate that, for each open subset G of E and $\alpha < \inf\{\|h\|_H : h \notin G\}$, there is an $M \in (0,\infty)$ with the property that

$$(*) \qquad \mathbb{P}\left(\frac{S_n}{\Lambda} \notin G\right) \leq \exp\left[-\frac{\alpha^2 \Lambda^2}{2n}\right] \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } \Lambda \geq M\sqrt{n}.$$

To check (*), first note that the distribution of S_n under $\mathbb{P}$ is the same as that of $x \rightsquigarrow n^{\frac{1}{2}}x$ under $\mathcal{W}$ and therefore that $\mathbb{P}\left(\frac{\tilde{S}_n}{\Lambda} \notin G\right) = \mathcal{W}_{\frac{n}{\Lambda^2}}(G\complement)$. Hence, (*) is really just an application of the upper bound in (8.4.2). Given (*), I proceed in very much the same way as I did at the analogous place in § 1.5. Namely, for any $\beta \in (1,2)$,

$$\begin{aligned}\varlimsup_{n\to\infty} \|\tilde{S}_n - \overline{B_H(0,1)}\|_E &\le \varlimsup_{m\to\infty} \max_{\beta^{m-1}\le n\le\beta^m} \|\tilde{S}_n - \overline{B_H(0,1)}\|_E \\ &\le \varlimsup_{m\to\infty} \max_{\beta^{m-1}\le n\le\beta^m} \frac{\|S_n - \overline{B_H(0,\Lambda_{[\beta^{m-1}]})}\|_E}{\Lambda_n} \\ &\le \varlimsup_{m\to\infty} \max_{1\le n\le\beta^m} \left\|\frac{S_n}{\Lambda_{[\beta^{m-1}]}} - \overline{B_H(0,1)}\right\|_E.\end{aligned}$$

At this point in § 1.5 (cf. the proof of Lemma 1.5.3), I applied Lévy's reflection principle to get rid of the "max". However, Lévy's argument works only for $\mathbb{R}$-valued random variables, and so here I will replace his estimate by one based on the idea in Exercise 1.4.25.

Lemma 8.4.5. *Let $\{Y_m : m \ge 1\}$ be mutually independent, E-valued random variables, and set $S_n = \sum_{m=1}^n Y_m$ for $n \ge 1$. Then, for any closed $F \subseteq E$ and $\delta > 0$,*

$$\mathbb{P}\left(\max_{1\le m\le n} \|S_m - F\|_E \ge 2\delta\right) \le \frac{\mathbb{P}(\|S_n - F\|_E \ge \delta)}{1 - \max_{1\le m\le n} \mathbb{P}(\|S_n - S_m\|_E \ge \delta)}.$$

Proof: Set

$$A_m = \{\|S_m - F\|_E \ge 2\delta \text{ and } \|S_k - F\|_E < 2\delta \text{ for } 1 \le k < m\}.$$

Following the hint for Exercise 1.4.25, observe that

$$\begin{aligned}&\mathbb{P}\left(\max_{1\le m\le n} \|S_m - F\|_E \ge 2\delta\right) \min_{1\le m\le n} \mathbb{P}(\|S_n - S_m\|_E < \delta) \\ &\le \sum_{m=1}^n \mathbb{P}\big(A_m \cap \{\|S_n - S_m\|_E < \delta\}\big) \le \sum_{m=1}^n \mathbb{P}\big(A_m \cap \{\|S_n - F\|_E \ge \delta\}\big),\end{aligned}$$

which, because the A_m's are disjoint, is dominated by $\mathbb{P}\big(\|S_n - F\|_E \ge \delta\big)$. □

Applying the preceding to the situation at hand, we see that

$$\begin{aligned}&\mathbb{P}\left(\max_{1\le n\le\beta^m} \left\|\frac{S_n}{\Lambda_{[\beta^{m-1}]}} - \overline{B_H(0,1)}\right\|_E \ge 2\delta\right) \\ &\qquad \le \frac{\mathbb{P}\left(\left\|\frac{S_{[\beta^m]}}{\Lambda_{[\beta_{m-1}]}} - \overline{B_H(0,1)}\right\|_E \ge \delta\right)}{1 - \max_{1\le n\le\beta^m} \mathbb{P}\big(\|S_n\|_E \ge \delta\Lambda_{[\beta^{m-1}]}\big)}.\end{aligned}$$

After combining this with the estimate in (*), it is an easy matter to show that, for each $\delta > 0$ there is a $\beta \in (1,2)$ such that

$$\sum_{m=1}^{\infty} \mathbb{P}\left(\max_{\beta^{m-1} \le n \le \beta^m} \left\| \frac{S_n}{\Lambda_{[\beta^{m-1}]}} - \overline{B_H(0,1)} \right\|_E \ge 2\delta \right) < \infty,$$

from which it should be clear why $\overline{\lim}_{n\to\infty} \|\tilde{S}_n - \overline{B_H(0,1)}\|_E = 0$ $\mathbb{P}$-almost surely.

The proof that, $\mathbb{P}$-almost surely, $\underline{\lim}_{n\to\infty} \|\tilde{S}_n - h\|_E = 0$ for all $h \in \overline{B_H(0,1)}$ differs in no substantive way from the proof of the analogous assertion in the second part of Theorem 1.5.9. Namely, because $\overline{B_H(0,1)}$ is separable, it suffices to work with one $h \in \overline{B_H(0,1)}$ at a time. Furthermore, just as I did there, I can reduce the problem to showing that, for each $k \ge 2$, $\epsilon > 0$, and h with $\|h\|_H < 1$,

$$\sum_{m=1}^{\infty} \mathbb{P}\Big(\big\|\tilde{S}_{k^m - k^{m-1}} - h\big\|_E < \epsilon \Big) = \infty.$$

But, if $\|h\|_H < \alpha < 1$, then (8.4.2) says that

$$\mathbb{P}\Big(\big\|\tilde{S}_{k^m - k^{m-1}} - h\big\|_E < \epsilon \Big) = \mathcal{W}_{\frac{k^m - k^{m-1}}{\Lambda^2_{k^m - k^{m-1}}}}\big(B_E(h,\epsilon)\big) \ge e^{-\alpha^2 \log_{(2)}(k^m - k^{m-1})}$$

for all large enough m's.

Exercises for § 8.4

EXERCISE 8.4.6. Show that the Σ in Corollary 8.4.3 is $\frac{1}{2}$ in the case of the classical abstract Wiener space $\big(\mathbf{H}^1(\mathbb{R}^N), \Theta(\mathbb{R}^N), \mathcal{W}^{(N)}\big)$ and therefore that

$$\lim_{R\to\infty} R^{-2} \log \mathcal{W}^{(N)}\big(\|\boldsymbol{\theta}\|_{\Theta(\mathbb{R}^N)} \ge R\big) = -2.$$

Next, show that

$$\lim_{R\to\infty} R^{-2} \log \mathcal{W}^{(N)}\left(\sup_{\tau\in[0,t]} |\boldsymbol{\theta}(\tau)| \ge R \right) = -\frac{1}{2t}$$

and that

$$\lim_{R\to\infty} R^{-2} \log \mathcal{W}^{(N)}\left(\sup_{\tau\in[0,t]} |\boldsymbol{\theta}(\tau)| \ge R \,\middle|\, \boldsymbol{\theta}(t) = 0 \right) = -\frac{2}{t}.$$

Finally, show that

$$\lim_{R\to\infty} R^{-1} \log \mathcal{W}^{(N)}\left(\int_0^t |\boldsymbol{\theta}(\tau)|^2 \, d\tau \ge R \right) = -\frac{\pi^2}{8t^2}$$

and that

$$\lim_{R\to\infty} R^{-1}\log \mathcal{W}^{(N)}\left(\int_0^t |\boldsymbol{\theta}(\tau)|^2\,d\tau \geq R\,\middle|\,\boldsymbol{\theta}(t)=0\right) = -\frac{\pi^2}{2t^2}.$$

Hint: In each case after the first, Brownian scaling can be used to reduce the problem to the case when $t=1$, and the challenge is to find the optimal constant C for which $\|h\|_E \leq C\|h\|_H$, $h\in H$ for the appropriate abstract Wiener space $(E,H,\mathcal{W})$. In the second case $E = C_0\big([0,1]:\mathbb{R}^N\big) \equiv \big\{\boldsymbol{\theta}\restriction[0,1]:\boldsymbol{\theta}\in\Theta(\mathbb{R}^N)\big\}$ and $H=\big\{\boldsymbol{\eta}\restriction[0,1]:\boldsymbol{\eta}\in\mathbf{H}^1(\mathbb{R}^N)\big\}$, in the third (cf. part (**ii**) of Exercise 8.3.21) $E=\Theta_1(\mathbb{R}^N)$ and $H=\mathbf{H}^1_1(\mathbb{R}^N)$, in the fourth $E=L^2\big([0,1];\mathbb{R}^N\big)$ and $H=\{\boldsymbol{\eta}\restriction[0,1]:\boldsymbol{\eta}\in\mathbf{H}^1(\mathbb{R}^N)\}$, and in the fifth $E=L^2\big([0,1];\mathbb{R}^N\big)$ and $H=\mathbf{H}^1_1(\mathbb{R}^N)$. The optimization problems when $E=\Theta(\mathbb{R}^N)$ or $C_0\big([0,1];\mathbb{R}^N\big)$ are rather easy consequences of $|\boldsymbol{\eta}(t)|\leq t^{\frac12}\|\boldsymbol{\eta}\|_{\mathbf{H}^1(\mathbb{R}^N)}$. When $E=\Theta_1(\mathbb{R}^N)$, one should start with the observation that if $\boldsymbol{\eta}\in\mathbf{H}^1_1(\mathbb{R}^N)$, then $2\|\boldsymbol{\eta}\|_{\mathrm{u}}\leq\|\dot{\boldsymbol{\eta}}\|_{L^1([0,1];\mathbb{R}^N)}\leq\|\boldsymbol{\eta}\|_{\mathbf{H}^1_1(\mathbb{R}^N)}$. In the final two cases, one can either use elementary variational calculus or one can use make use of, respectively the orthonormal bases $\big\{2^{\frac12}\sin\big(n+\frac12\big)\pi\tau:n\geq 0\big\}$ and $\{2^{\frac12}\sin n\pi\tau:n\geq1\}$ in $L^2\big([0,1];\mathbb{R}\big)$.

EXERCISE 8.4.7. Suppose that $f\in C\big(E;\mathbb{R}\big)$, and show, as a consequence of Theorem 8.4.4, that

$$\varliminf_{n\to\infty} f\big(\tilde{S}_n\big) = \min\{f(h):\|h\|_H\leq1\} \text{ and } \varlimsup_{n\to\infty} f\big(\tilde{S}_n\big) = \max\{f(h):\|h\|_H\leq 1\}$$

$\mathcal{W}^{\mathbb{N}}$-almost surely.

§ 8.5 Euclidean Free Fields

In this section I will give a very cursory introduction to a family of abstract Wiener spaces which played an important role in the attempt to give a mathematically rigorous construction of quantum fields. From the physical standpoint, the fields treated here are "trivial" in the sense that they model "free" (i.e., non-interacting) fields. Nonetheless, they are interesting from a mathematical standpoint, and, if nothing else, show how profoundly properties of a process are effected by the dimension of its parameter set.

I begin with the case when the parameter set is one-dimensional and the resulting process can be seen as a minor variant of Brownian motion. As we will see, the intractability of the higher dimensional analogs increases with the number of dimensions.

§ 8.5.1. The Ornstein–Uhlenbeck Process. Given $\mathbf{x}\in\mathbb{R}^N$ and $\boldsymbol{\theta}\in\Theta(\mathbb{R}^N)$, consider the integral equation

$$\mathbf{U}(t,\mathbf{x},\boldsymbol{\theta}) = \mathbf{x}+\boldsymbol{\theta}(t)-\frac12\int_0^t \mathbf{U}(\tau,\mathbf{x},\boldsymbol{\theta})\,d\tau,\quad t\geq 0. \tag{8.5.1}$$

A completely elementary argument (e.g., via Gronwall's inequality) shows that, for each $\mathbf{x}$ and $\boldsymbol{\theta}$, there is at most one solution. Furthermore, integration by parts allows one to check that if

$$\mathbf{U}(t,\mathbf{0},\boldsymbol{\theta}) = e^{-\frac{t}{2}} \int_0^t e^{\frac{\tau}{2}}\, d\boldsymbol{\theta}(\tau),$$

where the integral is taken in the sense of Riemann-Stieltjes, then

$$\mathbf{U}(t,\mathbf{x},\boldsymbol{\theta}) = e^{-\frac{t}{2}}\mathbf{x} + \mathbf{U}(t,\mathbf{0},\boldsymbol{\theta})$$

is one, and therefore the one and only, solution.

The stochastic process $\{\mathbf{U}(t,\mathbf{x}) : t \geq 0\}$ under $\mathcal{W}^{(N)}$ was introduced by L. Ornstein and G. Uhlenbeck* and is known as the **Ornstein–Uhlenbeck process** starting from $\mathbf{x}$. From our immediate point of view, its importance is that it leads to a completely tractable example of a free field.

Intuitively, $\mathbf{U}(t,\mathbf{0},\boldsymbol{\theta})$ is a Brownian motion which is subjected to a linear restoring force. Thus, locally it should behave very much like a Brownian motion. However, over long time intervals, it should feel the effect of the restoring force, which is always pushing it back toward the origin. To see how these intuitive ideas are reflected in the distribution of $\{\mathbf{U}(t,\mathbf{0},\boldsymbol{\theta}) : t \geq 0\}$, I begin by using Exercise 8.2.13 to identify $\big(\mathbf{e},\mathbf{U}(t,\mathbf{0})\big)_{\mathbb{R}^N}$ as $e^{-\frac{t}{2}}\mathcal{I}(\mathbf{h}^t_{\mathbf{e}})$ for each $\mathbf{e} \in \mathbb{S}^{N-1}$, where $\mathbf{h}^t_{\mathbf{e}}(\tau) = 2\big(e^{\frac{t\wedge\tau}{2}} - 1\big)\mathbf{e}$. Hence, the span of $\big\{\big(\boldsymbol{\xi},\mathbf{U}(t,\mathbf{0})\big)_{\mathbb{R}^N} : t \geq 0 \;\&\; \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ is a Gaussian family in $L^2(\mathcal{W}^{(N)};\mathbb{R})$, and

$$\mathbb{E}^{\mathcal{W}^{(N)}}\big[\mathbf{U}(s,\mathbf{0}) \otimes \mathbf{U}(t,\mathbf{0})\big] = \big(e^{-\frac{|t-s|}{2}} - e^{-\frac{s+t}{2}}\big)\mathbf{I}.$$

The key to understanding the process $\{\mathbf{U}(t,\mathbf{0}) : t \geq 0\}$ is the observation that it has the same distribution as the process $\big\{e^{-\frac{t}{2}}\mathbf{B}\big(e^t - 1\big) : t \geq 0\big\}$, where $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion, a fact which follows immediately from the observation that they are Gaussian families with the same covariance structure. In particular, by combining this with the Law of the Iterated Logarithm proved in Exercise 4.3.15, we see that, for each $\mathbf{e} \in \mathbb{S}^{N-1}$,

$$\varlimsup_{t\to\infty} \frac{\big(\mathbf{e},\mathbf{U}(t,\mathbf{x})\big)_{\mathbb{R}^N}}{\sqrt{2\log t}} = 1 = -\varliminf_{t\to\infty} \frac{\big(\mathbf{e},\mathbf{U}(t,\mathbf{x})\big)_{\mathbb{R}^N}}{\sqrt{2\log t}} \tag{8.5.2}$$

$\mathcal{W}^{(N)}$-almost surely, which confirms the suspicion that the restoring force dampens the Brownian excursions out toward infinity.

* In their article "On the theory of Brownian motion," *Phys. Reviews* **36** (3), L. Ornstein & G. Uhlenbeck introduced this process in an attempt to reconcile some of the more disturbing properties of Wiener paths with physical reality.

A second indication that $\mathbf{U}(\,\cdot\,,\mathbf{x})$ tends to spend more time than Brownian paths do near the origin is that its distribution at time t will be $\gamma_{e^{-\frac{t}{2}}\mathbf{x},(1-e^{-t})\mathbf{I}}$, and so, as distinguished from Brownian motion itself, its distribution as time t tends to a limit, namely $\gamma_{\mathbf{0},\mathbf{I}}$. This observation suggests that it might be interesting to look at an *ancient* Ornstein–Uhlenbeck process, one that already has been running for an infinite amount of time. To be more precise, since the distribution of an ancient Ornstein–Uhlenbeck at time 0 would be $\gamma_{\mathbf{0},\mathbf{I}}$, what we should look at is the process which we get by making the $\mathbf{x}$ in $\mathbf{U}(\,\cdot\,,\mathbf{x},\boldsymbol{\theta})$ a standard normal random variable. Thus, I will say that a stochastic process $\{\mathbf{U}_A(t) : t \geq 0\}$ is an **ancient Ornstein–Uhlenbeck process** if its distribution is that of $\{\mathbf{U}(t,\mathbf{x},\boldsymbol{\theta}) : t \geq 0\}$ under $\gamma_{\mathbf{0},\mathbf{I}} \times \mathcal{W}^{(N)}$.

If $\{\mathbf{U}_A(t) : t \geq 0\}$ is an ancient Ornstein–Uhlenbeck process, then it is clear that $\big\{\big(\boldsymbol{\xi},\mathbf{U}_A(t)\big)_{\mathbb{R}^N} : t \geq 0 \;\&\; \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ spans a Gaussian family with covariance

$$\mathbb{E}^{\mathbb{P}}\big[\mathbf{U}_A(s) \otimes \mathbf{U}_A(t)\big] = e^{-\frac{|t-s|}{2}}\mathbf{I}.$$

As a consequence, we see that if $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion, then $\big\{e^{-\frac{t}{2}}\mathbf{B}\big(e^t\big) : t \geq 0\big\}$ is an ancient Ornstein–Uhlenbeck process. In addition, as we suspected, the ancient Ornstein–Uhlenbeck process is a **stationary process** in the sense that, for each $T > 0$, the distribution of $\{\mathbf{U}_A(t+T) : t \geq 0\}$ is the same as that of $\{\mathbf{U}_A(t) : t \geq 0\}$, which can checked either by using the preceding represtation in terms of Brownian motion or by observing its covariance is a function of $t - s$.

In fact, even more is true: it is time reversible in the sense that, for each $T > 0$, $\{\mathbf{U}_A(t) : t \in [0,T]\}$ has the same distribution as $\{\mathbf{U}_A(T-t) : t \in [0,T]\}$. This observation suggests that we can give the ancient Ornstein-Uhlenbeck its past by running it backwards. That is, define $\mathbf{U}_R : [0,\infty) \times \mathbb{R}^N \times \Theta(\mathbb{R}^N)^2 \longrightarrow \mathbb{R}^N$ by

$$\mathbf{U}_R(t,\mathbf{x},\boldsymbol{\theta}_+,\boldsymbol{\theta}_-) = \begin{cases} \mathbf{U}(t,\mathbf{x},\boldsymbol{\theta}_+) & \text{if } t \geq 0 \\ \mathbf{U}(-t,\mathbf{x},\boldsymbol{\theta}_-) & \text{if } t < 0, \end{cases}$$

and consider the process $\{\mathbf{U}_R(t,\mathbf{x},\boldsymbol{\theta}_+,\boldsymbol{\theta}_-) : t \in \mathbb{R}\}$ under $\gamma_{\mathbf{0},\mathbf{I}} \times \mathcal{W}^{(N)} \times \mathcal{W}^{(N)}$. This process also spans a Gaussian family, and it is still true that

$$\text{(8.5.3)} \quad \mathbb{E}^{\gamma_{\mathbf{0},\mathbf{I}} \times \mathcal{W}^{(N)} \times \mathcal{W}^{(N)}}\big[\mathbf{U}_R(s) \otimes \mathbf{U}_R(t)\big] = u(s,t)\mathbf{I}, \text{ where } u(s,t) \equiv e^{-\frac{|t-s|}{2}},$$

only now for all $s,\, t \in \mathbb{R}$. One advantage of having added the past is that the statement of reversibility takes a more appealing form. Namely, $\{\mathbf{U}_R(t) : t \in \mathbb{R}\}$ is **reversible** in the sense that its distribution is the same whether one runs it forward or backward in time. That is, $\{\mathbf{U}_R(-t) : t \in \mathbb{R}\}$ has the same distribution as $\{\mathbf{U}_R(t) : t \in \mathbb{R}\}$. For this reason, I will say that $\{\mathbf{U}_R(t) : t \geq 0\}$ is a **reversible Ornstein–Uhlenbeck process** if its distribution is the same as that of $\{\mathbf{U}_R(t,\mathbf{x},\boldsymbol{\theta}_+,\boldsymbol{\theta}_-) : t \geq 0\}$ under $\gamma_{\mathbf{0},\mathbf{I}} \times \mathcal{W}^{(N)} \times \mathcal{W}^{(N)}$.

An alternative way to realize of a reversible Ornstein–Uhlenbeck process is to start with an $\mathbb{R}^N$-valued Brownian motion $\{\mathbf{B}(t) : t \geq 0\}$ and consider the process $\{e^{-\frac{t}{2}}\mathbf{B}(e^t) : t \in \mathbb{R}\}$. Clearly $\left\{\left(\boldsymbol{\xi}, e^{-\frac{t}{2}}\mathbf{B}(e^t)\right)_{\mathbb{R}^N} : (t, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^N\right\}$ is a Gaussian family with covariance given by (8.5.3). It is amusing to observe that, when one uses this realization, the reversibility of the Orstein–Uhlenbeck process is equivalent to the time inversion invariance (cf. Exercise 4.3.11) of the original Brownian motion.

§ **8.5.2. Ornstein–Uhlenbeck as an Abstract Wiener Space.** So far, my treatment of the Ornstein–Uhlenbeck process has been based on its relationship to Brownian motion. Here we will look at it as an abstract Wiener space.

Begin with the one-sided process $\{\mathbf{U}(t, \mathbf{0}) : t \geq 0\}$. Seeing as this process has the same distribution as $\left\{e^{-\frac{t}{2}}\mathbf{B}\left(e^t - 1\right) : t \geq 0\right\}$, it is reasonably clear that the Hilbert space associated with this process should be the space $\mathbf{H}^U(\mathbb{R}^N)$ of functions $\mathbf{h}^U(t) = e^{-\frac{t}{2}}\mathbf{h}\left(e^t - 1\right)$, $\mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N)$. Thus, define the map $F^U : \mathbf{H}^1(\mathbb{R}^N) \longrightarrow \mathbf{H}^U(\mathbb{R}^N)$ accordingly, and introduce the Hilbert norm $\|\cdot\|_{\mathbf{H}^U(\mathbb{R}^N)}$ on $\mathbf{H}^U(\mathbb{R}^N)$ which makes F^U into an isometry. Equivalently,

$$\begin{aligned}\|\mathbf{h}^U\|^2_{\mathbf{H}^U(\mathbb{R}^N)} &= \int_{[0,\infty)} \left[\frac{d}{ds}\Big((1+s)^{\frac{1}{2}}\mathbf{h}^U\big(\log(1+s)\big)\Big)\right]^2 ds \\ &= \|\dot{\mathbf{h}}^U\|^2_{L^2([0,\infty);\mathbb{R}^N)} + \left(\dot{\mathbf{h}}^U, \mathbf{h}^U\right)_{L^2([0,\infty);\mathbb{R}^N)} + \tfrac{1}{4}\|\mathbf{h}^U\|^2_{L^2([0,\infty);\mathbb{R}^N)}.\end{aligned}$$

Note that

$$\left(\dot{\mathbf{h}}^U, \mathbf{h}^U\right)_{L^2([0,\infty);\mathbb{R}^N)} = \tfrac{1}{2}\int_{[0,\infty)} \frac{d}{dt}|\mathbf{h}^U(t)|^2\,dt = \tfrac{1}{2}\lim_{t\to\infty}|\mathbf{h}^U(t)|^2 = 0.$$

To check the final equality, observe that it is equivalent to $\lim_{t\to\infty} t^{-\frac{1}{2}}|\mathbf{h}(t)| = 0$ for $\mathbf{h} \in \mathbf{H}(\mathbb{R}^N)$. Hence, since $\sup_{t>0} t^{-\frac{1}{2}}|\mathbf{h}(t)| \leq \|\mathbf{h}\|_{\mathbf{H}^1(\mathbb{R}^N)}$ and $\lim_{t\to\infty} t^{-\frac{1}{2}}|\mathbf{h}(t)| = 0$ if $\dot{\mathbf{h}}$ has compact support, the same result is true for all $\mathbf{h} \in \mathbf{H}^1(\mathbb{R}^N)$. In particular,

$$\|\mathbf{h}^U\|_{\mathbf{H}^U(\mathbb{R}^N)} = \sqrt{\|\dot{\mathbf{h}}^U\|^2_{L^2([0,\infty);\mathbb{R}^N)} + \tfrac{1}{4}\|\mathbf{h}^U\|^2_{L^2([0,\infty);\mathbb{R}^N)}}.$$

If we were to follow the prescription in Theorem 8.3.1, we would next complete $\mathbf{H}^U(\mathbb{R}^N)$ with respect to the norm $\sup_{t\geq 0} e^{-\frac{t}{2}}|\mathbf{h}^U(t)|$. However, we already know from (8.5.2) that $\{\mathbf{U}(t, 0) : t \geq 0\}$ lives on $\Theta^U(\mathbb{R}^N)$, the space of $\boldsymbol{\theta} \in \Theta(\mathbb{R}^N)$ such that $\lim_{t\to\infty}(\log t)^{-1}|\boldsymbol{\theta}(t)| = 0$ with Banach norm

$$\|\boldsymbol{\theta}\| \equiv \sup_{t\geq 0}\big(\log(e+t)\big)^{-1}|\boldsymbol{\theta}(t)|,$$

and so we will adopt $\Theta^U(\mathbb{R}^N)$ as the Banach space for $\mathbf{H}^U(\mathbb{R}^N)$. Clearly, the dual space $\Theta^U(\mathbb{R}^N)^*$ of $\Theta^U(\mathbb{R}^N)$ can be identified with the space of $\mathbb{R}^N$-valued Borel measures $\boldsymbol{\lambda}$ on $[0,\infty)$ which give 0 mass to $\{0\}$ and satisfy $\|\boldsymbol{\lambda}\|_{\Lambda^U(\mathbb{R}^N)} \equiv \int_{[0,\infty)} \log(e+t)\,|\boldsymbol{\lambda}|(dt) < \infty$.

THEOREM 8.5.4. *Let $\mathcal{U}_0^{(N)} \in \mathbf{M}_1\big(\Theta^U(\mathbb{R}^N)\big)$ be the distribution of $\{\mathbf{U}(t,\mathbf{0}) : t \geq 0\}$ under $\mathcal{W}^{(N)}$. Then $\big(\mathbf{H}^U(\mathbb{R}^N), \Theta^U(\mathbb{R}^N), \mathcal{U}_0^{(N)}\big)$ is an abstract Wiener space.*

PROOF: Since $C_c^\infty\big((0,\infty);\mathbb{R}^N\big)$ is contained in $\mathbf{H}^U(\mathbb{R}^N)$ and is dense in $\Theta^U(\mathbb{R}^N)$, we know that $\mathbf{H}^U(\mathbb{R}^N)$ is dense in $\Theta^U(\mathbb{R}^N)$. In addition, because $\boldsymbol{\eta}^U(t) = e^{-\frac{t}{2}}\boldsymbol{\eta}(e^t-1)$ where $\boldsymbol{\eta} \in \mathbf{H}^1(\mathbb{R}^N)$ and $\|\boldsymbol{\eta}^U\|_{\mathbf{H}^U(\mathbb{R}^N)} = \|\boldsymbol{\eta}\|_{\mathbf{H}^1(\mathbb{R}^N)}$, $\|\boldsymbol{\eta}^U\|_{\mathrm{u}} \leq \|\boldsymbol{\eta}^U\|_{\mathbf{H}^U(\mathbb{R}^N)}$ follows from $|\boldsymbol{\eta}(t)| \leq t^{\frac{1}{2}}\|\boldsymbol{\eta}\|_{\mathbf{H}^1(\mathbb{R}^N)}$. Hence, $\mathbf{H}^U(\mathbb{R}^N)$ is continuously embedded in $\Theta^U(\mathbb{R}^N)$.

To complete the proof, remember our earlier calculation of the covariance of $\{\mathbf{U}(t;\mathbf{0}) : t \geq 0\}$, and use it to check that

$$\mathbb{E}^{\mathcal{U}_0^{(N)}}\big[\langle\boldsymbol{\theta},\boldsymbol{\lambda}\rangle^2\big] = \iint\limits_{[0,\infty)^2} u_0(s,t)\,\boldsymbol{\lambda}(ds)\cdot\boldsymbol{\lambda}(dt) \quad \text{where } u_0(s,t) \equiv e^{-\frac{|s-t|}{2}} - e^{-\frac{s+t}{2}}.$$

Hence, what I need to show is that if $\boldsymbol{\lambda} \in \Theta^U(\mathbb{R}^N)^* \longrightarrow \mathbf{h}^U_{\boldsymbol{\lambda}} \in \mathbf{H}^U(\mathbb{R}^N)$ is the map determined by $\langle\mathbf{h}^U,\boldsymbol{\lambda}\rangle = \big(\mathbf{h}^U,\mathbf{h}^U_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^U(\mathbb{R}^N)}$, then

$$\|\mathbf{h}^U_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}^U(\mathbb{R}^N)} = \iint\limits_{[0,\infty)^2} u_0(s,t)\,\boldsymbol{\lambda}(ds)\cdot\boldsymbol{\lambda}(dt). \tag{8.5.5}$$

In order to do this, we must first know how $\mathbf{h}^U_{\boldsymbol{\lambda}}$ is constructed from $\boldsymbol{\lambda}$. But if (8.5.5) is going to hold, then, by polarization,

$$\begin{aligned}\big(\mathbf{e},\mathbf{h}^U_{\boldsymbol{\lambda}}(\tau)\big)_{\mathbb{R}^N} = \langle\mathbf{h}^U_{\boldsymbol{\lambda}},\delta_\tau\mathbf{e}\rangle &= \iint\limits_{[0,\infty)^2} u_0(s,t)\,\delta_\tau(ds)\big(\mathbf{e},\boldsymbol{\lambda}(dt)\big)_{\mathbb{R}^N} \\ &= \left(\mathbf{e},\int_{[0,\infty)} u_0(\tau,t)\,\boldsymbol{\lambda}(dt)\right)_{\mathbb{R}^N}.\end{aligned}$$

Thus, one should guess that $\mathbf{h}^U_{\boldsymbol{\lambda}}(\tau) = \int_{[0,\infty)} u_0(\tau,t)\,\boldsymbol{\lambda}(dt)$ and must check that, with this choice, $\mathbf{h}^U_{\boldsymbol{\lambda}} \in \mathbf{H}^U(\mathbb{R}^N)$, (8.5.5) holds, and, for all $\mathbf{h}^U \in \mathbf{H}^U(\mathbb{R}^N)$, $\langle\mathbf{h}^U,\boldsymbol{\lambda}\rangle = \big(\mathbf{h}^U,\mathbf{h}^U_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^U(\mathbb{R}^N)}$.

The key to proving all these is the equality

$$\int_{[0,\infty)} \dot{\mathbf{h}}^U(\tau)\partial_\tau u_0(\tau,t)\,d\tau + \tfrac{1}{4}\int_{[0,\infty)} \mathbf{h}^U(\tau)u_0(\tau,t)\,d\tau = \mathbf{h}^U(t), \tag{*}$$

which is an elementary application of integration by parts. Applying (*) with $N=1$ to $h^U = u_0(\,\cdot\,,s)$, we see that

$$\int_{[0,\infty)} \partial_\tau u_0(s,\tau)\partial_\tau u_0(t,\tau)\,d\tau = u_0(s,t),$$

from which it follows easily both that $\mathbf{h}^U_{\boldsymbol{\lambda}} \in \mathbf{H}^U(\mathbb{R}^N)$ and that (8.5.5) holds. In addition, if $\mathbf{h}^U \in \mathbf{H}^U(\mathbb{R}^N)$, then $\langle \mathbf{h}^U, \boldsymbol{\lambda}\rangle = \big(\mathbf{h}^U, \mathbf{h}^U_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^U(\mathbb{R}^N)}$ follows from (*) after one integrates both sides of the preceding with respect to $\boldsymbol{\lambda}(dt)$. $\square$

I turn next to the reversible case. By the considerations in § 8.4.1, we know that the distribution $\mathcal{U}^{(N)}_R$ of $\{\mathbf{U}_R(t) : t \geq 0\}$ under $\gamma_{0,1} \times \mathcal{W}^{(N)} \times \mathcal{W}^{(N)}$ is a Borel measure on the space Banach space $\Theta^U(\mathbb{R};\mathbb{R}^N)$ of continuous $\boldsymbol{\theta} : \mathbb{R} \longrightarrow \mathbb{R}^N$ such that $\lim_{|t|\to\infty} (\log t)^{-1}|\boldsymbol{\theta}(t)| = 0$ with norm

$$\|\boldsymbol{\theta}\|_{\Theta^U(\mathbb{R};\mathbb{R}^N)} \equiv \sup_{t\in\mathbb{R}} \big(\log(e + |t|)\big)^{-1} |\boldsymbol{\theta}(t)| < \infty.$$

Furthermore, it should be clear that one can identify $\Theta^U(\mathbb{R};\mathbb{R}^N)^*$ with the space of $\mathbb{R}^N$-valued Borel measures $\boldsymbol{\lambda}$ on $\mathbb{R}$ satisfying

$$\|\boldsymbol{\lambda}\|_{\Lambda^U(\mathbb{R};\mathbb{R}^N)} \equiv \int_{\mathbb{R}} \log(e + |t|)\, |\boldsymbol{\lambda}|(dt) < \infty.$$

THEOREM 8.5.6. *Take $\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)$ to be the separable Hilbert space of absolutely continuous $\mathbf{h} : \mathbb{R} \longrightarrow \mathbb{R}^N$ satisfying*

$$\|\mathbf{h}\|_{\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)} \equiv \sqrt{\|\dot{\mathbf{h}}\|^2_{L^2(\mathbb{R}:\mathbb{R}^N)} + \tfrac{1}{4}\|\mathbf{h}\|^2_{L^2(\mathbb{R}:\mathbb{R}^N)}} < \infty.$$

Then $\big(\mathbf{H}^1(\mathbb{R};\mathbb{R}^N), \Theta^U(\mathbb{R};\mathbb{R}^N), \mathcal{U}^{(N)}_R\big)$ is an abstract Wiener space.

PROOF: Set $u(s,t) \equiv e^{-\frac{|s-t|}{2}}$, and let $\boldsymbol{\lambda} \in \Lambda^U(\mathbb{R};\mathbb{R}^N)$. By the same reasoning as I used in the preceding proof,

$$\langle \mathbf{h}, \boldsymbol{\lambda}\rangle = \big(\mathbf{h}, \mathbf{h}_{\boldsymbol{\lambda}}\big)_{\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)}$$

and

$$\|\mathbf{h}_{\boldsymbol{\lambda}}\|^2_{\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)} = \iint_{\mathbb{R}\times\mathbb{R}} u(s,t)\, \boldsymbol{\lambda}(ds) \cdot \boldsymbol{\lambda}(dt),$$

when $\mathbf{h}_{\boldsymbol{\lambda}}(\tau) = \int_{\mathbb{R}} u(\tau,t)\,\boldsymbol{\lambda}(dt)$. Hence, since $\big\{\big(\boldsymbol{\xi}, \boldsymbol{\theta}(t)\big)_{\mathbb{R}^N} : t \geq 0 \ \&\ \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ spans a Gaussian family in $L^2\big(\mathcal{U}^{(N)}_R;\mathbb{R}\big)$ and $u(s,t)\mathbf{I} = \mathbb{E}^{\mathcal{U}^{(N)}_R}\big[\boldsymbol{\theta}(s)\otimes\boldsymbol{\theta}(t)\big]$, the proof is complete. $\square$

§ **8.5.3. Higher Dimensional Free Fields.** Thinking *a la* Feynman, Theorem 8.5.6 is saying that $\mathcal{U}^{(N)}_R$ wants to be the measure on $H^1(\mathbb{R};\mathbb{R})$ given by

$$\frac{1}{(\sqrt{2\pi})^{\dim(\mathbf{H}^1(\mathbb{R};\mathbb{R}^N))}} \exp\left[-\frac{1}{2}\int_{\mathbb{R}} \left(|\dot{\mathbf{h}}(t)|^2 + \tfrac{1}{4}|\mathbf{h}(t)|^2\right) dt\right] \lambda_{\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)}(d\mathbf{h}),$$

where $\lambda_{\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)}$ is the Lebesgue measure on $\mathbf{H}^1(\mathbb{R};\mathbb{R}^N)$.

I am now going to look at the analogous situation when $N = 1$ but the parameter set $\mathbb{R}$ is replaced by $\mathbb{R}^\nu$ for some $\nu \geq 2$. That is, I want to look at the measure which Feynman would have written as

$$\frac{1}{(\sqrt{2\pi})^{\dim(H^1(\mathbb{R}^\nu;\mathbb{R}))}} \exp\left[-\frac{1}{2}\int_{\mathbb{R}^\nu} \left(|\nabla h(\mathbf{x}))|^2 + \tfrac{1}{4}|h(\mathbf{x})|^2\right)\, d\mathbf{x}\right] \lambda_{H^1(\mathbb{R}^\nu;\mathbb{R})}(dh),$$

where $H^1(\mathbb{R}^\nu;\mathbb{R})$ is the separable Hilbert space obtained by completing the Schwartz test function space $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$ with respect to the Hilbert norm

$$\|h\|_{H^1(\mathbb{R}^\nu;\mathbb{R})} \equiv \sqrt{\|\nabla h\|^2_{L^2(\mathbb{R}^\nu;\mathbb{R})} + \tfrac{1}{4}\|h\|^2_{L^2(\mathbb{R}^\nu;\mathbb{R})}}.$$

When $\nu = 1$ this is exactly the Hilbert space $H^1(\mathbb{R};\mathbb{R})$ described in Theorem 8.5.6 for $N = 1$. When $\nu \geq 2$, generic elements of $H^1(\mathbb{R}^\nu;\mathbb{R})$ are better than generic elements of $L^2(\mathbb{R}^\nu;\mathbb{R})$ but are not enough better to be continuous. In fact, they are not even well-defined pointwise, and matters get worse as ν gets larger. Thus, although Feynman's representation is already questionable when $\nu = 1$, its interpretation when $\nu \geq 2$ is even more fraught with difficulties. As we will see, these difficulties are reflected mathematically by the fact that in order to construct an abstract Wiener space for $H^1(\mathbb{R}^\nu;\mathbb{R})$ when $\nu \geq 2$, we will have to resort to Banach spaces whose elements are generalized functions (i.e., distributions in the sense of L. Schwartz).*

The approach which I will adopt is based on the following subterfuge. The space $H^1(\mathbb{R}^\nu;\mathbb{R})$ is one of a continuously graded family of spaces known as **Sobolev spaces**. Sobolev space are graded according to the number of derivatives "better or worse" than $L^2(\mathbb{R}^\nu;\mathbb{R})$ their elements are. To be more precise, for each $s \in \mathbb{R}$, define the **Bessel operator** B^s on $\mathscr{S}(\mathbb{R}^\nu;\mathbb{C})$ so that

$$\widehat{B^s\varphi}(\boldsymbol{\xi}) = \left(\tfrac{1}{4} + |\boldsymbol{\xi}|^2\right)^{-\frac{s}{2}}\hat{\varphi}(\boldsymbol{\xi}).$$

When $s = -2m$, it is clear that $B^s = \left(\frac{1}{4} - \Delta\right)^m$, and so, in general, it is reasonable to think of B^s as an operator which, depending on whether $s \leq 0$ or $s \geq 0$, involves taking or restoring derivatives of order $|s|$. In particular, $\|\varphi\|_{H^1(\mathbb{R}^\nu;\mathbb{R})} = \|B^{-1}\varphi\|_{L^2(\mathbb{R}^\nu;\mathbb{R})}$ for $\varphi \in \mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$. More generally, define the Sobolev space $H^s(\mathbb{R}^\nu;\mathbb{R})$ to be the separable Hilbert space obtained by completing $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$ with respect to

$$\|h\|_{H^s(\mathbb{R}^\nu;\mathbb{R})} \equiv \|B^{-s}h\|_{L^2(\mathbb{R}^\nu;\mathbb{R})} = \sqrt{\frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu} \left(\tfrac{1}{4} + |\boldsymbol{\xi}|^2\right)^s |\hat{h}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}}.$$

* The need to deal with generalized functions is the primary source of the difficulties which mathematicians have when they attempt to construct non-trivial quantum fields. Without going into any details, suffice it to say that in order to construct interacting (i.e., non-Gaussian) fields, one has to take non-linear functions of a Gaussian field. However, if the Gaussian field is distribution valued, it is not at all clear how to apply a non-linear function to it.

Obviously, $H^0(\mathbb{R}^\nu;\mathbb{R})$ is just $L^2(\mathbb{R}^\nu;\mathbb{R})$. When $s>0$, $H^s(\mathbb{R}^\nu;\mathbb{R})$ is a subspace of $L^2(\mathbb{R}^\nu;\mathbb{R})$, and the quality of its elements will improve as s gets larger. However, when $s<0$, some elements of $H^s(\mathbb{R}^\nu;\mathbb{R})$ will be strictly worse than elements of $L^2(\mathbb{R}^\nu;\mathbb{R})$, and their quality will deteriorate as s becomes more negative. Nonetheless, for every $s\in\mathbb{R}$, $H^s(\mathbb{R}^\nu;\mathbb{R})\subseteq\mathscr{S}'(\mathbb{R}^\nu;\mathbb{R})$, where $\mathscr{S}'(\mathbb{R}^\nu;\mathbb{R})$, whose elements are called real-valued **tempered distributions**, is the dual space of $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$. In fact, with a little effort, one can check that an alternative description of $H^s(\mathbb{R}^\nu;\mathbb{R})$ is as the subspace of $u\in\mathscr{S}'(\mathbb{R}^\nu;\mathbb{R})$ with the property that $B^{-s}u\in L^2(\mathbb{R}^\nu;\mathbb{R})$. Equivalently, $H^s(\mathbb{R}^\nu;\mathbb{R})$ is the isometric image in $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$ of $L^2(\mathbb{R}^\nu;\mathbb{R})$ under the map B^s, and, more generally, $H^{s_2}(\mathbb{R}^\nu;\mathbb{R})$ is the isometric image of $H^{s_1}(\mathbb{R}^\nu;\mathbb{R})$ under $B^{s_2-s_1}$. Thus, by Theorem 8.3.1, once we understand the abstract Wiener spaces for any one of the spaces $H^s(\mathbb{R}^\nu;\mathbb{R})$, understanding the abstract Wiener spaces for any of the others comes down to understanding the action of the Bessel operators, a task which, depending on what one wants to know, can be highly non-trivial.

LEMMA 8.5.7. *The space $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ is continuously embedded as a dense subspace of the separable Banach space $C_0(\mathbb{R}^\nu;\mathbb{R})$ whose elements are continuous functions which tend to 0 at infinity and whose norm is the uniform norm. Moreover, given a totally finite, signed Borel measure λ on $\mathbb{R}^\nu$, the function*

$$h_\lambda(\mathbf{x})\equiv K_\nu\int_{\mathbb{R}^\nu} e^{-\frac{|\mathbf{x}-\mathbf{y}|}{2}}\,\lambda(d\mathbf{y}),\quad \textit{with } K_\nu\equiv\frac{\pi^{\frac{1-\nu}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)},$$

is an element of $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$,

$$\|h_\lambda\|^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}=K_\nu\iint\limits_{\mathbb{R}^\nu\times\mathbb{R}^\nu} e^{-\frac{|\mathbf{x}-\mathbf{y}|}{2}}\,\lambda(d\mathbf{x})\lambda(d\mathbf{y}),$$

and

$$\langle h,\lambda\rangle=\big(h,h_\lambda\big)_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}\quad \textit{for each } h\in H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R}).$$

PROOF: To prove the initial assertion, use the Fourier inversion formula to write

$$h(\mathbf{x})=(2\pi)^{-\nu}\int_{\mathbb{R}^\nu} e^{-\sqrt{-1}(\mathbf{x},\boldsymbol{\xi})_{\mathbb{R}^\nu}}\,\hat{h}(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$

for $h\in\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$, and derive from this the estimate

$$\|h\|_{\mathrm{u}}\le(2\pi)^{-\frac{\nu}{2}}\left(\int_{\mathbb{R}^\nu}\left(\tfrac14+|\boldsymbol{\xi}|^2\right)^{-\frac{\nu+1}{2}}d\boldsymbol{\xi}\right)^{\frac12}\|h\|_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}.$$

Hence, since $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ is the completion of $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$ with respect to the norm $\|\cdot\|_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$, it is clear the $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ continuously embedded in

$C_0(\mathbb{R}^\nu;\mathbb{R})$. In addition, since $\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$ is dense in $C_0(\mathbb{R}^\nu;\mathbb{R})$, $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ is also.

To carry out the next step, let λ be given, and observe that the Fourier transform of $B^{\nu+1}\lambda$ is $\left(\frac{1}{4}+|\boldsymbol{\xi}|^2\right)^{-\frac{\nu+1}{2}}\hat{\lambda}(\boldsymbol{\xi})$ and therefore that

$$\begin{aligned} B^{\nu+1}\lambda(\mathbf{x}) &= \frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu} \frac{e^{-\sqrt{-1}(\mathbf{x},\boldsymbol{\xi})_{\mathbb{R}^\nu}}\hat{\lambda}(\boldsymbol{\xi})}{\left(\frac{1}{4}+|\boldsymbol{\xi}|^2\right)^{\frac{\nu+1}{2}}}\,d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu}\left(\int_{\mathbb{R}^\nu} \frac{e^{\sqrt{-1}(\mathbf{y}-\mathbf{x},\boldsymbol{\xi})_{\mathbb{R}^\nu}}}{\left(\frac{1}{4}+|\boldsymbol{\xi}|^2\right)^{\frac{\nu+1}{2}}}\,d\boldsymbol{\xi}\right)\lambda(d\mathbf{y}). \end{aligned}$$

Now use (3.3.19) (with $N=\nu$ and $t=\frac{1}{2}$) to see that

$$\frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu} \frac{e^{\sqrt{-1}(\mathbf{y}-\mathbf{x},\boldsymbol{\xi})_{\mathbb{R}^\nu}}}{\left(\frac{1}{4}+|\boldsymbol{\xi}|^2\right)^{\frac{\nu+1}{2}}}\,d\boldsymbol{\xi} = K_\nu e^{-\frac{|\mathbf{y}-\mathbf{x}|}{2}},$$

and thereby arrive at $h_\lambda = B^{\nu+1}\lambda$. In particular, this shows that

$$\|h_\lambda\|^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = \frac{1}{(2\pi)^\nu}\int_{\mathbb{R}^\nu}\frac{|\hat{\lambda}(\boldsymbol{\xi})|^2}{\left(\frac{1}{4}+|\boldsymbol{\xi}|^2\right)^{\frac{\nu+1}{2}}}\,d\boldsymbol{\xi} < \infty.$$

Now let $h\in\mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$, and use the preceding to justify

$$\langle h,\lambda\rangle = \langle B^{-\frac{\nu+1}{2}}h, B^{-\frac{\nu+1}{2}}B^{\nu+1}\lambda\rangle = \left(h,h_\lambda\right)_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}.$$

Since both sides are continuous with respect to convergence in $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$, we have now proved that $\langle h,\lambda\rangle = \left(h,h_\lambda\right)_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$ for all $h\in H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$. In particular,

$$\|h_\lambda\|^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = \langle h_\lambda,\lambda\rangle = K_\nu \iint\limits_{\mathbb{R}^\nu\times\mathbb{R}^\nu} e^{-\frac{|\mathbf{y}-\mathbf{x}|}{2}}\,\lambda(d\mathbf{x})\lambda(d\mathbf{y}). \qquad \square$$

THEOREM 8.5.8. *Let $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ be the space of continuous $\theta:\mathbb{R}^\nu\longrightarrow\mathbb{R}$ satisfying $\lim_{|\mathbf{x}|\to\infty}\left(\log(e+|\mathbf{x}|)\right)^{-1}|\theta(\mathbf{x})|=0$, and turn $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ into a separable Banach space with norm $\|\theta\|_{\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = \sup_{\mathbf{x}\in\mathbb{R}^N}\left(\log(e+|\mathbf{x}|)\right)^{-1}|\theta(\mathbf{x})|$. Then $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ is continuously embedded as a dense subspace of $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$, and there is a $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}\in\mathbf{M}_1\left(\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})\right)$ such that*

$$\left(H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R}), \Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R}), \mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}\right)$$

is an abstract Wiener space. Moreover, for each $\alpha\in\left(0,\frac{1}{2}\right)$, $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-almost every θ is Hölder continuous of order α and, for each $\alpha>\frac{1}{2}$, $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-almost no θ is anywhere Hölder continuous of order α.

PROOF: The initial part of the first assertion follows from the first part of Lemma 8.5.7 plus the essentially trivial fact that $C_0(\mathbb{R}^\nu;\mathbb{R})$ is continuously embedded as a dense subspace of $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$. Further, by the second part of that same lemma combined with Theorem 8.3.3, we will have proved the second part of the first assertion here once we show that, when $\{h_m : m \geq 0\}$ is an orthonormal basis in $H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$, the Wiener series $\sum_{m=0}^\infty \omega_m h_m$ converges in $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$ for $\gamma_{0,1}^{\mathbb{N}}$-almost every $\boldsymbol{\omega} = (\omega_0,\dots,\omega_m,\dots) \in \mathbb{R}^{\mathbb{N}}$. Thus, set $S_n(\boldsymbol{\omega}) = \sum_{m=0}^n \omega_m h_m$ for $n \geq 1$. More or less mimicking the steps outlined in Exercise 8.3.20, I will begin by showing that, for each $\alpha \in \big(0,\frac{1}{2}\big)$ and $R \in [1,\infty)$,

$$(*)\qquad \sup_{\mathbf{z}\in\mathbb{R}^\nu} \mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0}\sup_{\substack{\mathbf{x},\mathbf{y}\in Q(\mathbf{z},R)\\ \mathbf{x}\neq\mathbf{y}}} \frac{|S_n(\mathbf{y})-S_n(\mathbf{x})|}{|\mathbf{y}-\mathbf{x}|^\alpha}\right] < \infty,$$

where $Q(\mathbf{z},R) = \mathbf{z} + [-R,R)^\nu$. Indeed, by the argument given in that exercise combined with the higher dimensional analog of Kolmogorov's continuity criterion in Exercise 4.3.18, (*) will follow once we show that

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|S_n(\mathbf{y})-S_n(\mathbf{x})|^2\big] \leq C|\mathbf{y}-\mathbf{x}|, \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^\nu,$$

for some $C < \infty$. To this end, set $\lambda = \delta_{\mathbf{y}} - \delta_{\mathbf{x}}$, and apply Lemma 8.5.7 to check

$$\begin{aligned}\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\big[|S_n(\mathbf{y})-S_n(\mathbf{x})|^2\big] &= \sum_{m=0}^n \big(h_m,h_\lambda\big)^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}\\ &\leq \|h_\lambda\|^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = 2K_\nu\big(1-e^{-\frac{|\mathbf{y}-\mathbf{x}|}{2}}\big).\end{aligned}$$

Knowing (*), it becomes an easy matter to see that there exists a measurable $S : \mathbb{R}^\nu \times \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $\mathbf{x} \rightsquigarrow S(\mathbf{x},\boldsymbol{\omega})$ is continuous of each $\boldsymbol{\omega}$ and $S_n(\,\cdot\,,\boldsymbol{\omega}) \longrightarrow S(\,\cdot\,,\boldsymbol{\omega})$ uniformly on compacts for $\gamma_{0,1}^{\mathbb{N}}$-almost every $\boldsymbol{\omega}\in\mathbb{R}^{\mathbb{N}}$. In fact, because of (*), it suffices to check that $\lim_{n\to\infty} S_n(\mathbf{x})$ exists $\gamma_{0,1}^{\mathbb{N}}$-almost surely for each $\mathbf{x}\in\mathbb{R}^\nu$, and this follows immediately from Theorem 1.4.2 plus

$$\sum_{m=0}^\infty \mathrm{Var}\big(\omega_m h_m(\mathbf{x})\big) = \sum_{m=0}^\infty \big(h_m,h_{\delta_{\mathbf{x}}}\big)^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = \|h_{\delta_{\mathbf{x}}}\|^2_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})} = K_\nu.$$

Furthermore, again from (*), we know that, $\gamma_{0,1}^{\mathbb{N}}$-almost every $\boldsymbol{\omega}$, $\mathbf{x} \rightsquigarrow S(\mathbf{x},\boldsymbol{\omega})$ is α-Hölder continuous so long as $\alpha \in \big(0,\frac{1}{2}\big)$.

I must still check that, $\gamma_{0,1}^{\mathbb{N}}$-almost surely, the convergence of $S_n(\,\cdot\,,\boldsymbol{\omega})$ to $S(\,\cdot\,,\boldsymbol{\omega})$ is taking place in $\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})$, and, in view of the fact that we already know that, $\gamma_{0,1}^{\mathbb{N}}$-almost surely, it is taking place uniformly on compacts, this reduces to showing that

$$\lim_{|\mathbf{x}|\to\infty}\bigl(\log(e+|\mathbf{x}|)\bigr)^{-1}\sup_{n\geq 0}|S_n(\mathbf{x})| \longrightarrow 0 \quad \gamma_{0,1}^{\mathbb{N}}\text{-almost surely.}$$

For this purpose, observe that (*) says

$$\sup_{\mathbf{z}\in\mathbb{R}^\nu}\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{z},1)}\right]<\infty,$$

where $\|\cdot\|_{\mathrm{u},C}$ denotes the uniform norm over a set $C\subseteq\mathbb{R}^\nu$. At this point, I would like to apply Fernique's Theorem (Theorem 8.2.1) to the Banach space $\ell^\infty\bigl(\mathbb{N};C_{\mathrm{b}}(Q(\mathbf{z},1);\mathbb{R})\bigr)$ and thereby conclude that there exists an $\alpha>0$ such that

$$(**)\qquad B\equiv\sup_{\mathbf{z}\in\mathbb{R}^\nu}\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\exp\left(\alpha\sup_{n\geq 0}\|S_n\|^2_{\mathrm{u},Q(\mathbf{z},1)}\right)\right]<\infty.$$

However, $\ell^\infty\bigl(\mathbb{N};C_{\mathrm{b}}(Q(\mathbf{z},1);\mathbb{R})\bigr)$ is not separable. Nonetheless, there are two ways to get around this technicality. The first is to observe that the only place separability was used in the proof of Fernique's Theorem was at the beginning, where I used it to guarantee that $\mathcal{B}_E$ is generated by the maps $x\rightsquigarrow\langle x,x^*\rangle$ as x^* runs over E^* and therefore that the distribution of X is determined by the distribution of $\{\langle X,x^*\rangle : x^*\in E^*\}$. But, even though $\ell^\infty\bigl(\mathbb{N};C_{\mathrm{b}}(Q(\mathbf{z},1);\mathbb{R})\bigr)$ is not separable, one can easily check that it neverless possesses this property. The second way to deal with the problem is to apply his theorem to $\ell^\infty\bigl(\{0,\dots,N\};C_{\mathrm{b}}(Q(\mathbf{z},1);\mathbb{R})\bigr)$, which is separable, and to note that the resulting estimate can be made uniform in $N\in\mathbb{N}$. Either way, one arrives at (**).

Now set $\psi(t)=e^{\alpha t^2}-1$ for $t\geq 0$. Then $\psi^{-1}(s)=\sqrt{\alpha^{-1}\log(1+s)}$, and

$$\begin{aligned}\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{0},M)}&=\max\left\{\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{m},1)} : \mathbf{m}\in Q(\mathbf{0},M)\cap\mathbb{Z}^\nu\right\}\\&\leq\psi^{-1}\left(\sum_{\mathbf{m}\in Q(\mathbf{0},M)\cap\mathbb{Z}^\nu}\psi\left(\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{m},1)}\right)\right).\end{aligned}$$

Thus, because ψ^{-1} is concave, Jensen's inequality applies and yields

$$\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{0},M)}\right]\leq\psi^{-1}\bigl((2M)^\nu B\bigr),$$

and therefore

$$\begin{aligned}\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{|\mathbf{x}|\geq R}\sup_{n\geq 0}\frac{S_n(\mathbf{x})}{\log(e+|\mathbf{x}|)}\right]&\leq\sum_{m\geq(\log R)^{\frac14}}\frac{\mathbb{E}^{\gamma_{0,1}^{\mathbb{N}}}\left[\sup_{n\geq 0}\|S_n\|_{\mathrm{u},Q(\mathbf{0},e^{m^4})}\right]}{\log(e+e^{(m-1)^4})}\\&\leq\sum_{m\geq(\log R)^{\frac14}}\frac{\sqrt{\log(1+2^\nu e^{\nu(m+1)^4}B)}}{\sqrt{\alpha}\log(e+e^{(m-1)^4})}\longrightarrow 0\quad\text{as } R\to\infty.\end{aligned}$$

To complete the proof, I must show that, for any $\alpha > \frac{1}{2}$, $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-almost no θ is anywhere Hölder continuous of order α, and for this purpose I will proceed as in the proof of Theorem 4.3.4. Because the $\{\theta(\mathbf{x}+\mathbf{y}) : \mathbf{x} \in \mathbb{R}^\nu\}$ has the same $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-distribution for all $\mathbf{y}$, it suffices for me to show that, $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-almost surely, there is no $\mathbf{x} \in Q(\mathbf{0},1)$ at which θ is Hölder continuous of order $\alpha > \frac{1}{2}$. Now suppose that $\alpha \in \left(\frac{1}{2},1\right)$, and observe that, for any $L \in \mathbb{Z}^+$ and $\mathbf{e} \in \mathbb{S}^{\nu-1}$, the set $H(\alpha)$ of θ which are α-Hölder continuous at some $\mathbf{x} \in Q(\mathbf{0},1)$ is contained in

$$\bigcup_{M=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{m}\in Q(\mathbf{0},n)\cap\mathbb{Z}^\nu} \bigcap_{\ell=1}^{L} \left\{\theta : \left|\theta\left(\tfrac{\mathbf{m}+\ell\mathbf{e}}{n}\right) - \theta\left(\tfrac{\mathbf{m}+(\ell-1)\mathbf{e}}{n}\right)\right| \le \tfrac{M}{n^\alpha}\right\}.$$

Hence, again using translation invariance, we see that we need only show that, there is an $L \in \mathbb{Z}^+$ such that, for each $M \in \mathbb{Z}^+$,

$$n^\nu \mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}\left(\left\{\theta : \left|\theta\left(\tfrac{\ell\mathbf{e}}{n}\right) - \theta\left(\tfrac{(\ell-1)\mathbf{e}}{n}\right)\right| \le \tfrac{M}{n^\alpha},\ 1 \le \ell \le L\right\}\right)$$

tends to 0 as $n \to \infty$. To this end, set $U(t,\theta) = K_\nu^{-\frac{1}{2}}\theta(t\mathbf{e})$, and observe that the $\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}$-distribution of $\{U(t) : t \ge 0\}$ is that of an $\mathbb{R}$-valued ancient Ornstein–Uhlenbeck process. Thus, what I have to estimate is

$$\mathbb{P}\left(\left|e^{-\frac{\ell}{2n}}B\left(e^{\frac{\ell}{n}}\right) - e^{-\frac{\ell-1}{2n}}B\left(e^{\frac{\ell-1}{n}}\right)\right| \le \tfrac{M}{n^\alpha},\ 1 \le \ell \le L\right),$$

where $\left(B(t),\mathcal{F}_t,\mathbb{P}\right)$ is an $\mathbb{R}$-valued Brownian motion. But clearly this probability is dominated by the sum of

$$\mathbb{P}\left(\left|B\left(e^{\frac{\ell}{n}}\right) - B\left(e^{\frac{\ell-1}{n}}\right)\right| \le \tfrac{Me^{\frac{\ell}{2n}}}{2n^\alpha},\ 1 \le \ell \le L\right)$$

and

$$\mathbb{P}\left(\exists 1 \le \ell \le L\ \ \left(1 - e^{-\frac{1}{2n}}\right)\left|B\left(e^{\frac{\ell-1}{n}}\right)\right| \ge \tfrac{Me^{\frac{\ell}{2n}}}{2n^\alpha}\right).$$

The second of these is easily dominated by $2Le^{-\frac{M^2n^{2(1-\alpha)}}{8}}$, which, since $\alpha < 1$, means that it causes no problems. As for the first, one can use the independence of Brownian increments and Brownian scaling to dominate it by the Lth power of $\mathbb{P}\left(\left|B(1)-B\left(e^{-\frac{1}{n}}\right)\right| \le M(2n^\alpha)^{-1}\right)$. Hence, I can take any L such that $\left(\alpha-\frac{1}{2}\right)L > \nu$. $\square$

As a consequence of the preceding and Theorem 8.3.1, we have the following corollary.

COROLLARY 8.5.9. *Given $s \in \mathbb{R}$, set*

$$\Theta^s(\mathbb{R}^\nu;\mathbb{R}) = \left\{B^{s-\frac{\nu+1}{2}}\theta : \theta \in \Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})\right\},$$

$$\|\theta\|_{\Theta^s(\mathbb{R}^\nu;\mathbb{R})} = \|B^{\frac{\nu+1}{2}-s}\theta\|_{\Theta^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})},$$

and

$$\mathcal{W}_{H^s(\mathbb{R}^\nu;\mathbb{R})} = (B^{s-\frac{\nu+1}{2}})_*\mathcal{W}_{H^{\frac{\nu+1}{2}}(\mathbb{R}^\nu;\mathbb{R})}.$$

Then $\Theta^s(\mathbb{R}^\nu;\mathbb{R})$ is a separable Banach space in which $H^s(\mathbb{R}^\nu;\mathbb{R})$ is continuously embedded as a dense subspace, and $\big(H^s(\mathbb{R}^\nu;\mathbb{R}), \Theta^s(\mathbb{R}^\nu;\mathbb{R}), \mathcal{W}_{H^s(\mathbb{R}^\nu;\mathbb{R})}\big)$ is an abstract Wiener space.

Exercises for § 8.5

EXERCISE 8.5.10. In this exercise we will show how to use the Ornstein–Uhlenbeck process to prove **Poincaré's inequality**

$$\operatorname{Var}_{\gamma_{0,1}}(\varphi) = \|\varphi - \langle\varphi,\gamma_{0,1}\rangle\|^2_{L^2(\gamma_{0,1};\mathbb{R})} \le \|\varphi'\|^2_{L^2(\gamma_{0,1};\mathbb{R})} \tag{8.5.11}$$

for the standard Gaussian distribution on $\mathbb{R}$. I will outline the proof of (8.5.11) for $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$, but the estimate immediately extends to any $\varphi \in L^2(\gamma_{0,1};\mathbb{R})$ whose (distributional) first derivative is again in $L^2(\gamma_{0,1};\mathbb{R})$.

(**i**) For $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$, set

$$u_\varphi(t,x) = \mathbb{E}^{\mathcal{W}^{(1)}}\big[\varphi\big(U(t,x)\big)\big],$$

where $\{U(t,x) : t \ge 0\}$ is the one sided, $\mathbb{R}$-valued Ornstein-Uhlenbeck process starting at x. Show that $u'_\varphi(t,x) = e^{-\frac{t}{2}}u_{\varphi'}(t,x)$ and that

$$\lim_{t\searrow 0} u_\varphi(t,\,\cdot\,) = \varphi \quad\text{and}\quad \lim_{t\to\infty} u_\varphi(t,\,\cdot\,) = \langle\varphi,\gamma_{0,1}\rangle \quad \text{in } L^2(\gamma_{0,1};\mathbb{R}).$$

Show that another expression for u_φ is

$$u_\varphi(t,x) = \big(2\pi(1-e^{-t})\big)^{-\frac{1}{2}}\int_{\mathbb{R}} \varphi(y)\exp\left(-\frac{(y-e^{-\frac{t}{2}}x)^2}{2(1-e^{-t})}\right)dy.$$

Using this second expression, show that $u_\varphi(t,\,\cdot\,) \in \mathscr{S}(\mathbb{R};\mathbb{R})$ and that $t \in [0,\infty) \longmapsto u_\varphi(t,\,\cdot\,) \in \mathscr{S}(\mathbb{R};\mathbb{R})$ is continuous. In addition, show that $\dot{u}_\varphi(t,x) = \frac{1}{2}\big(u''_\varphi(t,x) - xu'_\varphi(t,x)\big)$.

(**ii**) For $\varphi_1, \varphi_2 \in C^2(\mathbb{R};\mathbb{R})$ whose second derivative is tempered, show that

$$\big(\varphi_1, \varphi_2'' - x\varphi_2\big)_{L^2(\gamma_{0,1};\mathbb{R})} = -\big(\varphi_1', \varphi_2'\big)_{L^2(\gamma_{0,1};\mathbb{R})},$$

and use this together with (**i**) to show that, for any $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$,

$$\langle u_\varphi(t,\,\cdot\,), \gamma_{0,1}\rangle = \langle \varphi, \gamma_{0,1}\rangle \text{ and } \frac{d}{dt}\|u_\varphi(t,\,\cdot\,)\|^2_{L^2(\gamma_{0,1};\mathbb{R})} = -e^{-t}\|u_{\varphi'}(t,\,\cdot\,)\|^2_{L^2(\gamma_{0,1};\mathbb{R})}.$$

Conclude that $\|u_\varphi(t,\,\cdot\,)\|_{L^2(\gamma_{0,1};\mathbb{R})} \le \|\varphi\|_{L^2(\gamma_{0,1};\mathbb{R})}$ and

$$\frac{d}{dt}\|u_\varphi(t,\,\cdot\,)\|^2_{L^2(\gamma_{0,1};\mathbb{R})} \ge -e^{-t}\|\varphi'\|^2_{L^2(\gamma_{0,1};\mathbb{R})}.$$

Finally, integrate the preceding inequality to arrive at (8.5.11).

Exercise 8.5.12. In this exercise I will outline how the ideas in Exercise 8.5.10 can be used to give another derivation of the logarithmic Sobolev inequality (2.4.38). Again, I restrict my attention to $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$, since the general case can be easily obtained from this by taking limits.

(**i**) Begin by showing that (2.4.38) for $\varphi \in \mathscr{S}(\mathbb{R};\mathbb{R})$ once one knows that

$$(*) \qquad \big\langle \varphi \log \varphi \big\rangle_{\gamma_{0,1}} \le \frac{1}{2}\left\langle \frac{(\varphi')^2}{\varphi}\right\rangle_{\gamma_{0,1}}$$

for uniformly positive $\varphi \in \mathbb{R} \oplus \mathscr{S}(\mathbb{R};\mathbb{R})$.

(**ii**) Given a uniformly postitive $\varphi \in \mathbb{R} \oplus \mathscr{S}(\mathbb{R};\mathbb{R})$, use the results in Exercise 8.5.10 to show that

$$\frac{d}{dt}\big\langle u_\varphi(t,\,\cdot\,)\log u_\varphi(t,\,\cdot\,)\big\rangle_{\gamma_{0,1}} = -\frac{e^{-t}}{2}\left\langle \frac{u_{\varphi'}(t,\,\cdot\,)^2}{u_\varphi(t,\,\cdot\,)}\right\rangle_{\gamma_{0,1}}.$$

(**iii**) Coninuing (**ii**), apply Schwarz's inequality to check that

$$\frac{u_{\varphi'}(t,x)^2}{u_\varphi(t,x)} \le u_{\frac{(\varphi')^2}{\varphi}}(t,x),$$

and combine this with (**ii**) to get

$$\frac{d}{dt}\big\langle u_\varphi(t,\,\cdot\,)\log u_\varphi(t,\,\cdot\,)\big\rangle_{\gamma_{0,1}} \ge -\frac{e^{-t}}{2}\left\langle \frac{(\varphi')^2}{\varphi}\right\rangle_{\gamma_{0,1}}.$$

Finally, integrate this to arrive at (*).

EXERCISE 8.5.13. Although it should be clear that the arguments given in Exercises 8.5.10 and 8.5.12 work equally well in $\mathbb{R}^N$ and yield (8.5.11) and (2.4.38) with $\gamma_{0,1}$ replaced by $\gamma_{\mathbf{0},\mathbf{I}}$ and $(\varphi')^2$ replaced by $|\nabla\varphi|^2$, it is significant that each of these inequalities for $\mathbb{R}$ implies its $\mathbb{R}^N$ analog. Indeed, show that Fubini's Theorem is all that one needs to pass to the higher dimensional results. The reason why this remark is significant is that it allows one to prove infinite dimensional versions of both Poincaré's inequality and the logarithmic Sobolev inequality, and both these play a crucial role in infinite dimensional analysis. In fact, Nelson's interest in hypercontractive estimates sprung from his brilliant insight that hypercontractive estimates would allow him to construct a non-trivial (i.e., non-Gaussian), translation invariant quantum field for $\mathbb{R}^2$.

EXERCISE 8.5.14. It is interesting to see what happens if one changes the sign of the second term on the right hand side of (8.5.1), thereby converting the centripetal force into a centrifugal one.

(**i**) Show that, for each $\boldsymbol{\theta} \in \Theta(\mathbb{R}^N)$, the unique solution to

$$\mathbf{V}(t,\boldsymbol{\theta}) = \boldsymbol{\theta}(t) + \tfrac{1}{2}\int_0^t \mathbf{V}(\tau,\boldsymbol{\theta})\,d\tau, \quad t \geq 0,$$

is

$$\mathbf{V}(t,\boldsymbol{\theta}) = e^{\frac{t}{2}}\int_0^t e^{-\frac{\tau}{2}}\,d\boldsymbol{\theta}(\tau),$$

where the integral is taken in the sense of Riemann–Stieltjes.

(**ii**) Show that $\big\{\big(\boldsymbol{\xi},\mathbf{V}(t,\,\cdot\,)\big)_{\mathbb{R}^N} : (t,\boldsymbol{\xi}) \in [0,\infty)\times\mathbb{R}^N\big\}$ under $\mathcal{W}^{(N)}$ is a Gaussian family with covariance

$$v(s,t) = e^{\frac{s+t}{2}} - e^{\frac{|t-s|}{2}}.$$

(**iii**) Let $\{\mathbf{B}(t) : t \geq 0\}$ be an $\mathbb{R}^N$-valued Brownian motion, and show that the distribution of

$$\big\{e^{\frac{t}{2}}\mathbf{B}\big(1-e^{-t}\big) : t \geq 0\big\}$$

is the $\mathcal{W}^{(N)}$-distribution of $\{\mathbf{V}(t) : t \geq 0\}$. Next, let $\Theta^V(\mathbb{R}^N)$ be the space of continuous $\boldsymbol{\theta} : [0,\infty) \longrightarrow \mathbb{R}^N$ with the properties that

$$\boldsymbol{\theta}(0) = \mathbf{0} = \lim_{t\to\infty} e^{-t}|\boldsymbol{\theta}(t)|,$$

and set $\|\boldsymbol{\theta}\|_{\Theta^V(\mathbb{R}^N)} \equiv \sup_{t\geq 0} e^{-t}|\boldsymbol{\theta}(t)|$. Show that $\big(\Theta^V(\mathbb{R}^N); \|\cdot\|_{\Theta^V(\mathbb{R}^N)}\big)$ is a separable Banach space and that there exists a unique $\mathcal{V}^{(N)} \in \mathbf{M}_1\big(\Theta^V(\mathbb{R}^N)\big)$ such that the distribution of $\{\boldsymbol{\theta}(t) : t \geq 0\}$ under $\mathcal{V}^{(N)}$ is the same as the distribution of $\{\mathbf{V}(t) : t \geq 0\}$ under $\mathcal{W}^{(N)}$.

(**iv**) Let $\mathbf{H}^V(\mathbb{R}^N)$ be the space of absolutely continuous $\mathbf{h} : [0,\infty) \longrightarrow \mathbb{R}^N$ with the properties that $\mathbf{h}(0) = \mathbf{0}$ and $\dot{\mathbf{h}} - \frac{1}{2}\mathbf{h} \in L^2\big([0,\infty);\mathbb{R}^N\big)$. Show that $\mathbf{H}^V(\mathbb{R}^N)$ with norm

$$\|\mathbf{h}\|_{\mathbf{H}^V(\mathbb{R}^N)} \equiv \big\|\dot{\mathbf{h}} - \tfrac{1}{2}\mathbf{h}\big\|_{L^2([0,\infty);\mathbb{R}^N)}$$

is a separable Hilbert space which is continuously embedded in $\Theta^V(\mathbb{R}^N)$ as a dense subspace. Finally, show that $\big(\mathbf{H}^V(\mathbb{R}^N), \Theta^V(\mathbb{R}^N), \mathcal{V}^{(N)}\big)$ is an abstract Wiener space.

(**v**) There is a subtlety here which is worth mentioning. Namely, show that $\mathbf{H}^U(\mathbb{R}^N)$ is isometrically embedded in $\mathbf{H}^V(\mathbb{R}^N)$. On the other hand, as distinguished from elements of $\mathbf{H}^U(\mathbb{R}^N)$, it is not true that $\|\dot{\boldsymbol{\eta}} - \frac{1}{2}\boldsymbol{\eta}\|^2_{L^2(R;\mathbb{R}^N)} = \|\dot{\boldsymbol{\eta}}\|^2_{L^2(\mathbb{R};\mathbb{R}^N)} + \frac{1}{4}\|\boldsymbol{\eta}\|^2_{L^2(\mathbb{R};\mathbb{R}^N)}$, the point being that whereas the elements $\mathbf{h}$ of $\mathbf{H}^V(\mathbb{R}^N)$ with $\dot{\mathbf{h}} \in C_{\mathrm{c}}\big((0,\infty);\mathbb{R}^N\big)$ are dense in $\mathbf{H}^U(\mathbb{R}^N)$, they are not dense in $\mathbf{H}^V(\mathbb{R}^N)$.

Exercise 8.5.15. Given $\mathbf{x} \in \mathbb{R}^\nu$ and a slowly increasing $\varphi \in C(\mathbb{R}^\nu;\mathbb{R})$, define $\tau_{\mathbf{x}}\varphi \in C(\mathbb{R}^\nu;\mathbb{R})$ so that $\tau_{\mathbf{x}}\varphi(\mathbf{y}) = \varphi(\mathbf{x}+\mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^\nu$. Next, extend $\tau_{\mathbf{x}}$ to $\mathscr{S}'(\mathbb{R}^\nu;\mathbb{R})$ so that $\langle \varphi, \tau_{\mathbf{x}} u\rangle = \langle \tau_{-\mathbf{x}}\varphi, u\rangle$ for $\varphi \in \mathscr{S}(\mathbb{R}^\nu;\mathbb{R})$, and check that this is a legitimate extension in the sense that it is consistent with the original definition when applied to u's which are slowly increasing, continuous functions. Finally, given $s \in \mathbb{R}$, define $\mathcal{O}_{\mathbf{x}} : H^s(\mathbb{R}^\nu;\mathbb{R}) \longrightarrow H^s(\mathbb{R}^\nu;\mathbb{R})$ by $\mathcal{O}_{\mathbf{x}} h = \tau_{\mathbf{x}} h$.

(**i**) Show that $B^s \circ \tau_{\mathbf{x}} = \tau_{\mathbf{x}} \circ B^s$ for all $s \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^\nu$.

(**ii**) Given $s \in \mathbb{R}$, define $\mathcal{O}_{\mathbf{x}} = \tau_{\mathbf{x}} \restriction H^s(\mathbb{R}^\nu;\mathbb{R})$, and show that $\mathcal{O}_{\mathbf{x}}$ is an orthogonal transformation.

(**iii**) Referring to Theorem 8.3.14 and Corollary 8.5.9, show that the measure preserving transformation $T_{\mathcal{O}_{\mathbf{x}}}$ which $\mathcal{O}_{\mathbf{x}}$ determines on $\big(\Theta^s(\mathbb{R}^\nu;\mathbb{R}), \mathcal{W}_{H^s(\mathbb{R}^\nu;\mathbb{R})}\big)$ is the restriction of $\tau_{\mathbf{x}}$ to $\Theta^s(\mathbb{R}^\nu;\mathbb{R})$.

(**iv**) If $\mathbf{x} \neq \mathbf{0}$, show that $T_{\mathcal{O}_{\mathbf{x}}}$ is erogodic on $\big(\Theta^s(\mathbb{R}^\nu;\mathbb{R}), \mathcal{W}_{H^s(\mathbb{R}^\nu;\mathbb{R})}\big)$.

§ 8.6 Brownian Motion on a Banach Space

In this concluding section I will discuss Brownian motion on a Banach space. More precisely, given a non-degenerate, centered, Gaussian measure $\mathcal{W}$ on a separable Banach space E, we will see that there exists an E-valued stochastic process $\{B(t) : t \geq 0\}$ with the properties that $B(0) = 0$, $t \rightsquigarrow B(t)$ is continuous, and, for all $0 \leq s < t$, $B(t) - B(s)$ is independent of $\sigma\big(\{B(\tau) : \tau \in [0,s]\}\big)$ and has distribution (cf. the notation in § 8.4) $\mathcal{W}_{t-s}$.

§ **8.6.1. Abstract Wiener Formulation.** Let $\mathcal{W}$ on E be as above, use H to denote its Cameron–Martin space, and take $H^1(H)$ to be the Hilbert space of absolutely continuous $h : [0,\infty) \longrightarrow H$ such that $h(0) = 0$ and $\|h\|_{H^1(H)} = \|\dot{h}\|_{L^2([0,\infty);H)} < \infty$. Finally, let $\Theta(E)$ be the space of continuous $\theta : [0,\infty) \longrightarrow$

E satisfying $\lim_{t\to\infty}\frac{\|\theta(t)\|_E}{t}=0$, and turn $\Theta(E)$ into a Banach space with norm $\|\theta\|_{\Theta(E)}=\sup_{t\ge 0}(1+t)^{-1}\|\theta(t)\|_E$. By exactly the same line of reasoning as I used when $E=\mathbb{R}^N$, one can show that $\Theta(E)$ is a separable Banach space in which $H^1(E)$ is continuously embedded as a dense subspace. My goal is to prove the following statement.

Theorem 8.6.1. *With $H^1(H)$ and $\Theta(E)$ as above, there is a unique $\mathcal{W}^{(E)}\in\mathbf{M}_1\big(\Theta(E)\big)$ such that $\big(H^1(H),\Theta(E),\mathcal{W}^{(E)}\big)$ is an abstract Wiener space.*

Choose an orthonormal basis $\{h^1_m:\,m\ge 0\}$ in $H^1(\mathbb{R})$, and, for $n\ge 0$, $t\ge 0$, and $\mathbf{x}=(x_0,\dots,x_m,\dots)\in E^{\mathbb{N}}$, set $S_n(t,\mathbf{x})=\sum_{m=0}^n h^1_m(t)x_m$. I will show that, $\mathcal{W}^{\mathbb{N}}$-almost surely, $\{S_n(\,\cdot\,,\mathbf{x}):\,n\ge 0\}$ converges in $\Theta(E)$, and, for the most part, the proof follows the same basic line of reasoning as that suggested in Exercise 8.3.20 when $E=\mathbb{R}^N$. However, there is a problem here which we did not encounter there. Namely, unless E is finite dimensional, bounded subsets will not necessarily be relatively compact in E. Hence, local uniform equicontinuity plus local boundedness is not sufficient to guarantee that a collection of E-valued paths is relatively compact in $C\big([0,\infty);E\big)$, and that is the reason why we have to work a little harder here.

Lemma 8.6.2. *For $\mathcal{W}^{\mathbb{N}}$-almost every $\mathbf{x}\in E^{\mathbb{N}}$, $\{S_n(\,\cdot\,,\mathbf{x}):\,n\ge 0\}$ is relatively compact in $\Theta(E)$.*

Proof: Choose $E_0\subseteq E$, as in Corollary 8.3.10, so that bounded subsets of E_0 are relatively compact in E and $\big(H,E_0,\mathcal{W}\restriction E_0\big)$ is again an abstract Wiener space. Without loss in generality, I will assume that $\|\cdot\|_E\le\|\cdot\|_{E_0}$, and, by Fernique's Theorem, we know that $C\equiv\mathbb{E}^{\mathcal{W}_0}\big[\|x\|^4_{E_0}\big]<\infty$.

Since $S_n(t,\mathbf{x})-S_n(s,\mathbf{x})=\sum_{m=0}^n\big(h^1_t-h^1_s,h^1_m\big)_{H^1(\mathbb{R})}x_m$, where $h^1_\tau=\,\cdot\wedge\tau$, the $\mathcal{W}_0^{\mathbb{N}}$-distribution of $S_n(t)-S_n(s)$ is $\mathcal{W}_{\epsilon_n}$, where $\epsilon_n^2=\sum_0^n\big(h^1_t-h^1_s,h^1_m\big)^2_{H^1(\mathbb{R})}\le t-s$. Hence, $\mathbb{E}^{\mathcal{W}^{\mathbb{N}}}\big[\|S_n(t)-S_n(s)\|^4_{E_0}\big]\le C(t-s)^2$. In addition, $\{\|S_n(t)-S_n(s)\|_{E_0}:\,n\ge 1\}$ is a submartingale, and so, by Doob's Inequality plus Kolmogorov's Continuity Criterion, there exists a $K<\infty$ such that, for each $T>0$,

$$(*)\qquad \mathbb{E}^{\mathcal{W}^{\mathbb{N}}}\left[\sup_{n\ge 0}\sup_{0\le s<t\le T}\frac{\|S_n(t)-S_n(s)\|_{E_0}}{(t-s)^{\frac18}}\right]\le KT^{\frac34}.$$

From (*) and $S_n(0)=0$, we know that, $\mathcal{W}^{\mathbb{N}}$-almost surely, $\{S_n(\,\cdot\,,\mathbf{x}):\,n\ge 0\}$ is uniformly $\|\cdot\|_{E_0}$-bounded and uniformly $\|\cdot\|_{E_0}$-equicontinuous on each interval $[0,T]$. Since this means that, for every $T>0$, $\{S_n(t,\mathbf{x}):\,n\ge 0\ \&\ t\in[0,T]\}$ is relatively compact in E and $\{S_n(\,\cdot\,,\mathbf{x})\restriction[0,T]:\,n\ge 0\}$ is uniformly $\|\cdot\|_E$-equicontinuous $\mathcal{W}^{\mathbb{N}}$-almost surely, the Ascoli–Arzela Theorem guarantees that, $\mathcal{W}^{\mathbb{N}}$-almost surely, $\{S_n(\,\cdot\,,\mathbf{x}):\,n\ge 0\}$ is relatively compact in $C\big([0,\infty);E\big)$ with the topology of uniform convergence on compacts. Thus, in order to complete the proof, all that we have to show is that, $\mathcal{W}^{\mathbb{N}}$-almost surely,

$$\lim_{T\to\infty}\sup_{n\geq 0}\sup_{t\geq T}\frac{\|S_n(t,\mathbf{x})\|_E}{t}=0.$$

But,

$$\sup_{t\geq 2^k}\frac{\|S_n(t,\mathbf{x})\|_E}{t}\leq\sum_{\ell\geq k}\sup_{2^\ell\leq t\leq 2^{\ell+1}}\frac{\|S_n(t,\mathbf{x})\|_E}{t}\leq\sum_{\ell\geq k}2^{-\frac{7\ell}{8}}\sup_{0\leq t\leq 2^{\ell+1}}\frac{\|S_n(t,\mathbf{x})\|_E}{t^{\frac{1}{8}}},$$

and therefore, by (*),

$$\mathbb{E}^{\mathcal{W}^{\mathbb{N}}}\left[\sup_{n\geq 0}\sup_{t\geq 2^k}\frac{\|S_n(t,\mathbf{x})\|_E}{t}\right]\leq\frac{2^{\frac{3}{4}}K}{2^{\frac{1}{8}}-1}2^{-\frac{k}{8}}.\quad\square$$

Now that we have the requisite compactness of $\{S_n : n\geq 0\}$, convergence comes to checking a criterion of the sort given in the following simple lemma.

Lemma 8.6.3. *Suppose that $\{\theta_n : n\geq 0\}$ is a relatively compact sequence in $\Theta(E)$. If $\lim_{n\to\infty}\langle\theta_n(t),x^*\rangle$ exists for each t in a dense subset of $[0,\infty)$ and x^* in a weak* dense subset of E^*, then $\{\theta_n : n\geq 0\}$ converges in $\Theta(E)$.*

Proof: For a relatively compact sequence to convergent, it is necessary and sufficient that every convergent subsequence have the same limit. Thus, suppose that θ and θ' are limit points of $\{\theta_n : n\geq 0\}$. Then, by hypothesis, $\langle\theta(t),x^*\rangle=\langle\theta'(t),x^*\rangle$ for t in a dense subset of $[0,\infty)$ and x^* in a weak* dense subset of E^*. But this means that the same equality holds for all $(t,x^*)\in[0,\infty)\times E^*$ and therefore that $\theta=\theta'$. $\square$

Proof of Theorem 8.6.1: In view of Lemmas 8.6.2 and 8.6.3 and the separability of E^* in the weak* topology, we will know that $\{S_n(\,\cdot\,,\mathbf{x}) : n\geq 0\}$ converges in $\Theta(E)$ for $\mathcal{W}^{\mathbb{N}}$-almost every $\mathbf{x}\in E^{\mathbb{N}}$ once we show that, for each $(t,x^*)\in[0,\infty)\times E^*$, $\{\langle S_n(t,\mathbf{x}),x^*\rangle : n\geq 0\}$ converges in $\mathbb{R}$ for $\mathcal{W}^{\mathbb{N}}$-almost every $\mathbf{x}\in E^{\mathbb{N}}$. But if $x^*\in E^*$, then $\langle S_n(t,\mathbf{x}),x^*\rangle=\sum_0^n\langle x_m,x^*\rangle h_m^1(t)$, the random variables $\mathbf{x}\rightsquigarrow\langle x_m,x^*\rangle h_m^1(t)$ are independent, centered Gaussians under $\mathcal{W}^{\mathbb{N}}$ with variance $\|h_{x^*}\|_H^2 h_m^1(t)^2$, and $\sum_0^\infty h_m^1(t)^2=\|h_t\|^2_{H^1(\mathbb{R})}=t$. Thus, by Theorem 1.4.2, we have the required convergence.

Next, define $B:[0,\infty)\times E^{\mathbb{N}}\longrightarrow E$ so that

$$B(t,\mathbf{x})=\begin{cases}\lim_{n\to\infty}S_n(t,\mathbf{x}) & \text{if } \{S_n(\,\cdot\,,\mathbf{x}) : n\geq 0\} \text{ converges in } \Theta(E)\\ 0 & \text{otherwise.}\end{cases}$$

Given $\lambda\in\Theta(E)^*$, determine $h_\lambda\in H^1(H)$ by $\big(h,h_\lambda\big)_{H^1(H)}=\langle h,\lambda\rangle$ for all $h\in H^1(H)$. I want to show that, under $\mathcal{W}^{\mathbb{N}}$, $\mathbf{x}\rightsquigarrow\langle B(\,\cdot\,,\mathbf{x}),\lambda\rangle$ is a centered Gaussian with variance $\|h_\lambda\|^2_{H^1(H)}$. To this end, define $x_m^* \in E^*$ so that* $\langle x, x_m^*\rangle = \langle h^1_m x, \lambda\rangle$ for $x \in E$. Then,

$$\langle B(\,\cdot\,,\mathbf{x}),\lambda\rangle = \lim_{n\to\infty}\langle S_n(\,\cdot\,,\mathbf{x}),\lambda\rangle = \lim_{n\to\infty}\sum_0^n\langle x_m, x_m^*\rangle \quad \mathcal{W}^{\mathbb{N}}\text{-almost surely.}$$

Hence, $\langle B(\,\cdot\,,\mathbf{x}),\lambda\rangle$ is certainly a centered Gaussian under $\mathcal{W}^{\mathbb{N}}$, and, because we are dealing with Gaussian random variables, almost sure convergence implies L^2-convergence. To compute its variance, choose an orthonormal basis $\{h_k : k \geq 0\}$ for H, and note that, for each $m \geq 0$,

$$\mathbb{E}^{\mathcal{W}^{\mathbb{N}}}\big[\langle x_m, x_m^*\rangle^2\big] = \|h_{x_m^*}\|_H^2 = \sum_{k=0}^\infty \langle h^1_m h_k, \lambda\rangle^2.$$

Thus, since $\{h^1_m h_k : (m,k) \in \mathbb{N}^2\}$ is an orthonormal basis in $H^1(H)$,

$$\mathbb{E}^{\mathcal{W}^{\mathbb{N}}}\big[\langle B(\,\cdot\,),\lambda\rangle^2\big] = \sum_{m,k=0}^\infty \langle h^1_m h_k,\lambda\rangle^2 = \sum_{m,k=0}^\infty \big(h^1_m h_k, h_\lambda\big)^2_{H^1(H)} = \|h_\lambda\|^2_{H^1(H)}.$$

Finally, to complete the proof, all that remains is to take $\mathcal{W}^{(E)}$ to be the $\mathcal{W}^{\mathbb{N}}$-distribution of $\mathbf{x} \rightsquigarrow B(\,\cdot\,,\mathbf{x})$. $\square$

§ **8.6.2. Brownian Formulation.** Let $(H, E, \mathcal{W})$ be an abstract Wiener space. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a non-decreasing family of sub σ-algebras $\{\mathcal{F}_t : t \geq 0\}$, and a measurable map $B : [0,\infty)\times\Omega \longrightarrow E$, say that the triple $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ is a **$\mathcal{W}$-Brownian motion** if

(1) B is $\{\mathcal{F}_t : t \geq 0\}$-progressively measurable,
(2) $B(0,\omega) = 0$ and $B(\,\cdot\,,\omega) \in C\big([0,\infty); E\big)$ for $\mathbb{P}$-almost every ω,
(3) $B(1)$ has distribution $\mathcal{W}$, and, for all $0 \leq s < t$, $B(t)-B(s)$ is independent of $\mathcal{F}_s$ and has the same distribution as $(t-s)^{\frac{1}{2}}B(1)$.

Lemma 8.6.4. *Suppose that $\{B(t) : t \geq 0\}$ satisfies conditions (1) and (2). Then $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion if and only if $\big(\langle B(t), x^*\rangle, \mathcal{F}_t, \mathbb{P}\big)$ is an $\mathbb{R}$-valued Brownian motion for each $x^* \in E^*$ with $\|h_{x^*}\|_H = 1$. In addition, if $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion, then the span $\mathfrak{G}(B)$ of $\{\langle B(t), x^*\rangle : (t, x^*) \in [0,\infty)\times E^*\}$ is a Gaussian family in $L^2(\mathbb{P};\mathbb{R})$ and*

$$\mathbb{E}^{\mathbb{P}}\big[\langle B(t_1), x_1^*\rangle\langle B(t_2), x_2^*\rangle\big] = (t_1 \wedge t_2)\big(h_{x_1^*}, h_{x_2^*}\big)_H. \tag{8.6.5}$$

Conversely, if $\mathfrak{G}(B)$ is a Gaussian family in $L^2(\mathbb{P};\mathbb{R})$ and (8.6.5) holds, then $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion when $\mathcal{F}_t = \sigma\big(\{B(\tau) : \tau \in [0,t]\}\big)$.

---

* Given $h^1 \in H^1(\mathbb{R})$ and $x \in E$, I use $h^1 x$ to denote the element θ of $\Theta(E)$ determined by $\theta(t) = h^1(t)x$.

PROOF: If $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion and $x^*\in E^*$ with $\|h_{x^*}\|_H=1$, then $\langle B(t),x^*\rangle-\langle B(s),x^*\rangle=\langle B(t)-B(s),x^*\rangle$ is independent of $\mathcal{F}_s$ and is a centered Gaussian with variance $(t-s)$. Thus, $\big(\langle B(t),x^*\rangle,\mathcal{F}_t,\mathbb{P}\big)$ is an $\mathbb{R}$-valued Brownian motion.

Next assume that $\big(\langle B(t),x^*\rangle,\mathcal{F}_t,\mathbb{P}\big)$ is an $\mathbb{R}$-valued Brownian motion for every x^* with $\|h_{x^*}\|_H=1$. Then $\langle B(t)-B(s),x^*\rangle$ is independent of $\mathcal{F}_s$ for every $x^*\in E^*$, and so, since $\mathcal{B}_E$ is generated by $\{\langle\,\cdot\,,x^*\rangle : x^*\in E^*\}$, $B(t)-B(s)$ is independent of $\mathcal{F}_s$. In addition, $\langle B(t)-B(s),x^*\rangle$ is a centered Gaussian with variance $(t-s)\|h_{x^*}\|_H^2$, and therefore $B(1)$ has distribution $\mathcal{W}$ and $B(t)-B(s)$ has the same distribution as $(t-s)^{\frac12}B(1)$. Thus, $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion.

Again assume that $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion. To prove that $\mathfrak{G}(B)$ is a Gaussian family for which (8.6.5) holds, it suffices to show that, for all $0\le t_1\le t_2$ and $x_1^*,x_2^*\in E^*$, $\langle B(t_1),x_1^*\rangle+\langle B(t_2),x_2^*\rangle$ is a centered Gaussian with covariance $t_1\|h_{x_1*}+h_{x_2^*}\|_H^2+(t_2-t_1)\|h_{x_2^*}\|_H^2$. Indeed, we would then know not only that $\mathfrak{G}(B)$ is a Gaussian family but also that the variance of $\langle B(t_1),x_1^*\rangle\pm\langle B(t_2),x_2^*\rangle$ is $t_1\|h_{x_1*}\pm h_{x_2^*}\|_H^2+(t_2-t_1)\|h_{x_2^*}\|_H^2$, from which (8.6.5) is immediate. But

$$\langle B(t_1),x_1^*\rangle+\langle B(t_2),x_2^*\rangle=\langle B(t_1),x_1^*+x_2^*\rangle+\langle B(t_2)-B(t_1),x_2^*\rangle,$$

and the terms on the right are independent, centered Gaussians, the first with variance $t_1\|h_{x_1^*}+h_{x_2^*}\|_H^2$ and the second with variance $(t_2-t_1)\|h_{x_2^*}\|_H^2$.

Finally, take $\mathcal{F}_t=\sigma\big(\{B(\tau) : \tau\in[0,t]\}\big)$, and assume that $\mathfrak{G}(B)$ is a Gaussian family satisfying (8.6.5). Given x^* with $\|h_{x^*}\|_H=1$ and $0\le s<t$, we know that $\langle B(t)-B(s),x^*\rangle=\langle B(t),x^*\rangle-\langle B(s),x^*\rangle$ is orthogonal in $L^2(\mathbb{P};\mathbb{R})$ to $\langle B(\tau),y^*\rangle$ for every $\tau\in[0,s]$ and $y^*\in E^*$. Hence, since $\mathcal{F}_s$ is generated by $\{\langle B(\tau),y^*\rangle : (\tau,y^*)\in[0,s]\times E^*\}$, we know that $\langle B(t)-B(s),x^*\rangle$ is independent of $\mathcal{F}_s$. In addition, $\langle B(t)-B(s),x^*\rangle$ is a centered Gaussian with variance $t-s$, and so we have proved that $\big(\langle B(t),x^*\rangle,\mathcal{F}_t,\mathbb{P}\big)$ is an $\mathbb{R}$-valued Brownian motion. Now apply the first part of the lemma to conclude that $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is a $\mathcal{W}$-Brownian motion. $\square$

THEOREM 8.6.6. *When $\Omega=\Theta(E)$, $\mathcal{F}=\mathcal{B}_E$, and $\mathcal{F}_t=\sigma\big(\{\theta(\tau) : \tau\in[0,t]\}\big)$, $\big(\theta(t),\mathcal{F}_t,\mathcal{W}^{(E)}\big)$ is a $\mathcal{W}$-Brownian motion. Conversely, if $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is any $\mathcal{W}$-Brownian motion, then $B(\,\cdot\,,\omega)\in\Theta(E)$ $\mathbb{P}$-almost surely and $\mathcal{W}^{(E)}$ is the $\mathbb{P}$-distribution of $\omega\rightsquigarrow B(\,\cdot\,,\omega)$.*

PROOF: To prove the first assertion, let $t_1,t_2\in[0,\infty)$ and $x_1^*,x_2^*\in E^*$ be given, and define $\lambda_i\in\Theta(E)^*$ so that $\langle\theta,\lambda_i\rangle=\langle\theta(t_i),x_i^*\rangle$ for $i\in\{1,2\}$. Then (cf. the notation in the proof of Theorem 8.6.1) $h_{\lambda_i}=h^1_{t_i}h_{x_i^*}$, and so

$$\mathbb{E}^{\mathcal{W}^{(E)}}\big[\langle\theta(t_1),x_1^*\rangle\langle\theta(t_2),x_2^*\rangle\big]=\big(h_{\lambda_1}h_{\lambda_2}\big)_{H^1(H)}=(t_1\wedge t_2)\big(h_{x_1^*},h_{x_2^*}\big)_H.$$

Starting from this, it is an easy matter to check that the span of $\{\langle\theta(t), x^*\rangle : (t, x^*) \in [0,\infty)\times E^*\}$ is a Gaussian family in $L^2(\mathcal{W}^{(E)};\mathbb{R})$ which satisfies (8.6.5).

To prove the converse, begin by observing that, because $\mathfrak{G}(B)$ is a Gaussian family satisfying (8.6.5), the distribution of $\omega \in \Omega \longmapsto B(\,\cdot\,,\omega) \in C\big([0,\infty);E\big)$ under $\mathbb{P}$ is the same as that of $\theta \in \Theta(E) \longmapsto \theta(\,\cdot\,) \in C\big([0,\infty);E\big)$ under $\mathcal{W}^{(E)}$. Hence

$$\mathbb{P}\left(\overline{\lim_{t\to\infty}}\frac{\|B(t)\|_E}{t} = 0\right) = \mathcal{W}^{(E)}\left(\overline{\lim_{t\to\infty}}\frac{\|\theta(t)\|_E}{t} = 0\right) = 1,$$

and so $B(\,\cdot\,,\omega) \in \Theta(E)$ $\mathbb{P}$-almost surely and the distribution of $\omega \rightsquigarrow B(\,\cdot\,,\omega)$ on $\Theta(E)$ is $\mathcal{W}^{(E)}$. $\square$

§ **8.6.3. Strassen's Theorem Revisited.** What I called Strassen's Theorem in § 8.4.2 is not the form in which Strassen himself presented it. Instead, his formulation was in terms of rescaled $\mathbb{R}$-valued Brownian motion, not partial sums of independent random variables. The true statement of Strassen Theorem is the following in the present setting.

THEOREM 8.6.7 (**Strassen**). *Given $\theta \in \Theta(E)$, define $\tilde\theta_n(t) = \frac{\theta(nt)}{\Lambda_n}$ for $n \ge 1$ and $t \in [0,\infty)$, where $\Lambda_n = \sqrt{2n\log_{(2)}(n\vee 3)}$. Then, for $\mathcal{W}^{(E)}$-almost every θ, the sequence $\{\tilde\theta_n : n \ge 0\}$ is relatively compact in $\Theta(E)$ and $\overline{B_{H^1(H)}(0,1)}$ s its set of limit points. Equivalently, for $\mathcal{W}^{(E)}$-almost every θ, $\overline{\lim}_{n\to\infty}\|\tilde\theta_n - \overline{B_{H^1(H)}(0,1)}\|_{\Theta(E)} = 0$ and, for each $h \in \overline{B_{H^1(H)}(0,1)}$, $\underline{\lim}_{n\to\infty}\|\tilde\theta_n - h\|_{\Theta(E)} = 0$.*

Not surprisingly, the proof differ only slightly from that of Theorem 8.4.4. In proving the $\mathcal{W}^{(E)}$-almost sure convergence of $\{\tilde\theta_n : n \ge 1\}$ to $\overline{\overline{B_{H^1(H)}(0,1)}}$, there are two new ingredients here. The first is the use of the Brownian scaling invariance (cf. Exercise 8.6.8), which says that the $\mathcal{W}^{(E)}$ is invariant under the scaling maps $S_\alpha : \Theta(E) \longrightarrow \Theta(E)$ given by $S_\alpha\theta = \alpha^{-\frac12}\theta(\alpha\,\cdot\,)$ for $\alpha > 0$ and is easily proved as a consequence of the fact these maps are isometric from $H^1(H)$ onto itself. The second new ingredient is the observation that, for any $R > 0$, $r \in (0,1]$, and $\theta \in \Theta(E)$, $\|\theta(r\,\cdot\,) - B_{H^1(H)}(0,R)\|_{\Theta(E)} \le \|\theta - B_{H^1(H)}(0,R)\|_{\Theta(E)}$. To see this, let $h \in B_{H^1(H)}(0,R)$ be given, and check that $h(r\,\cdot\,)$ is again in $B_H(0,R)$ and that $\|\theta(r\,\cdot\,) - h(r\,\cdot\,)\|_{\Theta(E)} \le \|\theta - h\|_{\Theta(E)}$. Taking these into account, one can now justify

$$\begin{aligned}
&\mathcal{W}^{(E)}\left(\max_{\beta^{m-1}\le n\le\beta^m}\big\|\tilde\theta_n - \overline{B_{H^1(H)}(0,1)}\big\|_{\Theta(E)} \ge \delta\right)\\
&= \mathcal{W}^{(E)}\left(\max_{\beta^{m-1}\le n\le\beta^m}\left\|\frac{\beta^{\frac m2}\theta(n\beta^{-m}\,\cdot\,)}{\Lambda_n} - \overline{B_{H^1(H)}(0,1)}\right\|_{\Theta(E)} \ge \delta\right)\\
&\le \mathcal{W}^{(E)}\left(\max_{\beta^{m-1}\le n\le\beta^m}\left\|\theta\big(\beta^{-m}n\,\cdot\,\big) - \overline{B_{H^1(H)}\left(0, \frac{\Lambda_{[\beta^{m-1}]}}{\beta^{\frac m2}}\right)}\right\|_{\Theta(E)} \ge \frac{\delta}{\beta^{\frac m2}\Lambda_{[\beta^{m-1}]}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{W}^{(E)}\left(\left\|\theta - \overline{B_{H^1(H)}\left(0, \frac{\Lambda_{[\beta^{m-1}]}}{\beta^{\frac{m}{2}}}\right)}\right\|_{\Theta(E)} \geq \frac{\delta}{\beta^{\frac{m}{2}}\Lambda_{[\beta^{m-1}]}}\right) \\
&= \mathcal{W}^{(E)}\Big(\big\|\beta^{\frac{m}{2}}\Lambda^{-1}_{[\beta^{m-1}]}\theta - \overline{B_{H^1(H)}(0,1)}\big\|_{\Theta(E)} \geq \delta\Big) \\
&= \mathcal{W}^{(E)}_{\beta^{\frac{m}{2}}\Lambda^{-1}_{[\beta^{m-1}]}}\big(\|\theta - \overline{B_{H^1(H)}(0,1)}\|_{\Theta(E)} \geq \delta\big)
\end{aligned}
$$

∎

for all $\beta \in (1,2)$ and $m \geq 1$. Armed with this information, one can simply repeat the argument given at the analogous place in the proof of Theorem 8.4.4.

The proof that, $\mathcal{W}^{(E)}$-almost surely, $\tilde{\theta}_n$ appoaches every $h \in C$ infinitely often also requires only minor modification. To begin, one remarks that if $A \subseteq \Theta(E)$ is relatively compact, then

$$
\lim_{T\to\infty} \sup_{\theta\in A} \sup_{t\notin[T^{-1},T]} \frac{\|\theta(t)\|_E}{1+t} = 0.
$$

Thus, since, by the preceding, for $\mathcal{W}^{(E)}$-almost every θ, $\overline{B_{H^1(H)}(0,1)} \cup \{\theta_n : n \geq 1\}$ is relatively compact in $\Theta(E)$, it suffices to prove that

$$
\varliminf_{n\to\infty} \sup_{t\in[k^{-1},k]} \frac{\big\|\big(\tilde{\theta}_n(t) - \tilde{\theta}_n(k^{-1})\big) - \big(h(t) - h(k^{-1})\big)\|_E}{1+t} = 0 \ \mathcal{W}^{(E)}\text{-almost surely}
$$

for each $h \in B_{H^1(H)}(0,1)$ and $k \geq 2$. Because, for a fixed $k \geq 2$, the random variables $\big(\tilde{\theta}_{k^{2m}} - \tilde{\theta}_{k^{2m}}(k^{-1})\big) \restriction [k^{-1},k]$, $m \geq 1$, are $\mathcal{W}^{(E)}$-independent random variables, we can use the Borel–Cantelli Lemma as in § 8.4.2 and thereby reduce the problem to showing that, if $\check{\theta}_{k^m}(t) = \tilde{\theta}_{k^m}(t + k^{-1}) - \tilde{\theta}_{k^m}(k^{-1})$, then

$$
\sum_{m=1}^{\infty} \mathcal{W}^{(E)}\big(\|\check{\theta}_{k^{2m}} - h\|_{\Theta(E)} \leq \delta\big) = \infty
$$

for each $\delta > 0$, $k \geq 2$, and $h \in B_{H^1(H)}(0,1)$. Finally, since ${\mathcal{W}^{(E)}}_{k^m\Lambda^{-1}_{k^{2m}}}$ is the $\mathcal{W}^{(E)}$ distribution of $\theta \rightsquigarrow \check{\theta}_{k^{2m}}$, the rest of the argument is the same as the one given in § 8.4.2.

Exercises for § 8.6

EXERCISE 8.6.8. Let $\big(H^1(H), \Theta(E), \mathcal{W}^{(E)}\big)$ be as in Theorem 8.6.1.

(**i**) Given $\alpha > 0$, define $S_\alpha : \Theta(E) \longrightarrow \Theta(E)$ so that $S_\alpha\theta(t) = \alpha^{-\frac{1}{2}}\theta(\alpha t)$, $t \in [0,\infty)$, and show that $(S_\alpha)_*\mathcal{W}^{(E)} = \mathcal{W}^{(E)}$. Again, this property is called **Brownian scaling invariance**.

(**ii**) Define $I : \Theta(E) \longrightarrow C\big([0,\infty); E\big)$ so the $I\theta(0) = 0$ and $I\theta(t) = t\theta(t^{-1})$ for $t > 0$. Show that I is an isometry from $\Theta(E)$ onto itself and that $I \restriction H^1(H)$ is an isometry on H onto itself. In addition, prove the **Brownian time inversion invariance** property: $I_*\mathcal{W}^{(E)} = \mathcal{W}^{(E)}$.

EXERCISE 8.6.9. Let $H^U(H)$ be the Hilbert space of absolutely continuous $h^U : \mathbb{R} \longrightarrow H$ with the property that

$$\|h\|_{H^U(H)} = \sqrt{\|\dot{h}^U\|^2_{L^2(\mathbb{R};H)} + \tfrac{1}{4}\|h^U\|^2_{L^2(\mathbb{R};H)}} < \infty,$$

and take $\Theta^U(E)$ to be the Banach space of continous $\theta^U : \mathbb{R} \longrightarrow E$ satisfying $\lim_{|t|\to\infty} \frac{\|\theta^U(t)\|}{\log t} = 0$ with norm $\|\theta^U\|_{\Theta^U(E)} = \sup_{t\in\mathbb{R}} \frac{\|\theta^U(t)}{\log(e+|t|)}$. If $F : \Theta(E) \longrightarrow C(\mathbb{R}; E)$ is given by $[F(\theta)](t) = e^{-\frac{t}{2}}\theta(e^t)$, show that F takes $\Theta(E)$ continuously into $\Theta^U(E)$ and that $\big(H^U(H), \Theta^U(E), \mathcal{U}^{(E)}\big)$ is an abstract Wiener space when $\mathcal{U}^{(E)}_R = F_*\mathcal{W}^{(E)}$. Of course, one should recognize the measure $\mathcal{U}^{(E)}_R$ as the distribtution of an E-valued, reversible, Ornstein–Uhlenbeck process.

EXERCISE 8.6.10. A particularly interesting case of the construction in Exercise 8.6.9 is when $H = H^1(\mathbb{R}^N)$ and $E = \Theta(\mathbb{R}^N)$. Working in that setting, define $\mathbf{B} : \mathbb{R} \times [0,\infty) \times \Theta^U\big(\Theta(E)\big) \longrightarrow \mathbb{R}^N$ by $\mathbf{B}\big((s,t),\theta\big) = [\theta(s)](t)$, and show that, for each $s \in \mathbb{R}$, $\big(\mathbf{B}(s,t), \mathcal{F}_{(s,t)}, \mathcal{U}^{\Theta(\mathbb{R}^N)}_R\big)$ is an $\mathbb{R}^N$-valued Brownian motion when $\mathcal{F}_{s,t} = \sigma\big(\{\mathbf{B}(s,\tau) : \tau \in [0,t]\}\big)$. Next, for each $t \in [0,\infty)$, show that the $\mathcal{U}^{\Theta(E)}_R$-distribution of $\theta \rightsquigarrow \mathbf{B}(\,\cdot\,,t)$ is that of $\sqrt{t}$ times a reverisble, $\mathbb{R}^N$-valued Ornstein–Uhlenbeck process.

EXERCISE 8.6.11. Continuing in the same setting as in the preceding, set $\sigma^2 = \mathbb{E}^{\mathcal{W}^{(E)}}\big[\|\theta\|^2_{\Theta(E)}\big]$, and combine the result in Exercise 8.2.12 with Brownian scaling invariance to show that

$$\mathcal{W}^{(E)}\left(\sup_{\tau\in[0,t]} \|\theta(t)\|_E \geq R\right) \leq K \exp\left[-\frac{R^2}{72\sigma^2 t}\right],$$

where K is the constant in Fernique's Theorem. Next, use this together with Theorem 8.4.4 and the reasoning in Exercise 4.3.16 to show that

$$\varlimsup_{t\to\infty} \frac{\|\theta(t)\|_E}{\sqrt{2t\log_{(2)} t}} = L = \varlimsup_{t\searrow 0} \frac{\|\theta(t)\|_E}{\sqrt{2t\log_{(2)} \frac{1}{t}}} \quad \mathcal{W}^{(E)}\text{-almost surely,}$$

where $L = \sup\big\{\|h\|_E : h \in \overline{B_H(0,1)}\big\}$.

EXERCISE 8.6.12. It should be recognized that Theorem 8.4.4 is an immediate corollary of Theorem 8.6.7. To see this, check that $\{\theta(n) : n \geq 1\}$ has the same distribution under $\mathcal{W}^{(E)}$ as $\{S_n : n \geq 1\}$ has under $\mathcal{W}^{\mathbb{N}}$ and that $\overline{B_H(0,1)} = \{h(1) : h \in \overline{B_{H^1(H)}}\}$, and use these to show that Theorem 8.4.4 follows from Theorem 8.6.7.

EXERCISE 8.6.13. For $\theta \in \Theta(E)$ and $n \in \mathbb{Z}^+$, define $\breve{\theta}_n \in \Theta(E)$ so that

$$\breve{\theta}_n(t) = \sqrt{\frac{n}{\log_{(2)}(n \vee 3)}}\,\theta\left(\frac{t}{n}\right), \quad t \in [0, \infty),$$

and show that, $\mathcal{W}^{(E)}$-almost surely, $\{\breve{\theta}_n : n \geq 1\}$ is relatively compact in $\Theta(E)$ and that $\overline{B_{H^1(H)}(0,1)}$ is the set of its limit points.

Hint: Referring to (**ii**) in Exercise 8.6.8, show that it suffices to prove these properties for the sequence $\{(I\theta)\breve{}_n : n \geq 1\}$. Next check that

$$\left\|(I\theta)\breve{}_n - Ih\right\|_{\Theta(E)} = \left\|\tilde{\theta}_n - h\right\|_{\Theta(E)} \quad \text{for } h \in H^1(H),$$

and use the Theorem 8.6.7 and fact the I is an isometry of $H^1(H)$ onto itself.