Chapter VII

Continuous Parameter Martingales

It turns out that many of the ideas and results introduced in § 5.2 can be easily transferred to the setting of processes depending on a *continuous parameter*. In addition, the resulting theory is intimately connected with Lévy processes, and particularly Brownian. In this chapter, I will give a brief introduction to this topic and some of the techniques to which it leads.*

§ **7.1 Continuous Parameter Martingales**

There is a huge number of annoying technicalities which have to be addressed in order to give a mathematically correct description of the continuous time theory of martingales. Fortunately, for the applications which I will give here, I can keep them to a minimum.

§ **7.1.1. Progressively Measurable Functions.** Let $(\Omega, \mathcal{F})$ be a measurable space and $\{\mathcal{F}_t : t \in [0,\infty)\}$ a non-decreasing family of sub-σ-algebras. I will say that a function X on $[0,\infty)\times\Omega$ into a measurable space $(E,\mathcal{B})$ is **progressively measurable** with respect to $\{\mathcal{F}_t : t \in [0,\infty)\}$ if $X \restriction [0,T]\times\Omega$ is $\mathcal{B}_{[0,T]}\times\mathcal{F}_T$-measurable for every $T \in [0,\infty)$. When E is a metric space, I will say that $X : [0,\infty)\times\Omega \longrightarrow E$ is **right-continuous** if $X(s,\omega) = \lim_{t\searrow s} X(t,\omega)$ for every $(s,\omega) \in [0,\infty)\times\Omega$ and will say that it is **continuous** if $X(\,\cdot\,,\omega)$ is continuous for all $\omega \in \Omega$.

Remark 7.1.1. The reader might have been expecting a slightly different definition of progressive measurability here. Namely, he might have thought that one would say that X is $\{\mathcal{F}_t : t \in [0,\infty)\}$-progressively measurable if it is $\mathcal{B}_{[0,\infty)}\times\mathcal{F}$-measurable and $\omega \in \Omega \longmapsto X(t,\omega) \in E$ is $\mathcal{F}_t$-measurable for each $t \in [0,\infty)$. Indeed, in extrapolating from the discrete parameter setting, this would be the first definition at which one would arrive. In fact, it was the notion with which Doob and Itô originally worked; and such functions were said by them to be **adapted** to $\{\mathcal{F}_t : t \in [0,\infty)\}$. However, it came to be realized that there are various problems with the notion of adaptedness. For example, even if X is adapted and $f : E \longrightarrow \mathbb{R}$ is a bounded, $\mathcal{B}$-measurable function, the

* A far more thorough treatment can be found in D. Revuz and M. Yor's treatise *Continuous Martingales and Brownian Motion* published by Springer–Verlag as volume #293 in their Grundlehren der Mathematishen Series.

function $(t,\omega) \rightsquigarrow Y(t,\omega) \equiv \int_0^t f\big(X(s,\omega)\big)\,ds \in \mathbb{R}$ need not be adapted. On the other hand, if X is progressively measurable, then Y will be also.

The following simple lemma should help to explain the virtue of progressive measurability and its relationship to adaptedness.

Lemma 7.1.2. *Let $\mathcal{PM}$ denote the set of $A \subseteq [0,\infty) \times \Omega$ with the property that $\big([0,t]\times\Omega\big) \cap A \in \mathcal{B}_{[0,t]} \times \mathcal{F}_t$ for every $t \geq 0$. Then $\mathcal{PM}$ is a sub-σ-algebra of $\mathcal{B}_{[0,\infty)} \times \mathcal{F}$ and X is progressively measurable if and only if it is $\mathcal{PM}$-measurable. Furthermore, if E is a separable metric space and $X : [0,\infty)\times\Omega \longrightarrow E$ is a right-continuous function, then X is progressively measurable if it is adapted.*

Proof: Checking that $\mathcal{PM}$ is a σ-algebra is easy. Furthermore, for any $X : [0,\infty)\times\Omega \longrightarrow E$, $T \in [0,\infty)$, and $\Gamma \in \mathcal{B}$,

$$\begin{aligned}&\big\{(t,\omega) \in [0,T]\times\Omega : X(t,\omega) \in \Gamma\big\}\\ &\qquad = \big([0,T]\times\Omega\big) \cap \big\{(t,\omega) \in [0,\infty)\times\Omega : X(t,\omega)\in\Gamma\big\},\end{aligned}$$

and so X is $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$-progressively measurable if and only if it is $\mathcal{PM}$-measurable. Hence, the first assertion has been proved.

Next, suppose that X is a right-continuous, adapted function. To see that X is progressively measurable, let $t \in [0,\infty)$ be given, and define

$$X_n^t(\tau,\omega) = X\left(\tfrac{[2^n\tau]+1}{2^n} \wedge t, \omega\right), \quad \text{for } (\tau,\omega) \in [0,\infty)\times\Omega \text{ and } n \in \mathbb{N}.$$

Obviously, X_n^t is $\mathcal{B}_{[0,t]}\times\mathcal{F}_t$-measurable for every $n \in \mathbb{N}$ and $X_n^t(\tau,\omega) \longrightarrow X(\tau,\omega)$ as $n \to \infty$ for every $(\tau,\omega) \in [0,t]\times\Omega$. Hence, $X \restriction [0,t]\times\Omega$ is $\mathcal{B}_{[0,t]}\times\mathcal{F}_t$, and so X is progressively measurable. $\square$

§ **7.1.2. Martingales, Definition and Examples.** Given a probability space $(\Omega,\mathcal{F},P)$ and a non-decreasing family of sub-σ-algebras $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$, I will say that $X : [0,\infty)\times\Omega \longrightarrow (-\infty,\infty]$ is a **submartingale with respect to** $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$ or, equivalently, that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is a **submartingale** if X is a right-continuous, progressively measurable function with the properties that $X(t)^-$ is $\mathbb{P}$-integrable for every $t \in [0,\infty)$ and

$$X(s) \leq \mathbb{E}^{\mathbb{P}}\big[X(t)\big|\mathcal{F}_s\big] \quad (\text{a.s.}, \mathbb{P}) \quad \text{for all} \quad 0 \leq s \leq t < \infty.$$

When both $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ and $\big(-X(t),\mathcal{F}_t,\mathbb{P}\big)$ are submartingales, I will say either that X is a **martingale with respect to** $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$ or simply that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is a **martingale**. Finally, if $Z : [0,\infty)\times\Omega \longrightarrow \mathbb{C}$ is a right-continuous, progressively measurable function, then $\big(Z(t),\mathcal{F}_t,\mathbb{P}\big)$ is said to be a **(complex) martingale** if both $\big(\mathfrak{Re}\,Z(t),\mathcal{F}_t,\mathbb{P}\big)$ and $\big(\mathfrak{Im}\,Z(t),\mathcal{F}_t,\mathbb{P}\big)$ are.

The next two results show that Lévy processes provide a rich source of continuous parameter martingales.

THEOREM 7.1.3. *Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ with $\hat{\mu}(\boldsymbol{\xi}) = e^{\ell_\mu(\boldsymbol{\xi})}$, where $\ell_\mu(\boldsymbol{\xi})$ equals*

$$\sqrt{-1}\,\big(\boldsymbol{\xi},\mathbf{m}\big)_{\mathbb{R}^N} - \big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big)\,M(d\mathbf{y}).$$

If $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and $\mathbf{Z}:[0,\infty)\times\Omega \longrightarrow \mathbb{R}^N$ is a $\mathcal{B}_{[0,\infty)}\times\mathcal{F}$-measurable map with the properties that $\mathbf{Z}(0,\omega)=\mathbf{0}$ and $\mathbf{Z}(\,\cdot\,,\omega)\in D(\mathbb{R}^N)$ for every $\omega\in\Omega$, then $\{\mathbf{Z}(t):t\ge 0\}$ is a Lévy process for μ if and only if, for each $\mathbf{x}\in\mathbb{R}^N$,

$$\Big(\exp\big(\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big),\mathcal{F}_t,\mathbb{P}\Big) \text{ is a martingale,} \tag{7.1.4}$$

where $\mathcal{F}_t = \sigma\big(\{\mathbf{Z}(\tau):\tau\in[0,t]\}\big)$.

PROOF: If $\{\mathbf{Z}(t):t\ge 0\}$ is a Lévy process for μ, then, because $\mathbf{Z}(t)-\mathbf{Z}(s)$ is independent of $\mathcal{F}_s$ and has characteristic function $e^{(t-s)\ell_\mu(\boldsymbol{\xi})}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{Z}(t)\big)_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big]\,\Big|\,\mathcal{F}_s\Big]\\
&\quad= \exp\big[\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{Z}(s)\big)_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})\big]\,e^{(s-t)\ell_\mu(\boldsymbol{\xi})}\mathbb{E}^P\Big[e^{\sqrt{-1}\left(\boldsymbol{\xi},\mathbf{Z}(t)-\mathbf{Z}(s)\right)_{\mathbb{R}^N}}\Big]\\
&\quad= \exp\big[\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{Z}(s)\big)_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})\big].
\end{aligned}$$

To prove the converse assertion, observe that the defining distributional property of a Lévy process for μ can be summarized as the statement that $\mathbf{Z}(0,\omega)=0$ and, for each $0\le s<t$, $\mathbf{Z}(t)-\mathbf{Z}(s)$ is independent of $\sigma\big(\{\mathbf{Z}(\tau):\tau\in[0,t]\}\big)$ and has distribution μ_{t-s}, where $\widehat{\mu_\tau}=e^{\tau\ell_\mu}$. Hence, since (7.1.4) implies that

$$\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{Z}(t)-\mathbf{Z}(s)\big)_{\mathbb{R}^N}\Big)\,\Big|\,\mathcal{F}_s\Big] = e^{(t-s)\ell_\mu(\boldsymbol{\xi})},\quad \boldsymbol{\xi}\in\mathbb{R}^N,$$

there is nothing more to do. $\square$

Another, and often more useful, way to capture the same result is to introduce the **Lévy operator**

$$\begin{aligned}
\mathcal{L}^\mu\varphi(x) &= \frac{1}{2}\mathrm{Trace}\big(\mathbf{C}\nabla^2\varphi(\mathbf{x})\big) + \big(\mathbf{m},\nabla\varphi(\mathbf{x})\big)_{\mathbb{R}^N}\\
&\quad+\int_{\mathbb{R}^N}\Big[\varphi(\mathbf{x}+\mathbf{y})-\varphi(\mathbf{x})-\mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\mathbf{y},\nabla\varphi(\mathbf{x})\big)_{\mathbb{R}^N}\Big]\,M(d\mathbf{y})
\end{aligned}\tag{7.1.5}$$

for $\varphi\in C^2_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$.

THEOREM 7.1.6. *Assume that $\mu \in \mathcal{I}(\mathbb{R}^N)$ and that $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ. Then, for every $F \in C_{\mathrm{b}}^{1,2}\big([0,\infty)\times\mathbb{R}^N;\mathbb{C}\big)$,*

$$\left(F\big(t,\mathbf{Z}(t)\big) - \int_0^t \big(\partial_\tau + \mathcal{L}_\mu\big)F\big(\tau,\mathbf{Z}(\tau)\big)\,d\tau, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale, where $\mathcal{F}_t = \sigma\big(\{\mathbf{Z}(\tau) : \tau \in [0,t]\}\big)$ and $\mathcal{L}^\mu$ is the operator described in (7.1.5). Conversely, if $\mathbf{Z}$ is a progressively measurable function satisfying $\mathbf{Z}(0,\omega) = \mathbf{0}$ and $\mathbf{Z}(\,\cdot\,,\omega) \in D(\mathbb{R}^N)$ for each $\omega \in \Theta$, and if

$$\left(\varphi\big(\mathbf{Z}(t)\big) - \int_0^t \mathcal{L}^\mu\varphi\big(\mathbf{Z}(\tau)\big)\,d\tau, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale for each $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^N;\mathbb{R})$, then $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ.

PROOF: Begin by noting that it suffices to handle the case when F is the restriction to $[0,\infty)\times\mathbb{R}^N$ of a an element of the Schwartz test function space $\mathscr{S}(\mathbb{R}\times\mathbb{R}^N;\mathbb{C})$. Indeed, because $\|\mathcal{L}^\mu\varphi\|_{\mathrm{u}} \leq C\|\varphi\|_{C_{\mathrm{b}}^2(\mathbb{R}^N;\mathbb{C})}$ for some $C < \infty$, the result for $F \in C_{\mathrm{b}}^{1,2}\big([0,\infty)\times\mathbb{R}^N;\mathbb{C}\big)$ follows, via an obvious approximation procedure, from the result for $F \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^N;\mathbb{C})$. Next observe that it suffices to treat $F \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. To see this, simply interpret the process $t \in [0,\infty) \longmapsto \big(t,\mathbf{Z}_\mu(t)\big) \in \mathbb{R}^{N+1}$ as a Lévy process for $\delta_1 \times \mu$.

Now let $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$ be given. The key to proving the required result is the identity

$$\frac{d}{dt}\varphi \star \breve{\mu}_t = (\mathcal{L}^\mu\varphi)\star\breve{\mu}_t, \tag{*}$$

where $\breve{\mu}_t$ is the distribution of $-\mathbf{x}$ under μ_t, the measure determined by $\widehat{\mu_t} = e^{t\ell_\mu}$. Given the computations preceding Theorem 3.2.22, the easiest way to check (*) is the work via Fourier transform and to use those computations to verify that

$$\frac{d}{dt}\widehat{\varphi\star\breve{\mu}_t}(\boldsymbol{\xi}) = \ell_\mu(-\boldsymbol{\xi})\hat{\varphi}(\boldsymbol{\xi})e^{t\ell_\mu(-\boldsymbol{\xi})} = \widehat{\mathcal{L}^\mu\varphi}(\boldsymbol{\xi})e^{t\ell_\mu(-\boldsymbol{\xi})},$$

which is equivalent to (*). To see how (*) applies, observe that

$$\mathbb{E}^{\mathbb{P}}\big[\varphi\big(\mathbf{Z}(t)\big)\,\big|\,\mathcal{F}_s\big] = \varphi\star\breve{\mu}_{t-s}\big(\mathbf{Z}(s)\big),$$

and therefore that, for any $A \in \mathcal{F}_s$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\varphi\big(\mathbf{Z}(t)\big),\,A\big] - \mathbb{E}^{\mathbb{P}}\big[\varphi\big(\mathbf{Z}(s)\big),\,A\big] &= \int_s^t \mathbb{E}^{\mathbb{P}}\big[(\mathcal{L}^\mu\varphi)\star\mu_{\tau-s}\big(\mathbf{Z}(s)\big),\,A\big]\,d\tau \\ &= \int_s^t \mathbb{E}^{\mathbb{P}}\big[\mathcal{L}^\mu\varphi\big(\mathbf{Z}(\tau)\big),\,A\big]\,d\tau = \mathbb{E}^{\mathbb{P}}\left[\int_s^t \mathcal{L}^\mu\varphi\big(\mathbf{Z}(\tau)\big)\,d\tau,\,A\right],\end{aligned}$$

which, after rearrangement, is the asserted martingale property.

To prove the converse assertion, again begin with the observation that, by an easy approximation procedure, one can prove the martingale property for all $\varphi \in C^2_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ as soon as one knows it for $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R})$. In particular, one can take $\varphi(\mathbf{x}) = e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}$, in which case $\mathcal{L}^\mu\varphi = \ell_\mu(\boldsymbol{\xi})\varphi$, and therefore, for any $A \in \mathcal{F}_s$, one gets that

$$u(t) \equiv \mathbb{E}^{\mathbb{P}}\left[\exp\Big(\sqrt{-1}\big(\boldsymbol{\xi},\mathbf{Z}(t)\big)_{\mathbb{R}^N}\Big)_{\mathbb{R}^N},\, A\right] = u(s) + \ell_\mu(\boldsymbol{\xi})\int_s^t u(\tau)\,d\tau.$$

Since this means that $u(t) = e^{(t-s)\ell_\mu(\boldsymbol{\xi})}u(s)$, it follows that $\{\mathbf{Z}(t) : t \ge 0\}$ satisfies (7.1.4) and is therefore a Lévy process for μ. $\square$

As an immediate consequence of the preceding we have the following characterizations of the distribution of a Lévy process. In the statement which follows, $\mathcal{F}_t$ is the σ-algebra over $D(\mathbb{R}^N)$ generated by $\{\boldsymbol{\psi}(\tau) : \tau \in [0,t]\}$.

THEOREM 7.1.7. *Given $\mu \in \mathcal{I}(\mathbb{R}^N)$, let $\mathbb{Q}^\mu \in \mathbf{M}_1\big(D(\mathbb{R}^N)\big)$ be the distribution of a Lévy process for μ. Then $\mathbb{Q}^\mu$ is the unique $\mathbb{P} \in \mathbf{M}_1\big(D(\mathbb{R}^N)\big)$ which satisfies either one of the properties that*

$$\left(\exp\Big[\sqrt{-1}\,\big(\boldsymbol{\xi},\boldsymbol{\psi}(t)\big)_{\mathbb{R}^N} + t\ell_\mu(\boldsymbol{\xi})\Big],\mathcal{F}_t,\mathbb{P}\right)$$

is a martingale with mean value 1 *for each* $\boldsymbol{\xi} \in \mathbb{R}^N$,

or

$$\left(\varphi\big(\boldsymbol{\psi}(t)\big) - \varphi(\mathbf{0}) - \int_0^t \mathcal{L}^\mu\varphi\big(\boldsymbol{\psi}(\tau)\big)\,d\tau,\mathcal{F}_t,\mathbb{P}\right)$$

is a martingale with mean value 0 *for each* $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R})$.

§ **7.1.3. Basic Results.** In this subsection I run through some of the results from § 5.2 which transfer immediately to the continuous parameter setting.

LEMMA 7.1.8. *Let the interval I and the function $f : I \longrightarrow \mathbb{R}\cup\{\infty\}$ be as in Corollary 5.2.10. If either $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is an I-valued martingale or $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is an I-valued submartingale and f is non-decreasing and bounded below, then $\big(f\circ X(t),\mathcal{F}_t,\mathbb{P}\big)$ is a submartingale.*

PROOF: The fact that the parameter is continuous plays no role here, and so this result is already covered by the argument in Corollary 5.2.10. $\square$

THEOREM 7.1.9 (**Doob's Inequality**). *Let $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ be a submartingale. Then, for every $\alpha \in (0,\infty)$ and $T \in [0,\infty)$,*

$$P\left(\sup_{t\in[0,T]} X(t) \ge \alpha\right) \le \frac{1}{\alpha}\mathbb{E}^P\left[X(T),\ \sup_{t\in[0,T]} X(t) \ge \alpha\right].$$

In particular, for non-negative submartingales and $T \in [0,\infty)$,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\in[0,T]} X(t)^p\right]^{\frac{1}{p}} \le \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}\big[X(T)^p\big]^{\frac{1}{p}}, \quad p \in (0,\infty).$$

Proof: Because of Exercise 1.4.18, I need only prove the first assertion. To this end, let $T \in (0,\infty)$ and $n \in \mathbb{N}$ be given, apply Theorem 5.2.1 to the discrete parameter submartingale $\left(X\left(\frac{mT}{2^n}\right), \mathcal{F}_{\frac{mT}{2^n}}, \mathbb{P}\right)$, and observe that

$$\sup\left\{X\left(\tfrac{mT}{2^n}\right) : 0 \le m \le 2^n\right\} \nearrow \sup_{t\in[0,T]} X(t) \quad \text{as } n\to\infty. \quad \square$$

Theorem 7.1.10 (**Doob's Martingale Convergence Theorem**). *Assume that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ be a $\mathbb{P}$-integrable submartingale. If*

$$\sup_{t\in[0,\infty)} \mathbb{E}^{\mathbb{P}}\big[X(t)^+\big] < \infty,$$

then there exists an $\mathcal{F}_\infty \equiv \bigvee_{t\ge 0}\mathcal{F}_t$-measurable $X = X(\infty) \in L^1(\mathbb{P};\mathbb{R})$ to which $X(t)$ converges $\mathbb{P}$-almost surely as $t\to\infty$. Moreover, when $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is either a non-negative submartingale or a martingale, the convergence takes place in $L^1(\mathbb{P};\mathbb{R})$ if and only if the family $\big\{X(t) : t\in[0,\infty)\big\}$ is uniformly $\mathbb{P}$-integrable, in which case $X(t) \le \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t]$ or $X(t) = \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t]$ (a.s., $\mathbb{P}$) for all $t\in[0,\infty)$, and

$$\mathbb{P}\left(\sup_{t\ge 0}|X(t)| \ge \alpha\right) \le \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[|X|, \sup_{t\ge 0}|X(t)| \ge \alpha\right]. \tag{7.1.11}$$

Finally, again when $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is either a non-negative submartingale or a martingale, for each $p\in(1,\infty)$ the family $\big\{|X(t)|^p : t\in[0,\infty)\big\}$ is uniformly $\mathbb{P}$-integrable if and only if $\sup_{t\in[0,\infty)}\|X(t)\|_{L^p(\mathbb{P})} < \infty$, in which case $X(t) \longrightarrow X$ in $L^p(\mathbb{P};\mathbb{R})$.

Proof: To prove the initial convergence assertion, note that, by Theorem 5.2.15 applied to the discrete parameter process $\big(X(n),\mathcal{F}_n,\mathbb{P}\big)$, there is an $\bigvee_{n\in\mathbb{N}}\mathcal{F}_n$-measurable $X \in L^1(\mathbb{P};\mathbb{R})$ to which $X(n)$ converges $\mathbb{P}$-almost surely. Hence, we need only check that $\lim_{t\to\infty} X(t)$ exists in $[-\infty,\infty]$ $\mathbb{P}$-almost surely. To this end, define $U^{(n)}_{[a,b]}(\omega)$ for $n\in\mathbb{N}$ and $a<b$ to be the precise number of times that the sequence $\big\{X\left(\frac{m}{2^n},\omega\right) : m\in\mathbb{N}\big\}$ upcrosses the interval $[a,b]$ (cf. the paragraph preceding Theorem 5.2.15), observe that $U^{(n)}_{[a,b]}(\omega)$ is non-decreasing as n increases, and set $U_{[a,b]}(\omega) = \lim_{n\to\infty} U^{(n)}_{[a,b]}(\omega)$. Note that if $U_{[a,b]}(\omega) < \infty$, then (by right-continuity), there is an $s\in[0,\infty)$ such that either $X(t,\omega) \le b$ for all $t \geq s$ or $X(t,\omega) \geq a$ for all $t \geq s$. Hence, we will know that $X(t,\omega)$ converges in $[-\infty,\infty]$ for $\mathbb{P}$-almost every $\omega \in \Omega$ as soon as we show that $\mathbb{E}^{\mathbb{P}}\big[U_{[a,b]}\big] < \infty$ for every pair $a < b$. In addition, by (5.2.16), we know that

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}\Big[U^{(n)}_{[a,b]}\Big] \leq \sup_{t\in[0,\infty)} \frac{\mathbb{E}^{\mathbb{P}}\big[(X(t)-a)^+\big]}{b-a} < \infty,$$

and so the required estimate follows from the Monotone Convergence Theorem.

Now assume that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is either a non-negative submartingale or a martingale. Given the preceding, it is clear that $X(t) \longrightarrow X$ in $L^1(\mathbb{P};\mathbb{P})$ if $\big\{X(t) : t \in [0,\infty)\big\}$ is uniformly $\mathbb{P}$-integrable. Conversely, suppose that $X(t) \longrightarrow X$ in $L^1(\mathbb{P};\mathbb{R})$. Then, for any $T \in [0,\infty)$,

$$(*) \qquad |X(T)| \leq \lim_{t\to\infty} \mathbb{E}^{\mathbb{P}}\big[|X(t)|\,\big|\,\mathcal{F}_T\big] = \mathbb{E}^{\mathbb{P}}\big[|X|\,\big|\,\mathcal{F}_T\big].$$

In particular, from Theorem 7.1.9,

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X(t)| \geq \alpha\right) \leq \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[|X|,\ \sup_{t\in[0,T]} |X(t)| \geq \alpha\right]$$

for every $T \in (0,\infty)$. Hence, (7.1.11) follows when one lets $T \to \infty$. But, again from (*),

$$\mathbb{E}^{\mathbb{P}}\big[|X(T)|,\ |X(T)| \geq \alpha\big] \leq \mathbb{E}^{\mathbb{P}}\big[|X|,\ |X(T)| \geq \alpha\big] \leq \mathbb{E}^{\mathbb{P}}\big[|X|,\ \sup_{t\geq 0} |X(t)| \geq \alpha\big],$$

and therefore, since, by (7.1.11), $\mathbb{P}\left(\sup_{t\geq 0} |X(t)| \geq \alpha\right) \longrightarrow 0$ as $\alpha \to \infty$, we can conclude that $\{X(t) : t \geq 0\}$ is uniformly $\mathbb{P}$-integrable.

Finally, if $\{X(T) : T \geq 0\}$ is bounded in $L^p(\mathbb{P};\mathbb{R})$ for some $p \in (1,\infty)$, then, by the last part of Theorem 7.1.9, $\sup_{t\geq 0} |X(t)|^p$ is $\mathbb{P}$-integrable and therefore $X(t) \longrightarrow X$ in $L^p(\mathbb{P};\mathbb{R})$. □

§ 7.1.4. Stopping Times and Stopping Theorems. A **stopping time** relative to a non-decreasing family $\{\mathcal{F}_t : t \geq 0\}$ of σ-algebras is a map $\zeta : \Omega \longrightarrow [0,\infty]$ with the property that $\{\zeta \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. Given a stopping time ζ, I will associate with it the σ-algebra $\mathcal{F}_\zeta$ consisting of those $A \subseteq \Omega$ such that $A \cap \{\zeta \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. Note that, because $\{\zeta < t\} = \bigcup_{n=0}^{\infty}\{\zeta \leq (1-2^{-n})t\}$, $\{\zeta < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Here are a few useful facts about stopping times.

Lemma 7.1.12. *Let ζ be a stopping time. Then ζ is $\mathcal{F}_\zeta$-measurable, and, for any progressively measurable function X with values in a measurable space $(E,\mathcal{B})$, the function $\omega \rightsquigarrow X(\zeta,\omega) \equiv X\big(\zeta(\omega),\omega\big)$ is $\mathcal{F}_\zeta$-measurable on $\{\zeta < \infty\}$ in the sense that $\big\{\omega : \zeta(\omega) < \infty\ \&\ X(\zeta,\omega) \in \Gamma\big\} \in \mathcal{F}_\zeta$ for all $\Gamma \in \mathcal{B}$. In addition, $f \circ \zeta$ is again a stopping time if $f : [0,\infty] \longrightarrow [0,\infty]$ is a non-decreasing, right continuous function satisfying $f(\tau) \geq \tau$ for all $\tau \in [0,\infty]$. Next, suppose that ζ_1 and ζ_2 are a pair of stopping times. Then $\zeta_1 + \zeta_2$, $\zeta_1 \wedge \zeta_2$, and $\zeta_1 \vee \zeta_2$ are all stopping times, and $\mathcal{F}_{\zeta_1\wedge\zeta_2} \subseteq \mathcal{F}_{\zeta_1} \cap \mathcal{F}_{\zeta_2}$. Finally, for any $A \in \mathcal{F}_{\zeta_1}$, $A \cap \{\zeta_1 \leq \zeta_2\} \in \mathcal{F}_{\zeta_1\wedge\zeta_2}$.*

Proof: Since $\{\zeta \leq s\} \cap \{\zeta \leq t\} = \{\zeta \leq s \wedge t\} \in \mathcal{F}_t$, it is clear that ζ is $\mathcal{F}_\zeta$-measurable. Next, suppose that X is a progressively measurable function. To prove that $X(\zeta)$ is $\mathcal{F}_\zeta$ measurable, begin by checking that $\{\omega : \big(\zeta(\omega), \omega\big) \in A\big\} \in \mathcal{F}_t$ for any $A \in \mathcal{B}_t \times \mathcal{F}_t$. Indeed, this is obvious when $A = [0, s] \times B$ for $s \in [0, t]$ and $B \in \mathcal{F}_t$ and, since these generate $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$, follows in general. Now, for any $t \geq 0$ and $\Gamma \in \mathcal{B}$,

$$A(t, \Gamma) \equiv \big\{(\tau, \omega) \in [0, \infty) \times \Omega : \big(\tau, X(\tau, \omega)\big) \in [0, t] \times \Gamma\big\} \in \mathcal{B}_{[0,t]} \times \mathcal{F}_t,$$

and therefore

$$\{X(\zeta) \in \Gamma\} \cap \{\zeta \leq t\} = \big\{\omega : \big(\zeta(\omega), \omega\big) \in A(t, \Gamma)\big\} \in \mathcal{F}_t.$$

As for $f \circ \zeta$ when f satisfies the stated conditions, simply note that $\{f \circ \zeta \leq t\} = \{\zeta \leq f^{-1}(t)\} \in \mathcal{F}_t$, where $f^{-1} \equiv \inf\{\tau : f(\tau) \geq t\} \leq t$.

Next suppose that ζ_1 and ζ_2 are two stopping times. It is trivial to see that $\zeta_1 \wedge \zeta_2$ and $\zeta_1 \vee \zeta_2$ are again stopping times. In addition, if $\mathbb{Q}$ denotes the set of rational numbers, then

$$\{\zeta_1 + \zeta_2 > t\} = \{\zeta_1 > t\} \cup \bigcup_{q \in \mathbb{Q} \cap [0,1]} \{\zeta_1 \geq qt \ \&\ \zeta_2 > (1 - q)t\} \in \mathcal{F}_t.$$

Thus, $\zeta_1 + \zeta_2$ is a stopping time. To prove the final assertions, begin with the observation that if $\zeta_1 \leq \zeta_2$, then $A \cap \{\zeta_2 \leq t\} = \big(A \cap \{\zeta_1 \leq t\}\big) \cap \{\zeta_2 \leq t\} \in \mathcal{F}_t$ for all $A \in \mathcal{F}_{\zeta_1}$ and $t \geq 0$, and therefore $\mathcal{F}_{\zeta_1} \subseteq \mathcal{F}_{\zeta_2}$. Next, for any ζ_1 and ζ_2, $\{\zeta_1 \leq \zeta_2\} \in \mathcal{F}_{\zeta_2}$ since

$$\{\zeta_1 > \zeta_2\} \cap \{\zeta_2 \leq t\} = \bigcup_{q \in \mathbb{Q} \cap [0,1]} \{\zeta_1 > qt\} \cap \{\zeta_2 \leq qt\} \in \mathcal{F}_t.$$

Finally, if $A \in \mathcal{F}_{\zeta_1}$, then

$$\big(A \cap \{\zeta_1 \leq \zeta_2\}\big) \cap \{\zeta_1 \wedge \zeta_2 \leq t\} = \big(A \cap \{\zeta_1 \leq t\}\big) \cap \{\zeta_1 \leq t \wedge \zeta_2\},$$

and therefore, since $A \cap \{\zeta_1 \leq t\} \in \mathcal{F}_t$ and $\{\zeta_1 \leq t \wedge \zeta_2\} \in \mathcal{F}_{t \wedge \zeta_2} \subseteq \mathcal{F}_t$, we have that $A \cap \{\zeta_1 \leq \zeta_2\} \in \mathcal{F}_{\zeta_1 \wedge \zeta_2}$. $\square$

In order to prove the continuous parameter analog of Theorems 5.2.13 and 5.2.11 I will need the following uniform integrability result.

Lemma 7.1.13. *If $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is either a martingale or a non-negative, integrable submartingale, then, for each $T > 0$, the set*

$$\big\{X(\zeta) : \zeta \text{ is a stopping time dominated by } T\big\}$$

is uniformly $\mathbb{P}$-integrable. Furthermore, if, in addition, $\{X(t) : t \geq 0\}$ is uniformly $\mathbb{P}$-integrable, and (cf. Theorem 7.1.10) $X(\infty) = \lim_{t \to \infty} X(t)$ (a.s., $\mathbb{P}$), then $\big\{X(\zeta) : \zeta \text{ is a stopping time}\big\}$ is uniformly $\mathbb{P}$-integrable.

PROOF: Throughout, without loss in generality, I will assume that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is a non-negative, integrable submartingale.

Given a stopping time $\zeta \le T$, define $\zeta_n = \frac{[2^n\zeta]+1}{2^n}$ for $n \ge 0$. By Lemma 7.1.12, ζ_n is again a stopping time. Thus, by Theorem 5.2.13 applied to the discrete parameter submartingale $\big(X(m2^{-n}),\mathcal{F}_{m2^{-n}},\mathbb{P}\big)$,

$$X(\zeta_n) \le \mathbb{E}^{\mathbb{P}}\big[X\big(2^{-n}([2^nT]+1\big)\,\big|\,\mathcal{F}_{\zeta_n}\big] \le \mathbb{E}^{\mathbb{P}}\big[X(T+1),\,\big|\,\mathcal{F}_{\zeta_n}\big],$$

and so

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X(\zeta_n),\,X(\zeta_n)\ge\alpha\big] &\le \mathbb{E}^{\mathbb{P}}\big[X(T+1),\,X(\zeta_n)\ge\alpha\big]\\ &\le \mathbb{E}^{\mathbb{P}}\left[X(T+1),\ \sup_{t\in[0,T+1]} X(t)\ge\alpha\right].\end{aligned}$$

Starting from here, noting that $\zeta_n \searrow \zeta$ as $n\to\infty$, and applying Fatou's Lemma, we arrive at

$$(*)\qquad \mathbb{E}^{\mathbb{P}}\big[X(\zeta),\,X(\zeta)>\alpha\big] \le \mathbb{E}^{\mathbb{P}}\left[X(T+1),\ \sup_{t\in[0,T+1]} X(t)\ge\alpha\right].$$

Hence, since, by Theorem 7.1.9, $\mathbb{P}\Big(\sup_{t\in[0,T+1]}X(t)\ge\alpha\Big)$ tends to 0 as $\alpha\to\infty$, this proves the first assertion. When $\{X(t):\,t\ge 0\}$ is uniformly integrable, we can replace (*) by

$$\mathbb{E}^{\mathbb{P}}\big[X(\zeta\wedge T),\,X(\zeta\wedge T)>\alpha\big] \le \mathbb{E}^{\mathbb{P}}\left[X(\infty),\,\sup_{t\ge 0}X(t)\ge\alpha\right]$$

for any stopping time ζ and $T>0$. Hence, after another application of Fatou's Lemma, we get

$$\mathbb{E}^{\mathbb{P}}\big[X(\zeta),\,X(\zeta)>\alpha\big] \le \mathbb{E}^{\mathbb{P}}\left[X(\infty),\,\sup_{t\ge 0}X(t)\ge\alpha\right].$$

At the same time, the first inequality in Theorem 7.1.9 can be replaced by

$$\mathbb{P}\left(\sup_{t\ge 0}X(t)\ge\alpha\right) \le \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}\left[X(\infty),\,\sup_{t\ge 0}X(t)\ge\alpha\right] \le \frac{1}{\alpha}\mathbb{E}^{\mathbb{P}}[X(\infty)],$$

and so the asserted uniform integrability follows. $\square$

It turns out that in the continuous time context, Doob's Stopping Time Theorem is most easily seen as a corollary of Hunt's. Thus, I will begin with Hunt's.

THEOREM 7.1.14 (**Hunt**). *Let $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ be either a non-negative, integrable submartingale or a martingale. If ζ_1 and ζ_2 are bounded stopping times and $\zeta_1 \le \zeta_2$, then $X(\zeta_1) \le \mathbb{E}^{\mathbb{P}}\big[X(\zeta_2)\,\big|\,\mathcal{F}_{\zeta_1}\big]$, and equality holds in the martingale case. Moreover, when $\{X(t) : t \ge 0\}$ is uniformly $\mathbb{P}$-integrable and $X(\infty) \equiv \lim_{t\to\infty} X(t)$, then the same result holds for arbitrary stopping times $\zeta_1 \le \zeta_2$.*

PROOF: Given $\zeta_1 \le \zeta_2 \le T$, define $(\zeta_i)_n = 2^{-n}\big([2^n \zeta_i] + 1\big)$ for $n \ge 0$, note that $(\zeta_i)_n$ is a $\{\mathcal{F}_{m2^{-n}} : m \ge 0\}$-stopping time and that $\mathcal{F}_{\zeta_1} \subseteq \mathcal{F}_{(\zeta_1)_n}$, and apply Theorem 5.2.13 to the discrete parameter submartingale $\big(X(m2^{-n}, \mathcal{F}_{m2^{-n}}, \mathbb{P}\big)$ in order to see that

$$\mathbb{E}^{\mathbb{P}}\Big[X\big((\zeta_1)_n\big),\, A\Big] \le \mathbb{E}^{\mathbb{P}}\Big[X\big((\zeta_2)_n\big),\, A\Big], \quad A \in \mathcal{F}_{\zeta_1},$$

with equality in the martingale case. Because of right continuity and Lemma 7.1.13, $X\big((\zeta_i)_n\big) \longrightarrow X(\zeta_i)$ in $L^1(\mathbb{P};\mathbb{R})$, and so we have now shown that $X(\zeta_1) \le \mathbb{E}^{\mathbb{P}}\big[X(\zeta_2)\,\big|\,\mathcal{F}_{\zeta_1}\big]$, with equality in the martingale case.

When $\{X(t) : t \ge 0\}$ is uniformly $\mathbb{P}$-integrable and $\zeta_1 \le \zeta_2$ are unbounded, $\{X(\zeta_i \wedge T) : T \ge 0\}$ is uniformly $\mathbb{P}$-integrable for $i \in \{1,2\}$. Hence, for any $A \in \mathcal{F}_{\zeta_1}$ and $0 \le t \le T$,

$$\mathbb{E}^{\mathbb{P}}\big[X(T \wedge \zeta_1),\, A \cap \{\zeta_1 \le t\}\big] \le \mathbb{E}^{\mathbb{P}}\big[X(T \wedge \zeta_2),\, A \cap \{\zeta_1 \le t\}\big],$$

with equality in the martingale case. Letting first T and then t tend to infinity, one gets the same relationship for $X(\zeta_1)$ and $X(\zeta_2)$, initially with $A \cap \{\zeta_1 < \infty\}$ and then, trivially, with A alone. $\square$

THEOREM 7.1.15 (**Doob's Stopping Time Theorem**). *If $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is either a non-negative, integrable submartingale or a martingale, then, for every stopping time ζ, $\big(X(t \wedge \zeta), \mathcal{F}_t, \mathbb{P}\big)$ is either an integrable submartingale or a martingale.*

PROOF: Given $0 \le s < t$ and $A \in \mathcal{F}_s$, note that $A \cap \{\zeta > s\} \in \mathcal{F}_{s\wedge\zeta}$ and therefore, by Hunt's Theorem applied to the stopping times $s \wedge \zeta$ and $t \wedge \zeta$, that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X(t\wedge\zeta),\, A\big] &= \mathbb{E}^{\mathbb{P}}\big[X(\zeta),\, A \cap \{\zeta \le s\}\big] + \mathbb{E}^{\mathbb{P}}\big[X(t \wedge \zeta),\, A \cap \{\zeta > s\}\big] \\ &\ge \mathbb{E}^{\mathbb{P}}\big[X(\zeta),\, A \cap \{\zeta \le s\}\big] + \mathbb{E}^{\mathbb{P}}\big[X(s \wedge \zeta),\, A \cap \{\zeta > s\}\big] = \mathbb{E}^{\mathbb{P}}\big[X(s \wedge \zeta),\, A\big],\end{aligned}$$

where the inequality is an equality in the martingale case. $\square$

To demonstrate just how powerful these results are, I give the following extension of the independent increment property of Lévy processes. In its statement, the maps $\delta_t : D(\mathbb{R}^N) \longrightarrow D(\mathbb{R}^N)$ for $t \in [0,\infty)$ are defined so that $\delta_t \boldsymbol{\psi}(\tau) = \boldsymbol{\psi}(\tau + t) - \boldsymbol{\psi}(t)$, $\tau \in [0,\infty)$. Also, $\mathcal{F}_t = \sigma\big(\{\boldsymbol{\psi}(\tau) : \tau \in [0,t]\}\big)$, ζ is a stopping time relative to $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$, and δ_ζ is the map on $\{\boldsymbol{\psi} : \zeta(\boldsymbol{\psi}) < \infty\}$ into $D(\mathbb{R}^N)$ given by $\delta_\zeta \boldsymbol{\psi} = \delta_{\zeta(\boldsymbol{\psi})}\boldsymbol{\psi}$.

THEOREM 7.1.16. *Given $\mu \in \mathcal{I}(\mathbb{R}^N)$, let $\mathbb{Q}^\mu \in \mathbf{M}_1\big(D(\mathbb{R}^N)\big)$ be the measure described in Theorem 7.1.6. Then for each stopping time ζ and $\mathcal{F}_{D(\mathbb{R}^N)} \times \mathcal{F}_\zeta$-measurable functions $F : D(\mathbb{R}^N) \times D(\mathbb{R}^N) \longrightarrow [0,\infty)$,*

$$\int_{\{\zeta<\infty\}} F\big(\delta_\zeta \boldsymbol{\psi}, \boldsymbol{\psi}\big)\,\mathbb{Q}^\mu(d\boldsymbol{\psi}) = \iint \mathbf{1}_{[0,\infty)}\big(\zeta(\boldsymbol{\psi}')\big) F(\boldsymbol{\psi}, \boldsymbol{\psi}')\,Q^\mu(d\boldsymbol{\psi}) Q^\mu(d\boldsymbol{\psi}').$$

PROOF: By elementary measure theory, all that we have to show is that, for each $B \in \mathcal{F}_\zeta$ contained in $\{\zeta < \infty\}$, $\mathbb{Q}^\mu\big((\delta_\zeta^{-1}\Gamma) \cap B\big) = \mathbb{Q}^\mu(\Gamma)\mathbb{Q}^\mu(B)$.

Let $B \in \mathcal{F}_\zeta$ contained in $\{\zeta < \infty\}$ with $\mathbb{Q}^\mu(B) > 0$ be given, choose $T > 0$ so that $\mathbb{Q}^\mu(B_T) > 0$ when $B_T = B \cap \{\zeta \le T\}$, and define $\mathbb{Q}_T \in \mathbf{M}_1\big(D(\mathbb{R}^N)\big)$ so that

$$\mathbb{Q}_T(\Gamma) = \frac{\mathbb{Q}^\mu\big((\delta_\zeta^{-1}\Gamma) \cap B_T\big)}{\mathbb{Q}^\mu(B_T)}.$$

If we show that $\mathbb{Q}_T = \mathbb{Q}^\mu$, then we will know that

$$\begin{aligned}\mathbb{Q}^\mu\big((\delta_\zeta^{-1}\Gamma) \cap B\big) &= \lim_{T\to\infty} \mathbb{Q}^\mu\big((\delta_\zeta^{-1}\Gamma) \cap B_T\big) \\ &= \mathbb{Q}^\mu(\Gamma) \lim_{T\to\infty} \mathbb{Q}^\mu(B_T) = \mathbb{Q}^\mu(\Gamma)\mathbb{Q}^\mu(B)\end{aligned}$$

and therefore will be done.

By Theorem 7.1.6, checking that $\mathbb{Q}_T = \mathbb{Q}^\mu$ comes down to showing that, for any $0 \le s < t$, $\boldsymbol{\xi} \in \mathbb{R}^N$, and $A \in \mathcal{F}_s$,

$$\mathbb{E}^{\mathbb{Q}_T}\Big[e^{\sqrt{-1}(\mathbf{x},\boldsymbol{\psi}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})},\, A\Big] = \mathbb{E}^{\mathbb{Q}_T}\Big[e^{\sqrt{-1}(\mathbf{x},\boldsymbol{\psi}(s))_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})},\, A\Big].$$

To this end, note that, by Theorem 7.1.14 applied to $s + \zeta \wedge T$ and $t + \zeta \wedge T$,

$$\begin{aligned}&\mathbb{Q}^\mu(B_T)\mathbb{E}^{\mathbb{Q}_T}\Big[e^{\sqrt{-1}(\mathbf{x},\boldsymbol{\psi}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})},\, A\Big] \\ &= \mathbb{E}^{\mathbb{Q}^\mu}\Big[e^{-\sqrt{-1}(\boldsymbol{\xi},\boldsymbol{\psi}(\zeta))_{\mathbb{R}^N} + \zeta\ell_\mu(\boldsymbol{\xi})} e^{\sqrt{-1}(\boldsymbol{\xi},\boldsymbol{\psi}(t+\zeta))_{\mathbb{R}^N} - (t+\zeta)\ell_\mu(\boldsymbol{\xi})},\, (\delta_\zeta^{-1}A) \cap B_T\Big] \\ &= \mathbb{E}^{\mathbb{Q}^\mu}\Big[e^{-\sqrt{-1}(\boldsymbol{\xi},\boldsymbol{\psi}(\zeta))_{\mathbb{R}^N} + \zeta\ell_\mu(\boldsymbol{\xi})} e^{\sqrt{-1}(\boldsymbol{\xi},\boldsymbol{\psi}(s+\zeta))_{\mathbb{R}^N} - (s+\zeta)\ell_\mu(\boldsymbol{\xi})},\, (\delta_\zeta^{-1}A) \cap B_T\Big] \\ &= \mathbb{Q}^\mu(B_T)\mathbb{E}^{\mathbb{Q}_T}\Big[e^{\sqrt{-1}(\mathbf{x},\boldsymbol{\psi}(s))_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})},\, A\Big],\end{aligned}$$

since $\boldsymbol{\psi} \rightsquigarrow e^{-\sqrt{-1}(\boldsymbol{\xi},\boldsymbol{\psi}(\zeta))_{\mathbb{R}^N} + \zeta\ell_\mu(\boldsymbol{\xi})}\mathbf{1}_A(\delta_\zeta\boldsymbol{\psi})\mathbf{1}_{B_T}(\boldsymbol{\psi})$ is $\mathcal{F}_{s+\zeta\wedge T}$-measurable. □

§ **7.1.5. An Integration by Parts Formula.** In this subsection I will derive a simple result which has many interesting applications.

THEOREM 7.1.17. *Suppose $V : [0,\infty) \times \Omega \longrightarrow \mathbb{C}$ is a right-continuous, progressively measurable function, and let $|V|(t,\omega) \in [0,\infty]$ denote the total variation of $V(\,\cdot\,,\omega)$ on the interval $[0,t]$. Then $|V| : [0,\infty) \times \Omega \longrightarrow [0,\infty]$ is a non-decreasing, progressively measurable function which is right-continuous on each* *interval* $[0,t)$ *for which* $|V|(t,\omega) < \infty$. *Next, suppose that* $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ *is a* $\mathbb{C}$*-valued martingale with the property that, for each* $(t,\omega) \in (0,\infty)\times\Omega$, *the product* $\|X(\,\cdot\,,\omega)\|_{[0,t]}|V|(t,\omega) < \infty$, *and define*

$$Y(t,\omega) = \begin{cases} \int_{(0,t]} X(s,\omega)\,V(ds,\omega) & \text{if} \quad |V|(t,\omega) < \infty \\ 0 & \text{otherwise} \end{cases}$$

where, in the case when $|V|(t,\omega) < \infty$, *the integral is the Lebesgue integral of* $X(\,\cdot\,,\omega)$ *on* $[0,t]$ *with respect to the* $\mathbb{C}$*-valued measure determined by* $V(\,\cdot\,,\omega)$. *If*

$$\mathbb{E}^{\mathbb{P}}\Big[\|X\|_{[0,T]}\Big(|V|(T) + \big|V(0)\big|\Big)\Big] < \infty \quad \text{for all } T \in (0,\infty),$$

then $\big(X(t)V(t) - Y(t),\mathcal{F}_t,\mathbb{P}\big)$ *is a martingale.*

PROOF: Without loss in generality, I will assume that both X and V are $\mathbb{R}$-valued. To see that $|V|$ is $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$-progressively measurable, simply observe that, by right-continuity,

$$|V|(t,\omega) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{[2^n t]} \left|V\left(\tfrac{k+1}{2^n} \wedge t,\omega\right) - V\left(\tfrac{k}{2^n},\omega\right)\right|;$$

and to see that $|V|(\,\cdot\,,\omega)$ is right-continuous on $[0,t)$ whenever $|V|(t,\omega) < \infty$, recall that the magnitude of the jumps (from the right and left) of the variation of a function coincide with those of the function itself.

I turn now to the second part. Certainly Y is $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$-progressively measurable. In addition, because $\|X(\,\cdot\,,\omega)\|_{[0,t]}|V|(t,\omega) < \infty$ for all $(t,\omega) \in [0,\infty)\times\Omega$, for any $\omega \in \Omega$ one has that

$$Y(t,\omega) = 0 \quad \text{or} \quad Y(t,\omega) = \int_{(0,t]} X(s,\omega)\,V(ds,\omega) \quad \text{for all } t \in [0,\infty);$$

and so, in either case, $Y(\,\cdot\,,\omega)$ is right-continuous and $Y(t,\omega) - Y(s,\omega)$ can be computed as

$$\lim_{n\to\infty} \sum_{k=[2^n s]}^{[2^n t]} X\left(\tfrac{k+1}{2^n} \wedge t,\omega\right)\Big(V\left(\tfrac{k+1}{2^n} \wedge t,\omega\right) - V\left(\tfrac{k}{2^n} \vee s,\omega\right)\Big).$$

In fact, under the stated integrability condition, the convergence in the preceding takes place in $L^1(\mathbb{P};\mathbb{R})$ for every $t \in [0,\infty)$; and therefore, for any $0 \le s \le t < \infty$

and $A \in \mathcal{F}_s$:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[Y(t)-Y(s),\,A\big]\\
&= \lim_{n\to\infty} \sum_{k=[2^n s]}^{[2^n t]} \mathbb{E}^{\mathbb{P}}\left[X\left(\tfrac{k+1}{2^n}\wedge t,\omega\right)\left(V\left(\tfrac{k+1}{2^n}\wedge t,\omega\right)-V\left(\tfrac{k}{2^n}\vee s,\omega\right)\right),\,A\right]\\
&= \lim_{n\to\infty} \sum_{k=[2^n s]}^{[2^n t]} \mathbb{E}^{\mathbb{P}}\left[X(t)\left(V\left(\tfrac{k+1}{2^n}\wedge t,\omega\right)-V\left(\tfrac{k}{2^n}\vee s,\omega\right)\right),\,A\right]\\
&= \mathbb{E}^{\mathbb{P}}\Big[X(t)\big(V(t)-V(s)\big),\,A\Big] = \mathbb{E}^{\mathbb{P}}\Big[X(t)V(t)-X(s)V(s),\,A\Big],
\end{aligned}
$$

and clearly this is equivalent to the asserted martingale property. □

We will make frequent practical applications of Theorem 7.1.17 below, but here I will show that it enables us to prove that there is an important dichotomy between continuous martingales and functions of bounded variation. However, before doing so, I need to make a small, technical digression.

A function $\zeta : \Omega \longrightarrow [0,\infty]$ is an **extended stopping time** relative to $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$ if $\{\zeta < t\} \in \mathcal{F}_t$ for every $t \in (0,\infty)$. Since $\{\zeta < t\} \in \mathcal{F}_t$ for any stopping time ζ, it is clear that every stopping time is an extended stopping time. On the other hand, not every extended stopping time is a stopping time. To wit, if $X : [0,\infty)\times\Omega \longrightarrow \mathbb{R}$ is a right-continuous, progressively measurable function relative to $\big\{\big\{\sigma\big(X(\tau) : \tau \in [0,t]\big\}\big) : t \ge 0\big\}$, then $\zeta = \inf\{t \ge 0 : X(t) > 1\}$ will always be an extended stopping time but will seldom be a stopping time.

Lemma 7.1.18. *For each $t \ge 0$, set $\mathcal{F}_{t+} = \bigcap_{\tau > t}\mathcal{F}_\tau$. Then $\zeta : \Omega \longrightarrow [0,\infty]$ is an extended stopping time if and only if it is a stopping time relative to $\{\mathcal{F}_{t+} : t \ge 0\}$. Moreover, if $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is either a non-negative, integrable submartingale or a martingale, then so is $\big(X(t),\mathcal{F}_{t+},\mathbb{P}\big)$. In particular, if ζ is an extended stopping time, then $\big(X(t\wedge\zeta),\mathcal{F}_{t+},\mathbb{P}\big)$ is a non-negative, integrable submartingale or a martingale.*

Proof: The first assertion is immediate from $\{\zeta \le t\} = \bigcap_{\tau>t}\{\zeta < \tau\}$. To prove the second assertion, apply right continuity and the first uniform integrability result in Lemma 7.1.13 to see that if $0 \le s < t$ and $A \in \mathcal{F}_{s+}$ then

$$
\mathbb{E}^{\mathbb{P}}\big[X(s),\,A\big] = \lim_{\tau\searrow s}\mathbb{E}^{\mathbb{P}}\big[X(\tau),\,A\big] \le \mathbb{E}^{\mathbb{P}}\big[X(t),\,A\big],
$$

where the inequality is an equality in the martingale case. □

Theorem 7.1.19. *Suppose that $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ is a continuous martingale, and let $|X|(t,\omega) = \mathrm{var}_{[0,t]}\big(X(\,\cdot\,,\omega)\big)$ denote the variation of $X(\,\cdot\,,\omega)\restriction[0,t]$. Then*

$$
\mathbb{P}\big(\exists t > 0\ 0 < |X|(t,\omega) < \infty\big) = 0.
$$

Equivalently, for $\mathbb{P}$-almost every ω and all $t > 0$, either $X(\tau,\omega) = X(0,\omega)$ for $\tau \in [0,t]$ or $|X|(t,\omega) = \infty$.

PROOF: Without loss in generality, I will assume that $X(0,\omega) \equiv 0$. Given $R > 0$, let $\zeta_R(\omega) = \sup\{t \geq 0 : |X|(t,\omega) \leq R\}$, and set $X_R(t) = X(t \wedge \zeta_R)$. Then ζ_R is an extended stopping time, and so, by Lemma 7.1.18, $(X_R(t), \mathcal{F}_{t+}, \mathbb{P})$ is a bounded martingale. Hence, by Theorem 7.1.17,

$$\left(X_R(t)^2 - \int_0^t X_R(\tau)\, X_R(d\tau), \mathcal{F}_{t+}, \mathbb{P} \right)$$

is also a martingale, and so

$$\mathbb{E}^{\mathbb{P}}\big[X_R(t)^2\big] = \mathbb{E}^{\mathbb{P}}\left[\int_0^t X_R(\tau)\, X_R(d\tau)\right].$$

On the other hand, since $X_R(\,\cdot\,)$ is continuous, and therefore, by Fubini's Theorem,

$$X_R(t)^2 = \iint\limits_{[0,t]^2} X_R(d\tau_1) X_R(d\tau_2) = 2\int_0^t X_R(\tau)\, X_R(d\tau),$$

we also know that

$$\mathbb{E}^{\mathbb{P}}\big[X_R(t)^2\big] = 2\mathbb{E}^{\mathbb{P}}\left[\int_0^t X_R(\tau)\, X_R(d\tau)\right].$$

Hence, $\mathbb{E}^{\mathbb{P}}\big[X_R(t)^2\big] = 0$ for all $t > 0$, which means that $X_R(\,\cdot\,) \equiv 0$ $\mathbb{P}$-almost surely. $\square$

The preceding result leads immediately to the following analog of the uniqueness statement in Lemma 5.2.12.

COROLLARY 7.1.20. *Let $X : \Omega \longrightarrow \mathbb{R}$ be a right continuous, progressively measurable function. Then, up to a $\mathbb{P}$-null set, there is at most one continuous, progressively measurable $A : \Omega \longrightarrow R$ such that $A(0,\omega) = 0$, $A(\,\cdot\,,\omega)$ is of locally bounded variation for $\mathbb{P}$-almost every $\omega \in \Omega$, and $\big(X(t) - A(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale.*

The role of continuity here seems minor, but it is crucial. Namely, continuity was used in Theorem 7.1.19 only when I wanted to know that $X_R(t)^2 = \int_0^t X_R(\tau)\, X_R(d\tau)$. On the other hand, it is critical. Namely, if $\{N(t) : t \geq 0\}$ is the simple Poisson process in § 4.2 and $\mathcal{F}_t = \sigma\big(\{N(\tau) : \tau \in [0,t]\}\big)$, then it is easy to check that $\big(N(t) - t, \mathcal{F}_t, \mathbb{P}\big)$ is a martingale, all of whose paths are of locally bounded variation.

Exercises for § 7.1

EXERCISE 7.1.21. The definition of stopping times and their associated σ-algebras which I have adopted is due to E.B. Dynkin. Earlier, less ubiquitous but more transparent, definitions appear in the work of Doob and Hunt under the name of *optional stopping times*. To explain these earlier definitions, let E be a Polish space and Ψ a non-empty collection of right continuous paths $\psi : [0,\infty) \longrightarrow E$ with the property that for all $\psi \in \Psi$ and $t \in [0,\infty)$, the stopped path ψ^t given by $\psi^t(\tau) = \psi(t \wedge \tau)$ is again in Ψ. Similarly, given a function $\zeta : \Psi \longrightarrow [0,\infty]$, define ψ^ζ so that $\psi^\zeta(t) = \psi\big(t \wedge \zeta(\psi)\big)$. Finally, for each $t \in [0,\infty)$, define the σ-algebras $\mathcal{F}_t$ over Ψ to be the one generated by $\{\psi(\tau) : \tau \in [0,t]\}$, and take $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. In terms of these quantities, an **optional stopping time** is an $\mathcal{F}$-measurable map $\zeta : \Psi \longrightarrow [0,\infty]$ such that $\zeta(\psi) \leq t \implies \zeta(\psi) = \zeta(\psi^t)$, and the associated σ-algebra is $\sigma\big(\{\psi^\zeta(t) : t \geq 0\}\big)$. The goal of this exercise is to show that ζ is an optional stopping time if and only if it is a stopping time and that its associated σ-algebra is $\mathcal{F}_\zeta$.

(**i**) It is an easy matter (cf. Exercise 4.1.9) to check that $f : \Omega \longrightarrow \mathbb{R}$ is $\mathcal{F}$-measurable if and only if there exists a $\mathcal{B}^{\mathbb{Z}^+}$-measurable $F : E^{\mathbb{Z}^+} \longrightarrow \mathbb{R}$ and a sequence $\{t_m : m \in \mathbb{Z}^+\}$ such that $f(\psi) = F\big(\psi(t_1), \dots, \psi(t_m), \dots\big)$, from which it is clear that a $\mathcal{F}$-measurable f will be $\mathcal{F}_t$-measurable for some $t \in [0,\infty)$ if and only if $f(\psi) = f(\psi^t)$. Use this to show that every optional stopping time is a stopping time.

(**ii**) Show that $\zeta : \Psi \longrightarrow [0,\infty]$ is a stopping time relative to $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$ if and only if it is $\mathcal{F}$-measurable and, for each $t \in [0,\infty)$, $\{\psi : \zeta(\psi) \leq t\} = \{\psi : \zeta(\psi^t) \leq t\}$. In addition, if ζ is a stopping time, show that $\zeta(\psi) < \infty \implies \zeta(\psi) = \zeta(\psi^\zeta)$, and therefore that $\zeta(\psi) \leq t \implies \zeta(\psi) = \zeta(\psi^t)$ for all $t \in [0,\infty)$. Thus, ζ is an optional stopping time if and only if it is a stopping time.

Hint: In proving the second part, check that $\{\zeta = t\} \in \mathcal{F}_t$, and conclude that $\mathbf{1}_{\{t\}}\big(\zeta(\psi)\big) = \mathbf{1}_{\{t\}}\big(\zeta(\psi^t)\big)$ for all $(t,\psi) \in [0,\infty) \times \Psi$.

(**iii**) If ζ is a stopping time, show that $\mathcal{F}_\zeta = \sigma\big(\{\psi^\zeta(t) : t \geq 0\}\big)$. Besides having intuitive value, this shows that, at least in the situation here, $\mathcal{F}_\zeta$ is countably generated.

Hint: Using right continuity, first show that $\psi \rightsquigarrow \psi^\zeta$ is $\mathcal{F}$-measurable. Next, given a $\mathcal{B}$-measurable $f : E \longrightarrow \mathbb{R}$ and $t \in [0,\infty)$, use (**ii**) to show that

$$\mathbf{1}_{[0,t]}\big(\zeta(\psi)\big) f\big(\psi^\zeta(\tau)\big) = \mathbf{1}_{[0,t]}\big(\zeta(\psi^t)\big) f\Big(\psi\big(\tau \wedge \zeta(\psi^t)\big)\Big), \quad \tau \in [0,\infty),$$

and conclude that $\sigma\big(\{\psi^\zeta(t) : t \geq 0\}\big) \subseteq \mathcal{F}_\zeta$. To prove the opposite inclusion, show that if $f : \Psi \longrightarrow \mathbb{R}$ is $\mathcal{F}_\zeta$-measurable, then, for each $t \in [0,\infty)$, $\mathbf{1}_{\{t\}}\big(\zeta(\psi)\big) f(\psi) = \mathbf{1}_{\{t\}}\big(\zeta(\psi^t)\big) f(\psi^t)$, and thereby arrive at $f(\psi) = f(\psi^\zeta)$. Finally, use this together with Exercise 4.1.9 to show that f is $\sigma\big(\{\psi^\zeta(t) : t \geq 0\}\big)$-measurable.

EXERCISE 7.1.22. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t : t \in [0,\infty)\}$ is non-decreasing family of sub σ-algebras of $\mathcal{F}$. Denote by $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}_t}$ the completions of $\mathcal{F}$ and $\mathcal{F}_t$ with respect to $\mathbb{P}$. If $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is a submartingale or martingale, show that $\big(X(t), \overline{\mathcal{F}_t}, \mathbb{P}\big)$ is also.

EXERCISE 7.1.23. Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ be given as in Exercise 3.2.23, and extend ℓ_μ to $\mathbb{C}^N$ accordingly. If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ, show that (7.1.4) continues to hold for all $\boldsymbol{\xi} \in \mathbb{C}^N$.

EXERCISE 7.1.24. In Exercise 3.3.12, we discussed one-sided stable laws, and in Exercise 4.3.12 we showed that $\mathbb{P}\big(\max_{\tau\in[0,t]} B(\tau) \geq a\big) = 2\mathbb{P}\big(B(t) \geq a\big)$, where $\{B(t) : t \geq 0\}$ is an $\mathbb{R}$-valued Brownian motion. In this exercise, we will examine the relationship between these two.

(**i**) Set $\zeta^a(\psi) = \inf\{t \geq 0 : \psi(t) \geq a\}$, and show that the result in Exercise 4.3.12 can be rewritten as

$$\mathcal{W}^{(1)}\big(\zeta^a \leq t\big) = \sqrt{\frac{2}{\pi t}} \int_{at^{-\frac{1}{2}}}^{\infty} e^{\frac{y^2}{2}}\, dy.$$

Now use the results in Exercise 3.3.14 (especially, (3.3.16)) to conclude that the $\mathcal{W}^{(1)}$-distribution of ζ^a is $\nu^{\frac{1}{2}}_{2^{\frac{1}{2}}a}$, the one-sided $\frac{1}{2}$-stable law "at time $2^{\frac{1}{2}}a$."

(**ii**) Here is another, more conceptual way, to understand the conclusion drawn in (**i**) that the $\mathcal{W}^{(1)}$-distribution is a one-sided $\frac{1}{2}$-stable law. Namely, begin by showing that if $\psi(0) = 0$ and $\zeta^a(\psi) < \infty$, then $\zeta^{a+b}(\psi) = \zeta^a(\psi) + \zeta^b\big(\delta_{\zeta^a}\psi\big)$. As an application of Theorem 7.1.16, conclude from this that if β_a denotes the $\mathcal{W}^{(1)}$-distribution of ζ^a, then $\beta_{a+b} = \beta_a \star \beta_b$. In particular, this means that $\beta \equiv \beta_1$ is infinitely divisible and that $\widehat{\beta_a} = e^{a\ell_\beta}$, where ℓ_β is the exponent appearing in the Lévy–Khinchine formula for $\hat{\beta}$.

(**iii**) Next, use Brownian scaling to see that, for all $\lambda > 0$, $\zeta^{\lambda a}$ has the same $\mathcal{W}^{(1)}$-distribution as $\lambda^2\zeta^a$, and use this together with part (**iii**) of Exercise 3.3.12 to see that the distribution of ζ^1 is $\nu_c^{\frac{1}{2}}$ for some $c > 0$.

(**iv**) Although we know from (**i**) that the constant c must be $2^{\frac{1}{2}}$, here is an easier way to find it. Use Exercise 7.1.23 to see that $\big(e^{\lambda\psi(t)-\frac{1}{2}\lambda^2 t}, \mathcal{F}_t, \mathcal{W}^{(1)}\big)$ for every $\lambda \in \mathbb{R}$, and apply Doob's Stopping Time Theorem and the fact that $\mathcal{W}^{(1)}(\zeta^a < \infty) = 1$ to verify the identity $\mathbb{E}^{\mathcal{W}^{(1)}}\big[e^{-\frac{1}{2}\lambda^2\zeta^a}\big] = e^{-\lambda a}$ for $\lambda > 0$. Hence, the Laplace transform of $\nu_c^{\frac{1}{2}}$ is $e^{-\sqrt{2\lambda}}$, which, by the calculation in part (**iii**) of Exercise 3.3.12 means that $c = 2^{\frac{1}{2}}$. Of course, this calculation makes the preceding parts of this exercise unnecessary. Nonetheless, it is interesting to see the Brownian explanation for the properties of the one-sided, $\frac{1}{2}$-stable laws.

Exercise 7.1.25. An important corollary of Theorem 7.1.16 is the following formula. Working in the setting of that theorem, show that, for any stopping time ζ and $t \in (0,\infty)$ and $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$,

$$\mathbb{Q}^\mu\big(\{\boldsymbol{\psi} :\, \boldsymbol{\psi}(t) \in \Gamma \;\&\; \zeta(\boldsymbol{\psi}) \le t\}\big) = \mathbb{E}^{\mathbb{Q}^\mu}\Big[\mu_{t-\zeta}\big(\Gamma - \boldsymbol{\psi}(\zeta)\big),\, \zeta \le t\Big],$$

where, as usual, μ_τ is determined by $\widehat{\mu_\tau} = e^{\tau \ell_\mu}$. As a consequence,

$$\mathbb{Q}^\mu\big(\{\boldsymbol{\psi} :\, \boldsymbol{\psi}(t) \in \Gamma \;\&\; \zeta(\boldsymbol{\psi}) > t\}\big) = \mu_t(\Gamma) - \mathbb{E}^{\mathbb{Q}^\mu}\Big[\mu_{t-\zeta}\big(\Gamma - \boldsymbol{\psi}(\zeta)\big),\, \zeta \le t\Big],$$

which is a quite general, generic statement of what is called a **Duhamel's formula**.

§ 7.2 Brownian Motion and Martingales

In this section we will see that continuous martingales and Brownian motion are intimately related concepts. In addition, we will find that martingale theory, and especially Doob's and Hunt's Stopping Time Theorems, provides a powerful tool with which to study Brownian paths.

§ 7.2.1. Lévy's Characterization of Brownian Motion. When applied to $\mu = \gamma_{0,\mathbf{I}}$, Theorem 7.1.6 says that a progressively measurable function $\mathbf{B} : [0,\infty)\times\Omega \longrightarrow \mathbb{R}^N$ with $\mathbf{B}(0,\omega) = \mathbf{0}$ and $\mathbf{B}(\,\cdot\,,\omega) \in D(\mathbb{R}^N)$ is a Brownian motion if and only if

$$\left(\varphi\big(\mathbf{B}(t)\big) - \int_0^t \tfrac{1}{2}\Delta\varphi\big(\mathbf{B}(\tau)\big)\,d\tau, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale for all $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N;\mathbb{R})$. In this subsection we, following Lévy,† will give another martingale characterization of Brownian motion, this time involving many fewer test functions. On the other hand, we will have to assume ahead of time that $\mathbf{B}(\,\cdot\,,\omega) \in C(\mathbb{R}^N)$ every $\omega \in \Omega$.

Theorem 7.2.1 (**Lévy**). *Let $\mathbf{B} : [0,\infty)\times\Omega \longrightarrow \mathbb{R}^N$ be a progressively measurable function satisfying $\mathbf{B}(0,\omega) = \mathbf{0}$ and $\mathbf{B}(\,\cdot\,,\omega) \in C(\mathbb{R}^N)$ for every $\omega \in \Omega$. Then $\big(\mathbf{B}(t), \mathcal{F}_t, \mathbb{P}\big)$ is a Brownian motion if and only if*

$$\left(\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N} + \big(\boldsymbol{\eta}, \mathbf{B}(t)\big)^2_{\mathbb{R}^N} - \frac{t|\boldsymbol{\eta}|^2}{2}, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

† Lévy's Theorem is Theorem 11.9 in Chapter VII of Doob's *Stochastic Processes*, publ. by J. Wiley (1953). Doob uses a clever but somewhat opaque Central Limit argument. The argument which given here is far simpler and is adapted from the one introduced by H. Kunita and S. Watanabe in their article "On square integrable martingales," *Nagoya Math. J.* **30** (1967).

PROOF: First suppose that $\big(\mathbf{B}(t),\mathcal{F}_t,\mathbb{P}\big)$ is a Brownian motion. Then, because $\mathbf{B}(t)-\mathbf{B}(s)$ is independent of $\mathcal{F}_s$ and has distribution $\gamma_{\mathbf{0},\mathbf{I}}$,

$$\mathbb{E}^{\mathbb{P}}\big[\mathbf{B}(t)-\mathbf{B}(s)\,\big|\,\mathcal{F}_s\big]=\mathbf{0}\quad\text{and}\quad\mathbb{E}^{\mathbb{P}}\big[\mathbf{B}(t)\otimes\mathbf{B}(t)-\mathbf{B}(s)\otimes\mathbf{B}(s)\,\big|\,\mathcal{F}_s\big]=(t-s)\mathbf{I}.$$

Hence, the necessity is obvious.

To prove the sufficiency, Theorem 7.1.3 says that it is enough to prove that

$$\begin{aligned}(*)\qquad &\mathbb{E}^{\mathbb{P}}\left[\exp\Big[\sqrt{-1}\,\big(\boldsymbol{\xi},\mathbf{B}(t)\big)_{\mathbb{R}^N}+\tfrac{t|\boldsymbol{\xi}|^2}{2}\Big],\,A\right]\\ &\qquad=\mathbb{E}^{\mathbb{P}}\left[\exp\Big[\sqrt{-1}\,\big(\boldsymbol{\xi},\mathbf{B}(s)\big)_{\mathbb{R}^N}+\tfrac{s|\boldsymbol{\xi}|^2}{2}\Big],\,A\right]\end{aligned}$$

for $0\le s<t$ and $A\in\mathcal{F}_s$. The challenge is to learn how to do this by taking full advantage of the assumed continuity. To this end, let $\epsilon\in(0,1]$ be given, set $\zeta_0\equiv s$, and use induction to define

$$\zeta_n=\Big(\inf\Big\{t\ge\zeta_{n-1}:\,\big|\mathbf{B}(t)-\mathbf{B}\big(\zeta_{n-1}\big)\big|\ge\epsilon\Big\}\Big)\wedge\big(\zeta_{n-1}+\epsilon\big)\wedge t$$

for $n\in\mathbb{Z}^+$. Proceeding by induction, one can easily check that $\{\zeta_n:\,n\ge 0\}$ is a non-decreasing sequence of $[s,t]$-valued stopping times. Hence, by Theorem 7.1.14 and our assumption,

$$(**)\qquad \mathbb{E}^{\mathbb{P}}\Big[\Delta_n\,\Big|\,\mathcal{F}_{\zeta_{n-1}}\Big]=0=\mathbb{E}^{\mathbb{P}}\Big[\Delta_n^2-\delta_n\,\Big|\,\mathcal{F}_{\zeta_{n-1}}\Big],$$

where

$$\begin{aligned}\Delta_n(\omega)&\equiv\Big(\boldsymbol{\xi},\mathbf{B}\big(\zeta_n(\omega),\omega\big)-\mathbf{B}\big(\zeta_{n-1}(\omega),\omega\big)\Big)_{\mathbb{R}^N}\\ \delta_n(\omega)&\equiv|\boldsymbol{\xi}|^2\big(\zeta_n(\omega)-\zeta_{n-1}(\omega)\big).\end{aligned}$$

Moreover, because $\mathbf{B}(\,\cdot\,,\omega)$ is continuous, we know that, for each $\omega\in\Omega$, $|\Delta_n(\omega)|\le\epsilon|\boldsymbol{\xi}|$, $\delta_n(\omega)\le\epsilon|\boldsymbol{\xi}|^2$, and $\zeta_n(\omega)=t$ for all but a finite number of n's. In particular, we can write the difference between the left and the right sides of (*) as the sum over $n\in\mathbb{Z}^+$ of $\mathbb{E}^{\mathbb{P}}\big[D_nM_n,\,A\big]$, where

$$\begin{aligned}D_n&\equiv\exp\Big[\sqrt{-1}\,\Delta_n+\tfrac{\delta_n}{2}\Big]-1\\ M_n&\equiv\exp\Big[\sqrt{-1}\,\big(\boldsymbol{\xi},\mathbf{B}(\zeta_{n-1})\big)_{\mathbb{R}^N}+\tfrac{|\boldsymbol{\xi}|^2}{2}\zeta_{n-1}\Big].\end{aligned}$$

By Taylor's Theorem,

$$\left|D_n-\Big(\sqrt{-1}\,\Delta_n+\tfrac{\delta_n}{2}\Big)-\tfrac{1}{2}\Big(\sqrt{-1}\,\Delta_n+\tfrac{\delta_n}{2}\Big)^2\right|\le\tfrac{1}{6}e^{\frac{|\boldsymbol{\xi}|^2}{2}}\left|\sqrt{-1}\,\Delta_n+\tfrac{\delta_n}{2}\right|^3.$$

Hence, after rearranging terms, we see that $D_n = \sqrt{-1}\,\Delta_n - \frac{1}{2}\big(\Delta_n^2 - \delta_n\big) + E_n$, where, by our estimates on Δ_n and δ_n,

$$|E_n| \le \tfrac{1}{2}|\Delta_n\delta_n| + \tfrac{\delta_n^2}{8} + \tfrac{2}{3}e^{\frac{|\boldsymbol{\xi}|^2}{2}}\Big(|\Delta_n|^3 + \tfrac{\delta_n^3}{8}\Big) \le \epsilon\big(1+|\boldsymbol{\xi}|^2\big)e^{\frac{|\boldsymbol{\xi}|^2}{2}}\big(\Delta_n^2 + \delta_n\big);$$

and so, after taking (**) into account,we arrive at

$$\begin{aligned}\left|\sum_1^\infty \mathbb{E}^{\mathbb{P}}\big[D_n M_m,\, A\big]\right| &= \left|\sum_1^\infty \mathbb{E}^{\mathbb{P}}\big[E_n M_n,\, A\big]\right| \\ &\le 2\epsilon\big(1+|\boldsymbol{\xi}|^2\big)e^{\frac{|\boldsymbol{\xi}|^2}{2}}\sum_1^\infty \mathbb{E}^{\mathbb{P}}\big[\delta_n|M_n|,\, A\big] \le 2\epsilon\big(1+|\boldsymbol{\xi}|^2\big)(t-s)e^{\frac{|\boldsymbol{\xi}|^2}{2}(1+t)}.\end{aligned}$$

In other words, we have now proved that, for every $\epsilon \in (0,1]$, the difference between the two sides of (*) is dominated by $2\epsilon(1+|\boldsymbol{\xi}|^2)(t-s)e^{\frac{|\boldsymbol{\xi}|^2}{2}(1+t)}$, and so the equality in (*) has been established. □

As in Theorem 7.1.19, the subtlety here is in the use of the continuity assumption. Indeed, the same example which demonstrates its importance there, does so again here. Namely, if $\{N(t) : t \ge 0\}$ is a simple Poisson process and $X(t) = N(t) - t$, then both $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ and $\big(X(t)^2 - t, \mathcal{F}_t, \mathbb{P}\big)$ are martingales, but $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is certainly not a Brownian motion.

§ **7.2.2. Doob–Meyer Decomposition, an Easy Case.** The continuous parameter analog of Lemma 5.2.12 is a highly non-trivial result, one which was proved by P.A. Meyer and led him to his profound analysis of stochastic processes. Nonetheless, there is an important case in which Meyer's result is relatively easy to prove, and that is the case proved in this subsection. However, before getting to that result, there is a rather fussy matter to be delt with.

LEMMA 7.2.2. *For each $n \in \mathbb{N}$, let $X_n : [0,\infty) \longrightarrow \mathbb{R}$ be a right continuous, progressively measurable function with the property that $X_n(\,\cdot\,,\omega)$ is continuous for $\mathbb{P}$-almost every $\omega \in \Omega$. If*

$$\lim_{m\to\infty}\sup_{n>m}\|X_n(\,\cdot\,,\omega) - X_m(\,\cdot\,,\omega)\|_{[0,t]} = 0 \text{ (a.s., } \mathbb{P}) \textit{ for each } t \in (0,\infty),$$

then there is a right continuous, progressively measurable $X : [0,\infty) \longrightarrow \mathbb{R}$ such that $X(\,\cdot\,,\omega)$ is continuous and $X_n(\,\cdot\,,\omega) \longrightarrow X(\,\cdot\,,\omega)$ uniformly on compacts for $\mathbb{P}$-almost every $\omega \in \Omega$.

PROOF: Set $A = \{(t,\omega) : \lim_{m\to\infty}\sup_{n>m}\|X_n(\,\cdot\,,\omega) - X_m(\,\cdot\,,\omega)\|_{[0,t]} = 0\}$. Then A is progressively measurable. Next, define $\zeta(\omega) = \sup\{t \ge 0 : (t,\omega) \in A\}$, and note that $\{\zeta < t\} \in \mathcal{F}_t$ for each $t \in (0,\infty)$. Finally, set $B = \{(t,\omega) : \zeta(\omega) \ge t\}$. Then, B is again progressively measurable. To see this, first note that

$$\{(\tau,\omega) \in [0,t]\times\Omega : \tau\wedge\zeta(\omega) < s\} = \begin{cases} \Omega \in \mathcal{F}_t & \text{if } t \le s \\ \{\zeta < s\} \in \mathcal{F}_t & \text{if } t > s, \end{cases}$$

and so $(\tau,\omega) \rightsquigarrow \tau\wedge\zeta(\omega)$, and therefore also $(\tau,\omega) \rightsquigarrow \tau\wedge\zeta(\omega)-\tau$, is a progressively measurable function. Hence, since $B = \{(\tau,\omega) : \tau \wedge \zeta(\omega) - \tau \geq 0\}$, B is progressively measurable.

Now define

$$X(t,\omega) = \begin{cases} \lim_{n\to\infty} X_n(t,\omega) & \text{if } (t,\omega) \in A \\ 0 & \text{if } (t,\omega) \in B \setminus A \\ X\big(\zeta(\omega),\omega\big) & \text{if } (t,\omega) \notin B. \end{cases}$$

Clearly $X(\,\cdot\,,\omega)$ is right continuous. Moreover, because $\zeta = \infty$ (a.s., $\mathbb{P}$), $X(\,\cdot\,,\omega)$ is continuous and $X_n(\,\cdot\,,\omega) \longrightarrow X(\,\cdot\,,\omega)$ uniformly on compacts for $\mathbb{P}$-almost every $\omega \in \Omega$. Thus, it only remains to check that X is progressively measurable. For this purpose, let $\Gamma \in \mathcal{B}_{\mathbb{R}}$ be given, and set $C = \{(t,\omega) : X(t,\omega) \in \Gamma\}$. Because A and the X_n's are progressively measurable, it is clear that $C \cap A$ is progressively measurable. Similarly, because $B \setminus A$ is progressively measurable and $C \cap (B \setminus A)$ equals $B \setminus A$ or $\emptyset$ depending on whether $0 \in \Gamma$ or $0 \notin \Gamma$, $C \cap (B \setminus A)$, and therefore $C \cap B$, are progressively measurable. Hence, we now know that $X \restriction B$ is progressively measurable. Finally, we showed earlier that $(t,\omega) \rightsquigarrow t \wedge \zeta(\omega)$ is a progressively measurable, and therefore so is $(t,\omega) \in [0,\infty) \times \Omega \longmapsto \big(t \wedge \zeta(\omega),\omega\big) \in B$. Thus, because $X(t,\omega) = X\big(t\wedge\zeta(\omega),\omega\big)$, we are done. $\square$

Theorem 7.2.3. *Let $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ be an $\mathbb{R}$-valued, square integrable martingale with the property that $X(\,\cdot\,,\omega)$ is continuous for $\mathbb{P}$-almost every $\omega \in \Omega$. Then there is a $\mathbb{P}$-almost surely unique progressively measurable function $\langle X\rangle : [0,\infty)\times\Omega \longrightarrow [0,\infty)$ such that $\langle X\rangle(0,\omega) = 0$ and $\langle X\rangle(\,\cdot\,,\omega)$ is continuous and non-decreasing for $\mathbb{P}$-almost every $\omega \in \Omega$, and $\big(X(t)^2 - \langle X\rangle(t),\mathcal{F}_t,\mathbb{P}\big)$ is a martingale.*

Proof: The uniqueness is an immediate consequence of Corollary 7.1.20.

The proof of existence, which is based on a suggestion I got from K. Itô, is very much like that of Theorem 7.2.1. Without loss in generality, I will assume that $X(0) \equiv 0$.

I begin by reducing to the case when X is $\mathbb{P}$-almost surely bounded. To this end, suppose that we know the result in this case. Given a general X and $n \in \mathbb{N}$, define $\zeta_n = \inf\{t \geq 0 : |X(t)| \geq n\}$, and $X_n(t) = X(t\wedge\zeta_n)$. Then, $|X_n(\,\cdot\,,\omega)| \leq n$ and, by Doob's Inequaltiy, $\zeta_n(\omega) \nearrow \infty$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Moreover, by Corollary 7.1.15, $\big(X_n(t),\mathcal{F}_t,\mathbb{P}\big)$ is a martingale. Thus, by our assumption, for each n, we know $\langle X_n\rangle$ exists. In addition, by Corollary 7.1.15 and uniqueness, we know (cf. Exercise 7.2.10) that, $\mathbb{P}$-almost surely, $\langle X_m\rangle(t) = \langle X_n\rangle(t\wedge\zeta_m)$ for all $m \leq n$ and $t \geq 0$. Now define $\langle X\rangle$ so that $\langle X\rangle(t) = \langle X_n\rangle(t)$ for $\zeta_n \leq t < \zeta_{n+1}$. Then $\langle X\rangle$ is progressively measurable and right continuous, $\langle X\rangle(0) = 0$, and, $\mathbb{P}$-almost surely, $\langle X\rangle$ is continuous and non-decreasing. Furthermore, $\big(X(t\wedge\zeta_n)^2 - \langle X\rangle(t\wedge\zeta_n),\mathcal{F}_t,\mathbb{P}\big)$ is a martingale for each $n \in \mathbb{N}$. Finally, note that, by Doob's Inequality,

$$\mathbb{E}^{\mathbb{P}}\big[\|\langle X\rangle\|_{[0,t]}\big] \le \mathbb{E}^{\mathbb{P}}\big[\|X\|_{[0,t]}^2\big] \le 4\mathbb{E}^{\mathbb{P}}\big[|X(t)|^2\big],$$

and so, as $n \to \infty$, $X(t \wedge \zeta_n)^2 - \langle X\rangle(t \wedge \zeta_n) \longrightarrow X(t)^2 - \langle X\rangle(t)$ in $L^1(\mathbb{P};\mathbb{R})$. Hence, $\big(X(t)^2 - \langle X\rangle(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale.

I now assume that $|X(\,\cdot\,,\omega)| \le C < \infty$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Next, for each $n \in \mathbb{N}$, use induction to define $\{\zeta_{k,n} : k \ge 0\}$ so that $\zeta_{0,n} = 0$, $\zeta_{k,0} = k$, and, for $(k,n) \in (\mathbb{Z}^+)^2$, $\zeta_{k,n}$ is equal to

$$\Big(\min\{\zeta_{\ell,n-1} : \zeta_{\ell,n-1} > \zeta_{k-1,n}\}\Big) \wedge \Big(\inf\big\{t \ge \zeta_{k-1,n} : |X(t) - X(\zeta_{k-1,n})| \ge \tfrac{1}{n}\big\}\Big).$$

Working by induction, one sees that, for each $n \in \mathbb{N}$, $\{\zeta_{k,n} : k \ge 0\}$ is a non-decreasing sequence of bounded stopping times. Moreover, because $X(\,\cdot\,,\omega)$ is $\mathbb{P}$-almost surely continuous, we know that $\lim_{k\to\infty} \zeta_{k,n}(\omega) = \infty$ for each $n \in \mathbb{N}$ and $\mathbb{P}$-almost every $\omega \in \Omega$. Finally, the sequences $\{\zeta_{k,n} : k \ge 0\}$ are nested in the sense that $\{\zeta_{k,n-1} : k \ge 0\} \subseteq \{\zeta_{k,n} : k \ge 0\}$ for each $n \in \mathbb{Z}^+$.

Set $X_{k,n} = X\big(\zeta_{k,n}\big)$ and, for $k \ge 1$, $\Delta_{k,n}(t) = X\big(t \wedge \zeta_{k,n}\big) - X\big(t \wedge \zeta_{k-1,n}\big)$. Then $X(t)^2 = 2M_n(t) + \langle X\rangle_n(t)$, where

$$M_n(t) = \sum_{k=1}^{\infty} X_{k-1,n}\Delta_{k,n}(t) \quad \text{and} \quad \langle X\rangle_n(t) = \sum_{k=1}^{\infty} \Delta_{k,n}(t)^2.$$

Of course, for $\mathbb{P}$-almost every $\omega \in \Omega$, all but a finite number of terms in each of these sums vanish. In fact, $M_0(t) = \sum_{1\le k\le [t]} X_{k-1,0}\Delta_{k,0}(t)$ for each $t \ge 0$, and it is an easy matter to check that $\big(M_0(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathbb{P}$-almost surely continuous martingale. At the same time, one should observe that $\langle X\rangle_n(s) \le \langle X\rangle_n(t)$ if $s \ge 0$ and $t - s > \frac{1}{n}$.

I now want to show that $\big(M_n(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathbb{P}$-almost surely continuous martingale for all $n \in \mathbb{N}$. To this end, first observe that, for each $(k,n) \in \mathbb{Z}^+ \times \mathbb{N}$, $\big(X_{k-1,n}\Delta_{k,n}(t), \mathcal{F}_t, \mathbb{P}\big)$ is a continuous martingale. Indeed, if $0 \le s < t$ and $A \in \mathcal{F}_s$, then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(t),\, A\big] = \mathbb{E}^{\mathbb{P}}&\big[X_{k-1,n}\Delta_{k,n}(t),\, A \cap \{\zeta_{k-1,n} \le s\}\big] \\ &+ \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(t),\, A \cap \{\zeta_{k-1,n} > s\}\big].\end{aligned}$$

Next, check that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(t),\, A \cap \{\zeta_{k-1,n} \le s\}\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\big(X(\zeta_{k,n}) - X(\zeta_{k-1,n})\big),\, A \cap \{\zeta_{k,n} \le s\}\big] \\ &\qquad + \mathbb{E}^{\mathbb{P}}\Big[X_{k-1,n}\Big(X\big((t \wedge \zeta_{k,n}) \vee s\big) - X(\zeta_{k-1,n})\big)\Big),\, A \cap \{\zeta_{k-1,n} \le s < \zeta_{k,n}\}\Big] \\ &= \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(s),\, A \cap \{\zeta_{k,n} \le s\}\big] \\ &\qquad + \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\big(X(s) - X(\zeta_{k-1,n})\big),\, A \cap \{\zeta_{k-1,n} \le s < \zeta_{k,n}\}\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(s),\, A \cap \{\zeta_{k-1,n} \le s\}\big],\end{aligned}$$

where, in the passage to the second to last equality, I have used the fact that $X_{k-1,n}\mathbf{1}_A\mathbf{1}_{[\zeta_{k-1,n},\zeta_{k,n})}(s)$ is $\mathcal{F}_s$-measurable and applied Theorem 7.1.14; and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(t),\, A\cap\{\zeta_{k-1,n}>s\}\big]\\
&= \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\big(X(t\wedge\zeta_{k,n})-X(t\wedge\zeta_{k-1,n})\big),\, A\cap\{s<\zeta_{k-1,n}\le t\}\big]\\
&= \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\big(X(t)-X(t)\big),\, A\cap\{s<\zeta_{k-1,n}\le t\}\big]\\
&= 0 = \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(s),\, A\cap\{\zeta_{k-1,n}>s\}\big],
\end{aligned}$$

where I have used the fact that $X_{k-1,n}\mathbf{1}_A\mathbf{1}_{(s,t]}(\zeta_{k-1,n})$ is $\mathcal{F}_{t\wedge\zeta_{k-1,n}}$-measurable and again applied Theorem 7.1.14 in getting the second to last line. After combining these, one sees that $\mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(t),\, A\big] = \mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}\Delta_{k,n}(s),\, A\big]$, which means that $\big(X_{k-1,n}\Delta_{k,n}(t),\mathcal{F}_t,\mathbb{P}\big)$ is a continuous martingale as was claimed.

Given the preceding, it is clear that, for each n and ℓ, $\big(M_n(t\wedge\zeta_{\ell,n}),\mathcal{F}_t,\mathbb{P}\big)$ is a $\mathbb{P}$-almost surely continuous, square integrable martingale. In addition, for $k\neq k'$, $X_{k-1}\Delta_{k,n}(t\wedge\zeta_{\ell,n})$ is orthogonal to $X_{k'-1}\Delta_{k',n}(t\wedge\zeta_{\ell,n})$ in $L^2(\mathbb{P};\mathbb{R})$. Thus

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le\tau\le t\wedge\zeta_{\ell,n}} M_n(\tau)^2\right] \le 4\mathbb{E}^{\mathbb{P}}\big[M_n(t\wedge\zeta_{\ell,n})^2\big]\\
&\quad = 4\sum_{k=1}^{\ell}\mathbb{E}^{\mathbb{P}}\big[X_{k-1,n}^2\Delta_{k,n}(t\wedge\zeta_{\ell,n})^2\big] \le 4C^2\sum_{k=1}^{\ell}\mathbb{E}^{\mathbb{P}}\big[\Delta_{k,n}(t\wedge\zeta_{\ell,n})^2\big]\\
&\quad = 4C^2\mathbb{E}^{\mathbb{P}}\big[X(t\wedge\zeta_{\ell,n})^2\big] \le 4C^2\mathbb{E}^{\mathbb{P}}\big[X(t)^2\big],
\end{aligned}$$

from which it is easy to see that $\big(M_n(t),\mathcal{F}_t,\mathbb{P}\big)$ is a square integrable martingale.

I will now show that $\lim_{m\to\infty}\sup_{n>m}\|M_n-M_m\|_{[0,t]}=0$ $\mathbb{P}$-almost surely and in $L^2(\mathbb{P};\mathbb{R})$ for each $t\in[0,\infty)$. To this end, define $Y^{(m)}_{k-1,n}$ so that $Y^{(m)}_{k-1,n}(\omega) = X_{k-1,n}(\omega)-X_{\ell-1,m}(\omega)$ when $\zeta_{\ell-1,m}(\omega)\le\zeta_{k-1,n}(\omega)<\zeta_{\ell,m}(\omega)$. Then $Y^{(m)}_{k-1,n}$ is $\mathcal{F}_{k-1,n}$-measurable, $|Y^{(m)}_{k-1,n}|\le\frac{1}{m}$, and $M_n-M_m=\sum_{k=1}^{\infty}Y^{(m)}_{k-1,n}\Delta_{k,n}$. Hence, by the same reasoning as above,

$$\mathbb{E}^{\mathbb{P}}\big[\|M_n-M_m\|^2_{[0,t]}\big] \le 4\sum_{k=1}^{\infty}\mathbb{E}^{\mathbb{P}}\big[(Y^{(m)}_{k-1,n})^2\Delta_{k,n}(t)^2\big] \le \frac{4}{m^2}\mathbb{E}^{\mathbb{P}}\big[X(t)^2\big],$$

which is more than enough to get the asserted convergence result.

Given the preceding, we can now apply Lemma 7.2.2 to produce a right continuous, progressively measure, $\mathbb{P}$-almost surely continuous $M:[0,\infty)\times\Omega\longrightarrow\mathbb{R}$ to which $\{M_n : n\ge 1\}$ converges uniformly on compacts, both $\mathbb{P}$-almost surely and in $L^2(\mathbb{P};\mathbb{R})$. In particular, $\big(M(t),\mathcal{F}_t,\mathbb{P}\big)$ is a square integrable martingale.

Finally, set $\langle X\rangle = (X^2 - 2M)^+$. Obviously, $\langle X\rangle = X^2 - 2M$ (a.s., $\mathbb{P}$), and $\langle X\rangle$ is right continuous, progressively measurable, and $\mathbb{P}$-almost surely continuous. In addition, because, $\mathbb{P}$-almost surely, $\langle X\rangle_n \longrightarrow \langle X\rangle$ uniformly on compacts and $\langle X\rangle_n(s) \le \langle X\rangle_n(t)$ when $t - s > \frac{1}{n}$, it follows that $\langle X\rangle(\,\cdot\,,\omega)$ is non-decreasing for $\mathbb{P}$-almost every $\omega \in \Omega$. $\square$

Remark 7.2.4. The reader may be wondering why I insisted on complicating the preceding statement and proof by insisting that $\langle X\rangle$ be progressively measurable with respect to the original family of σ-algebras $\{\mathcal{F}_t : t \in [0,\infty)\}$. Indeed, Exercise 7.1.22 shows that I could have replaced all the σ-algebras with their completions, and, if I had done so, there would have been no reason not to have taken $X(\,\cdot\,,\omega)$ to be continuous and $\langle X\rangle(\,\cdot\,,\omega)$ to continuous and non-decreasing for every $\omega \in \Omega$. However, there is a price to be paid for completing σ-algebras. In the first place, when one does, all statements become dependent on the particular $\mathbb{P}$ with which one is dealing. Secondly, because completed σ-algebras are nearly never countably generated, certain desirable properties can be lost by introducing them. See, for example, Theorem 9.2.1.

By combining Theorem 7.2.3 with Theorem 7.2.1, one can show that, up to time re-parametrization, all continuous martingales are Brownian motions. In order to avoid technical difficulties, I will prove this only in the easiest case.

Corollary 7.2.5. *Let $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ be a continuous, square integrable martingale with the properties that, for $\mathbb{P}$-almost every $\omega \in \Omega$, $\langle X\rangle(\,\cdot\,,\omega)$ is strictly increasing and $\lim_{t\to\infty}\langle X\rangle(t,\omega) = \infty$. Then there exists a Brownian motion $\big(B(t),\mathcal{F}'_t,\mathbb{P}\big)$ such that $X(t) = X(0) + B\big(\langle X\rangle(t)\big)$, $t \in [0,\infty)$ $\mathbb{P}$-almost surely. In particular,*

$$\varlimsup_{t\to\infty} \frac{X(t)}{\sqrt{2\langle X\rangle(t)\log_{(2)}\langle X\rangle(t)}} = 1 = -\varliminf_{t\to\infty} \frac{X(t)}{\sqrt{2\langle X\rangle(t)\log_{(2)}\langle X\rangle(t)}}$$

$\mathbb{P}$-almost surely.

Proof: Clearly, given the first part, the last assertion is a trivial application of Exercise 4.3.15.

After replacing $\mathcal{F}$ and the $\mathcal{F}_t$'s by their completions and applying Exercise 7.1.22, I may and will assume that $X(0,\omega) = 0$, $X(\,\cdot\,,\omega)$ is continuous, $\langle X\rangle(\,\cdot\,,\omega)$ is continuous and strictly increasing, and $\lim_{t\to\infty}\langle X\rangle(t,\omega) = \infty$ for every $\omega \in \Omega$. Next, for each $(t,\omega) \in [0,\infty)$, set $\zeta_t(\omega) = \langle X\rangle^{-1}(t,\omega)$, where $\langle X\rangle^{-1}(\,\cdot\,,\omega)$ is the inverse of $\langle X\rangle(\,\cdot\,,\omega)$. Clearly, for each $\omega \in \Omega$, $t \rightsquigarrow \zeta_t(\omega)$ is a continuous, strictly increasing function which tends to infinity as $t \to \infty$. Moreover, because $\langle X\rangle$ is progressively measurable, ζ_t is a stopping time for each $t \in [0,\infty)$. Now set $B(t) = X(\zeta_t)$. Since it is obvious that $X(t) = B\big(\langle X\rangle(t)\big)$, all that we have to show is that $\big(B(t),\mathcal{F}'_t,\mathbb{P}\big)$ is a Brownian motion for some non-decreasing family $\{\mathcal{F}'_t : t \ge 0\}$ of sub σ-algebras.

Trivially $B(0,\omega) = 0$ and $B(\cdot\,,\omega)$ is continuous for all $\omega \in \Omega$. In addition, $B(t)$ is $\mathcal{F}_{\zeta_t}$-measurable, and so B is progressively measurable with respect to $\{\mathcal{F}_{\zeta_t} : t \geq 0\}$. Thus, by Theorem 7.2.1, we will be done once we show that $\big(B(t), \mathcal{F}_{\zeta_t}, \mathbb{P}\big)$ and $\big(B(t)^2 - t, \mathcal{F}_{\zeta_t}, \mathbb{P}\big)$ are martingales. To this end, first observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\sup_{\tau\in[0,\zeta_t]} X(\tau)^2\right] &= \lim_{T\to\infty}\mathbb{E}^{\mathbb{P}}\left[\sup_{\tau\in[0,T\wedge\zeta_t]} X(\tau)^2\right]\\ &\leq 4\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}}\big[X(T\wedge\zeta_t)^2\big] \leq 4\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(T\wedge\zeta_t)\big] \leq 4t.\end{aligned}$$

Thus, $\lim_{T\to\infty} X(T\wedge\zeta_t) \longrightarrow B(t)$ in $L^2(\mathbb{P};\mathbb{R})$. Now let $0 \leq s < t$ and $A \in \mathcal{F}_{\zeta_s}$ be given. Then, for each $T > 0$, $A_T \equiv A \cap \{\zeta_s \leq T\} \in \mathcal{F}_{T\wedge\zeta_s}$, and so, by Theorem 7.1.14,

$$\mathbb{E}^{\mathbb{P}}\big[X(T\wedge\zeta_t),\, A_T\big] = \mathbb{E}^{\mathbb{P}}\big[X(T\wedge\zeta_s),\, A_T\big]$$

and

$$\mathbb{E}^{\mathbb{P}}\big[X(T\wedge\zeta_t)^2 - \langle X\rangle(T\wedge\zeta_t),\, A_T\big] = \mathbb{E}^{\mathbb{P}}\big[X(T\wedge\zeta_s)^2 - \langle X\rangle(T\wedge\zeta_s),\, A_T\big].$$

Now let $T \to \infty$, and apply the preceding convergence assertion to get the desired conclusion. □

§ **7.2.3. Burkholder's Inequality Again.** In this subsection we will see what Burkholder's Inequality looks like in the continuous parameter setting, a result whose importance for the theory of stochastic integration is hard to overstate.

Theorem 7.2.6 (**Burkholder**). *Let $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ be a $\mathbb{P}$-almost surely continuous, square integrable martingale. Then, for each $p \in (1,\infty)$ and $t \in [0,\infty)$ (cf. (6.3.2))*

$$B_p^{-1}\|X(t) - X(0)\|_{L^p(\mathbb{P};\mathbb{R})} \leq \mathbb{E}^{\mathbb{P}}\big[\langle X(t)\rangle^{\frac{p}{2}}\big]^{\frac{1}{p}} \leq B_p\|X(t) - X(0)\|_{L^p(\mathbb{P};\mathbb{R})}. \tag{7.2.7}$$

Proof: After completing the σ-algebras if necessary, I may (cf. Exercise 7.1.22) and will assume that $X(\cdot\,,\omega)$ is continuous and that $\langle X\rangle(\cdot\,,\omega)$ is continuous and non-decreasing for every $\omega \in \Omega$. In addition, I may and will assume $X(0) = 0$. Finally, I will assume that X is bounded. To justify this last assumption, let $\zeta_n = \inf\{t \geq 0 : |X(t)| \geq n\}$, set $X_n(t) = X(t\wedge\zeta_n)$, and use Exercise 7.2.10 to see that one can take $\langle X_n\rangle = \langle X\rangle(t\wedge\zeta_n)$. Hence, if we know (7.2.7) for bounded martingales, then

$$B_p^{-1}\|X(t\wedge\zeta_n)\|_{L^p(\mathbb{P};\mathbb{R})} \leq \mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t\wedge\zeta_n)^{\frac{p}{2}}\big]^{\frac{1}{p}} \leq B_p\|X(t\wedge\zeta_n)\|_{L^p(\mathbb{P};\mathbb{R})}$$

for all $n \geq 1$. Since $\langle X \rangle$ is non-decreasing, we can apply Fatou's Lemma to the preceding and thereby get

$$\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})} \leq \varliminf_{n\to\infty} \|X(t\wedge\zeta_n)\|_{L^p(\mathbb{P};\mathbb{R})} \leq B_p \mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)^{\frac{p}{2}}\big]^{\frac{1}{p}},$$

which is the left hand side of (7.2.7). To get the right hand side, note that either $\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})} = \infty$, in which case there is nothing to do, or $\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})} < \infty$, in which case, by the second half of Theorem 7.1.9, $X(t\wedge\zeta_n) \longrightarrow X(t)$ in $L^p(\mathbb{P};\mathbb{R})$ and therefore

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)^{\frac{p}{2}}\big]^{\frac{1}{p}} &= \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t\wedge\zeta_n)^{\frac{p}{2}}\big]^{\frac{1}{p}} \\ &\leq B_p \lim_{n\to\infty} \|X(t\wedge\zeta_n)\|_{L^p(\mathbb{P};\mathbb{R})} = B_p\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})}.\end{aligned}$$

Proceeding under the above assumptions and referring to the notation in the proof of Theorem 7.2.3, begin by observing that, for any $t \in [0,\infty)$ and $n \in \mathbb{N}$, Theorem 7.1.14 shows that $\big(X(t\wedge\zeta_{k,n}), \mathcal{F}_{t\wedge\zeta_{k,n}}, \mathbb{P}\big)$ is a discrete parameter martingale indexed by $k \in \mathbb{N}$. In addition, $\zeta_{k,n} = t$ for all but a finite number of k's. Hence, by (6.3.7) applied to $\big(X(t\wedge\zeta_{k,n}), \mathcal{F}_{t\wedge\zeta_{k,n}}, \mathbb{P}\big)$,

$$B_p^{-1}\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})} \leq \mathbb{E}^{\mathbb{P}}\big[\langle X\rangle_n(t)^{\frac{p}{2}}\big]^{\frac{1}{p}} \leq B_p\|X(t)\|_{L^p(\mathbb{P};\mathbb{R})} \quad \text{for all } n \in \mathbb{N}.$$

In particular, this shows that $\sup_{n\geq 0} \|\langle X\rangle_n(t)\|_{L^p(\mathbb{P};\mathbb{R})} < \infty$ for every $p \in (1,\infty)$, and therefore, since $\langle X\rangle_n(t) \longrightarrow \langle X\rangle(t)$ (a.s., $\mathbb{P}$), this is more than enough to verify that $\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle_n(t)^{\frac{p}{2}}\big] \longrightarrow \mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)^{\frac{p}{2}}\big]$ for every $p \in (1,\infty)$. □

Exercises for § 7.2

Exercise 7.2.8. Let $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ be a square integrable, continuous martingale. Following the strategy used to prove Theorem 7.2.1, show that

$$\left(F\big(X(t)\big) - \int_0^t \tfrac{1}{2}\partial_x^2 F\big(X(\tau)\big)\langle X\rangle(d\tau), \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale for every $F \in C^2_{\mathrm{b}}(\mathbb{R};\mathbb{C})$.

Hint: Begin by using cut offs and mollification to reduce to the case when $F \in C^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{R})$. Next, given $s < t$ and $\epsilon > 0$, introduce the stopping times $\zeta_0 = s$ and

$$\zeta_n = \inf\{t \geq \zeta_{n-1} : |X(t) - X(\zeta_{n-1})| \geq \epsilon\} \wedge (\zeta_{n-1}+\epsilon) \wedge (\langle X\rangle(\zeta_{n-1}) + \epsilon) \wedge t$$

for $n \geq 1$. Now proceed as in the proof of Theorem 7.2.1.

EXERCISE 7.2.9. Let $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ be a continuous, square integrable martingale with $X(0) = 0$, and assume that there exists a non-decreasing function $A : [0, \infty) \longrightarrow [0, \infty)$ such that $\langle X \rangle(t) \le A(t)$ (a.s., $\mathbb{P}$) for each $t \in [0, \infty)$. The goal of this exercise is to show that $\big(E(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale when

$$E(t) = \exp\big[X(t) - \tfrac{1}{2}\langle X \rangle(t)\big].$$

(**i**) Given $R \in (0, \infty)$, set $\zeta_R = \inf\{t \ge 0 : |X(t)| \ge R\}$, and show that

$$\left(e^{X(t\wedge\zeta_R)} - \frac{1}{2}\int_0^{t\wedge\zeta_R} e^{X(\tau)}\, d\langle X \rangle, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale.

Hint: Choose $F \in C_c^\infty(\mathbb{R};\mathbb{R})$ so that $F(x) = e^x$ for $x \in [-2R, 2R]$, apply Exercise 7.2.8 to this F, and then use Doob's Stopping Time Theorem.

(**ii**) Apply Theorem 7.1.17 to the martingale in (**i**) and $e^{-\frac{1}{2}\langle X \rangle(t\wedge\zeta_R)}$ to show that $\big(E(t\wedge\zeta_R), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale.

(**iii**) By replacing X and R with $2X$ and $2R$ in (**ii**), show that

$$\mathbb{E}^{\mathbb{P}}\big[E(t\wedge\zeta_R)^2\big] \le e^{A(t)}\mathbb{E}^{\mathbb{P}}\big[e^{2X(t\wedge\zeta_R) - 2\langle X \rangle(t\wedge\zeta_R)}\big] = e^{A(t)}.$$

Conclude that $\{E(t\wedge\zeta_R) : R \in (0,\infty)\}$ is uniformly $\mathbb{P}$-integrable and therefore that $\big(E(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale.

EXERCISE 7.2.10. If $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is a $\mathbb{P}$-almost surely continuous, square integrable martingale, ζ is a stopping time, and $Y(t) = X(t\wedge\zeta)$, show that $\langle Y \rangle(t) = \langle X \rangle(t\wedge\zeta)$, $t \ge 0$, $\mathbb{P}$-almost surely.

EXERCISE 7.2.11. Continuing in the setting of Exercise 7.2.9, first show that, for every $\lambda \in \mathbb{R}$, $\big(E_\lambda(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale where

$$E_\lambda(t) = \exp\big[\lambda X(t) - \tfrac{\lambda^2}{2}\langle X \rangle(t)\big].$$

Next, use Doob's Inequality to see that, for each $\lambda \ge 0$,

$$\mathbb{P}\left(\sup_{\tau\in[0,t]} X(\tau) \ge R\right) \le \mathbb{P}\left(\sup_{\tau\in[0,t]} E_\lambda(\tau) \ge e^{\lambda R - \frac{\lambda^2}{2}A(t)}\right) \le e^{-\lambda R + \frac{\lambda^2}{2}A(t)}.$$

Starting from this, conclude that

$$\mathbb{P}\big(\|X\|_{[0,t]} \ge R\big) \le 2e^{-\frac{R^2}{2A(t)}}. \tag{7.2.12}$$

Finally, given this estimate, show that the conclusion in Exercise 7.2.8 continues to hold for any $F \in C^2(\mathbb{R};\mathbb{C})$ whose second derivative has at most exponential growth.

EXERCISE 7.2.13. Given a pair of square integrable, continuous martingales $\big(X(t),\mathcal{F}_t,\mathbb{P}\big)$ and $\big(Y(t),\mathcal{F}_t,\mathbb{P}\big)$, set $\langle X,Y\rangle = \frac{\langle X+Y\rangle-\langle X-Y\rangle}{4}$, and show that $\big(X(t)Y(t)-\langle X,Y\rangle(t),\mathcal{F}_t,\mathbb{P}\big)$ is a martingale. Further, show that $\langle X,Y\rangle$ is uniquely determined up the a $\mathbb{P}$-null set by this property together with the facts that $\langle X,Y\rangle(0,\omega)=0$ and $\langle X,Y\rangle(\,\cdot\,,\omega)$ is continuous and has locally bounded variation for $\mathbb{P}$-almost every $\omega\in\Omega$.

EXERCISE 7.2.14. Let $\big(\mathbf{B}(t),\mathcal{F}_t,\mathbb{P}\big)$ be an $\mathbb{R}^N$-valued Brownian motion. Given $f,\,g\in C^{1,2}_{\mathrm{b}}\big([0,\infty)\times\mathbb{R}^N;\mathbb{R}\big)$, set

$$X(t)=f\big(t,\mathbf{B}(t)\big)-\int_0^t\big(\partial_\tau+\tfrac12\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau,$$

$$Y(t)=g\big(t,\mathbf{B}(t)\big)-\int_0^t\big(\partial_\tau+\tfrac12\Delta\big)g\big(\tau,\mathbf{B}(\tau)\big)\,d\tau,$$

and show that

$$\langle X,Y\rangle(t)=\int_0^t\nabla f\cdot\nabla g\big(\tau,\mathbf{B}(\tau)\big)\,d\tau.$$

Hint: First reduce to the case when $f=g$. Second, write $X(t)^2$ as

$$\begin{aligned}f\big(t,\mathbf{B}(t)\big)^2-2X(t)&\int_0^t\big(\partial_\tau+\tfrac12\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau\\&-\left(\int_0^t\big(\partial_\tau+\tfrac12\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau\right)^2,\end{aligned}$$

and apply Theorem 7.1.17 to the second term.

§ 7.3 The Reflection Principle Revisited

In Exercise 4.3.12 we saw that Lévy's Reflection Principle (Theorem 1.4.13) has a sharpened version when applied to Brownian motion. In this section I will give another, more powerful, way of discussing the reflection principle for Brownian motion.

§ 7.3.1. Reflecting Symmetric Lévy Processes. In this subsection, μ will be used to denote a symmetric, infinitely divisible law. Equivalently (cf. Exercise 3.3.11), $\hat\mu=e^{\ell_\mu(\boldsymbol{\xi})}$, where

$$\ell_\mu(\boldsymbol{\xi})=-\frac12\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N}+\int_{\mathbb{R}^N}\Big(\cos\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}-1\Big)\,M(d\mathbf{y})$$

for some non-negative definite, symmetric $\mathbf{C}$ and symmetric Lévy measure M.

LEMMA 7.3.1. *Let $\{\mathbf{Z}(t) : t \geq 0\}$ be a Lévy process for μ, and set $\mathcal{F}_t = \sigma\big(\{\mathbf{Z}(\tau) : \tau \in [0,t]\}\big)$. If ζ is a stopping time relative to $\big\{\mathcal{F}_t : t \in [0,\infty)\big\}$ and*

$$\tilde{\mathbf{Z}}(t) \equiv 2\mathbf{Z}(t\wedge\zeta) - \mathbf{Z}(t) = \begin{cases} \mathbf{Z}(t) & \text{if } \zeta > t \\ 2\mathbf{Z}(\zeta) - \mathbf{Z}(t) & \text{if } \zeta \leq t, \end{cases}$$

then $\{\tilde{\mathbf{Z}(t)} : t \geq 0\}$ is again a Lévy process for μ.

PROOF: According to Theorem 7.1.3, all that we have to show is that

$$\Big(\exp\big[\sqrt{-1}\,(\boldsymbol{\xi}, \tilde{\mathbf{Z}}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big], \mathcal{F}_t, \mathbb{P}\Big)$$

is a martingale for all $\boldsymbol{\xi} \in \mathbb{R}^N$. Thus, let $0 \leq s < t$ and $A \in \mathcal{F}_s$ be given. Then, by Theorem 7.1.14 and the fact that $\ell_\mu(-\boldsymbol{\xi}) = \ell_\mu(\boldsymbol{\xi})$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\,(\boldsymbol{\xi}, \tilde{\mathbf{Z}}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta \leq s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[e^{2\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}(s\wedge\zeta))_{\mathbb{R}^N}} \exp\big[-\sqrt{-1}\,(\boldsymbol{\xi}, \mathbf{Z}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta \leq s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[e^{2\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}(s\wedge\zeta))_{\mathbb{R}^N}} \exp\big[-\sqrt{-1}\,\big(\boldsymbol{\xi}, \mathbf{Z}(s)\big)_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta \leq s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\,\big(\boldsymbol{\xi}, \tilde{\mathbf{Z}}(s)\big)_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta \leq s\}\Big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\,\big(\boldsymbol{\xi}, \tilde{\mathbf{Z}}(t)\big)_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta > s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[e^{2\sqrt{-1}(\boldsymbol{\xi},\mathbf{Z}(t\wedge\zeta))_{\mathbb{R}^N}} \exp\big[-\sqrt{-1}\,(\boldsymbol{\xi}, \mathbf{Z}(t))_{\mathbb{R}^N} - t\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta > s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{Z}(t\wedge\zeta)\big)_{\mathbb{R}^N} - (t\wedge\zeta)\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta > s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{Z}(s\wedge\zeta)\big)_{\mathbb{R}^N} - (s\wedge\zeta)\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta > s\}\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\exp\big[\sqrt{-1}\big(\boldsymbol{\xi}, \tilde{\mathbf{Z}}(s)\big)_{\mathbb{R}^N} - s\ell_\mu(\boldsymbol{\xi})\big],\, A \cap \{\zeta > s\}\Big]. \quad \square
\end{aligned}$$

Obviously, the process $\{\tilde{\mathbf{Z}}(t) : t \geq 0\}$ in Lemma 7.3.1 is the one obtained by reflecting (i.e., reversing the direction of $\{\mathbf{Z}(t) : t \geq 0\}$) at time ζ, and the lemma says that the distribution of the resulting process is the same as that of the original one. Most applications of this result are to situations when one knows or less precisely where the process is at the time when it is reflected. For example, suppose $N = 1$, $a \in (0,\infty)$, and $\zeta_a = \inf\{t \geq 0 : Z(t) \geq a\}$. Noting that, because $\tilde{Z}(t) = Z(t)$ for $t \leq \zeta_a$ and therefore that $\zeta_a = \inf\{t \geq 0 : \tilde{Z}(t) \geq a\}$, we have that

$$\begin{aligned}
\mathbb{P}\big(Z(t) \leq x \;\&\; \zeta_a \leq t\big) &= \mathbb{P}\big(2Z(\zeta_a) - Z(t) \leq x \;\&\; \zeta_a \leq t\big) \\
&= \mathbb{P}\big(Z(t) \geq 2Z(\zeta_a) - x \;\&\; \zeta_a \leq t\big).
\end{aligned}$$

Hence, if $x \le a$, and therefore $Z(t) \ge 2Z(\zeta_a) - x \implies \zeta_a \le t$ when $\zeta_a < \infty$,

$$\mathbb{P}\big(Z(t) \le x \ \&\ \zeta_a \le t\big) = \mathbb{P}\big(Z(t) \ge 2Z(\zeta_a) - x \ \&\ \zeta_a < \infty\big) \quad \text{for } x \le a.$$

Applying this when $x = a$ and using $\mathbb{P}\big(\zeta_a \le t\big) = \mathbb{P}\big(Z(t) \le a \ \&\ \zeta_a \le t\big) + \mathbb{P}\big(Z(t) > a\big)$, one gets $\mathbb{P}\big(\zeta_a \le t\big) \le 2\mathbb{P}\big(Z(t) \ge a\big)$, a conclusion which also could have been reached via Theorem 1.4.13.

§ **7.3.2. Reflected Brownian Motion.** The considerations in the preceding subsection are most interesting when applied to $\mathbb{R}$-valued Brownian motion. Thus, let $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ be an $\mathbb{R}$-valued Brownian motion. To appreciate the improvements that can be made in the calculations just made, again take $\zeta_a = \inf\{t \ge 0 : B(t) \ge a\}$ for some $a > 0$. Then, because Brownian paths are continuous, $\zeta_a < \infty \implies B(\zeta_a) = a$, and so, since $\mathbb{P}(\zeta_a < \infty) = 1$, we can say that

$$\text{(7.3.2)} \quad \mathbb{P}\big(B(t) \le x \ \&\ \zeta_a \le t\big) = \mathbb{P}\big(B(t) \ge 2a - x\big) \quad \text{for } (t,x) \in [0,\infty) \times (-\infty, a].$$

In particular, by taking $x = a$ and using $\mathbb{P}\big(B(t) \ge a\big) = \mathbb{P}\big(B(t) \ge a \ \&\ \zeta_a \le t\big)$ we recover the result in Exercise 4.3.12 that

$$\mathbb{P}\big(\zeta_a \le t\big) = 2\mathbb{P}\big(B(t) \ge a\big).$$

A more interesting application of Lemma 7.3.1 to Brownian motion is to the case when ζ is the exit time from an interval other than a half line.

Theorem 7.3.3. *Let $a_1 < 0 < a_2$ be given, define $\zeta^{(a_1,a_2)} = \inf\{t \ge 0 : B(t) \notin (a_1, a_2)\}$, and set $A_i(t) = \{\zeta^{(a_1,a_2)} \le t \ \&\ B(\zeta^{(a_1,a_2)}) = a_i\}$ for $i \in \{1,2\}$. Then, for $\Gamma \in \mathcal{B}_{[a_1,\infty)}$,*

$$\begin{aligned} 0 &\le \mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big) - \mathbb{P}\big(\{B(t) \in 2(a_2 - a_1) + \Gamma\} \cap A_1(t)\big) \\ &= \mathbb{P}\big(B(t) \in 2a_1 - \Gamma\big) - \mathbb{P}\big(B(t) \in 2(a_2 - a_1) + \Gamma\big), \end{aligned}$$

and, for $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$,

$$\begin{aligned} 0 &\le \mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_2(t)\big) - \mathbb{P}\big(\{B(t) \in -2(a_2 - a_1) + \Gamma\} \cap A_2(t)\big) \\ &= \mathbb{P}\big(B(t) \in 2a_2 - \Gamma\big) - \mathbb{P}\big(B(t) \in -2(a_2 - a_1) + \Gamma\big). \end{aligned}$$

Hence, for $\Gamma \in \mathcal{B}_{[a_1,\infty)}$, $\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big)$ equals

$$\sum_{m=1}^{\infty} \Big[\gamma_{0,t}\big(\Gamma - 2a_1 + 2(m-1)(a_2 - a_1)\big) - \gamma_{0,t}\big(\Gamma + 2m(a_2 - a_1)\big)\Big]$$

and, for $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$, $\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_2(t)\big)$ equals

$$\sum_{m=1}^{\infty} \Big[\gamma_{0,t}\big(\Gamma - 2a_2 - 2(m-1)(a_2 - a_1)\big) - \gamma_{0,t}\big(\Gamma - 2m(a_2 - a_1)\big)\Big],$$

where in both cases the convergence is uniform with respect t in compacts and $\Gamma \in \mathcal{B}_{(a_1,a_2)}$.

PROOF: Suppose $\Gamma \in \mathcal{B}_{[a_1,\infty)}$. Then, by Lemma 7.3.1,

$$\begin{aligned}\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big) &= \mathbb{P}\big(\{2a_1 - B(t) \in \Gamma\} \cap A_1(t)\big) \\ &= \mathbb{P}\big(B(t) \in 2a_1 - \Gamma\big) - \mathbb{P}\big(\{B(t) \in 2a_1 - \Gamma\} \cap A_2(t)\big),\end{aligned}$$

since $B(t) \in 2a_1 - \Gamma \implies B(t) \le a_1 \implies \zeta^{(a_1,a_2)} \le t$. Similarly

$$\begin{aligned}\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_2(t)\big) &= \mathbb{P}\big(\{2a_2 - B(t) \in \Gamma\} \cap A_1(t)\big) \\ &= \mathbb{P}\big(B(t) \in 2a_2 - \Gamma\big) - \mathbb{P}\big(\{B(t) \in 2a_2 - \Gamma\} \cap A_1(t)\big)\end{aligned}$$

when $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$. Hence, since $2a_1 - \Gamma \subseteq (-\infty, a_1] \subseteq (-\infty, a_2]$ if $\Gamma \in \mathcal{B}_{[a_1,\infty)}$,

$$\begin{aligned}\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big) = \mathbb{P}\big(B(t) \in 2a_1 - \Gamma\big) &- \mathbb{P}\big(B(t) \in 2(a_2 - a_1) + \Gamma\big) \\ &+ \mathbb{P}\big(\{B(t) \in 2(a_2 - a_1) + \Gamma\} \cap A_1(t)\big)\end{aligned}$$

when $\Gamma \in \mathcal{B}_{[a_1,\infty)}$. Similarly, when $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$,

$$\begin{aligned}\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_2(t)\big) = \mathbb{P}\big(B(t) \in 2a_2 - \Gamma\big) &- \mathbb{P}\big(B(t) \in -2(a_2 - a_1) + \Gamma\big) \\ &+ \mathbb{P}\big(\{B(t) \in -2(a_2 - a_1) + \Gamma\} \cap A_2(t)\big).\end{aligned}$$

To check that

$$\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big) - \mathbb{P}\big(\{B(t) \in 2(a_2 - a_1) + \Gamma\} \cap A_1(t)\big) \ge 0 \text{ when } \Gamma \in \mathcal{B}_{[a_1,\infty)},$$

first use Theorem 7.1.16 to see that

$$\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big) = \mathbb{E}^{\mathbb{P}}\big[\gamma_{0,t-\zeta^{(a_1,a_2)}}(\Gamma - a_1),\, A_1(t)\big].$$

Second, observe that, because $\Gamma \subseteq [a_1, \infty)$, $\gamma_{0,\tau}\big(2(a_2 - a_1) + \Gamma\big) \le \gamma_{0,\tau}(\Gamma)$ for all $\tau \ge 0$. The case when $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$ and $A_1(t)$ is replaced by $A_2(t)$ is handled in the same way.

Given the preceding, one can use induction to check that $\mathbb{P}\big(\{B(t) \in \Gamma\} \cap A_1(t)\big)$ equals

$$\begin{aligned}\sum_{m=1}^{M} \Big[\mathbb{P}\big(B(t) \in 2a_1 - 2(m-1)(a_2 - a_1) - \Gamma\big) - \mathbb{P}\big(B(t) \in 2m(a_2 - a_1) + \Gamma\big)\Big] \\ + \mathbb{P}\big(\{B(t) \in 2M(a_2 - a_1) + \Gamma\} \cap A_1(t)\big)\end{aligned}$$

for all $\Gamma \in \mathcal{B}_{[a_1,\infty)}$. The same line of reasoning applies when $\Gamma \in \mathcal{B}_{(-\infty,a_2]}$ and $A_1(t)$ is replaced by $A_2(t)$. $\square$

Perhaps the most useful consequence of the preceding is the following corollary.

Corollary 7.3.4. *Given a $c \in \mathbb{R}$ and an $r \in (0,\infty)$, set $I = (c-r, c+r)$ and*

$$P^I(t,x,\Gamma) = \mathbb{P}\big(\{x + B(t) \in \Gamma\} \cap \{\zeta^I > t\}\big), \quad x \in I \text{ and } \Gamma \in \mathcal{B}_I.$$

Then

$$P^I(s+t,x,\Gamma) = \int_I P^I(t,z,\Gamma)\, P^I(s,x,dz). \tag{7.3.5}$$

Next, set

$$\tilde{g}(t,x) = \sum_{m\in\mathbb{Z}} g(t, x+4m), \quad \text{where } g(t,x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}$$

and

$$p^{(-1,1)}(t,x,y) = \tilde{g}(t,y-x) - \tilde{g}(t,y+x+2) \quad \text{for } (t,x,y) \in (0,\infty)\times[-1,1]^2.$$

Then $p^{(-1,1)}$ is a smooth function which is symmetric in (x,y), strictly positive on $(0,\infty)\times(0,1)^2$, and vanishes when $x \in \{-1,1\}$. Finally, if

$$p^I(t,x,y) = r^{-1}p^{(-1,1)}\big(r^{-2}, r^{-1}(x-c), r^{-1}(y-c)\big), \quad (t,x,y) \in (0,\infty)\times I^2,$$

then

$$p^I(s+t,x,y) = \int_I p^I(s,x,z)p^I(t,z,y)\,dz, \tag{7.3.6}$$

and, for $(t,x) \in (0,\infty)\times I$, $P^I(t,x,dy) = p^I(t,x,y)\,dy$.

Proof: Begin by applying Theorem 7.1.16 to check that $P^I(s+t,x,\Gamma)$ equals

$$\begin{aligned}
&\mathcal{W}^{(1)}\big(\{x+\psi(s)+\delta_s\psi(t)\in\Gamma\}\cap\{x+\psi(s)+\delta_s\psi(\tau),\ \tau\in[0,t-s]\} \\
&\qquad\qquad \cap\{x+\psi(\sigma)\in I,\ \sigma\in[0,s]\}\big) \\
&= \mathbb{E}^{\mathcal{W}^{(1)}}\big[P^I\big(t,x+\psi(s),\Gamma\big),\ \{x+\psi(\sigma)\in I,\ \sigma\in[0,s]\}\big] \\
&= \int_I P^I(t,z,\Gamma)\,P^I(s,x,dz).
\end{aligned}$$

Next, set $a_1 = r^{-1}(c-x)-1$ and $a_2 = r^{-1}(x-x)+1$. Then

$$\begin{aligned}
P^I(t,x,\Gamma) &= \mathbb{P}\big(\{B(t)\in\Gamma-x\}\cap\{B(\tau)\in(ra_1,ra_2),\ \tau\in[0,t]\}\big) \\
&= \mathbb{P}\big(\{B(r^{-2}t)\in r^{-1}(\Gamma-x)\}\cap\{B(r^{-2}\tau)\in(a_1,a_2),\ \tau\in[0,t]\}\big) \\
&= \mathbb{P}\big(B(r^{-2}t)\in r^{-1}(\Gamma-x)\ \&\ \zeta^{(a_1,a_2)} > r^{-2}t\big) \\
&= \mathbb{P}\big(B(r^{-2}t)\in r^{-1}(\Gamma-x)\big) - \mathbb{P}\big(B(r^{-2}t)\in r^{-1}(\Gamma-x)\ \&\ \zeta^{(a_1,a_2)} \le r^{-2}t\big),
\end{aligned}$$

where, in the passage to the second line, I have used Brownian scaling. Now, use the last part of Theorem 7.3.3, the symmetry of $\gamma_{0,r^{-2}t}$, and elementary rearrangement of terms to arrive first at

$$P^I(t,x,\Gamma) = \sum_{m\in\mathbb{Z}} \Big[\gamma_{r^{-2}t}\big(4m + r^{-1}(\Gamma - x)\big) - \gamma_{r^{-2}t}\big(4m + 2 + r^{-1}(\Gamma + x - 2c)\big)\Big],$$

and then at $P^I(t,x,dy) = p^I(t,x,y)\,dy$. Given this and (7.3.5), (7.3.6) is obvious.

Turning to the properties of $p^{(-1,1)}(t,x,y)$, both its symmetry and smoothness are clear. In addition, as the density for $P^{(-1,1)}(t,x.\,\cdot\,)$, it is non-negative, and, because $x \rightsquigarrow \tilde{g}(t,x)$ is periodic with period 4, it is easy to see that $p^{(-1,1)}(t,\pm 1,y) = 0$. Thus, everything comes down to proving that $p^{(-1,1)}(t,x,y) > 0$ for $(t,x,y) \in (0,\infty)\times(-1,1)^2$. To this end, first observe that, after rearranging terms, one can write $p^{(-1,1)}(t,x,y)$ as

$$\begin{aligned} &g(t,y-x) - g(t,y+x) + g(t,2-x-y) \\ &\quad + \sum_{m=1}^{\infty}\Big[\big(g(t,y-x+4m) - g(t,y+x+2+4m)\big) \\ &\qquad\qquad + \big(g(t,y-x-4m) - g(t,y+x-2-4m)\big)\Big]. \end{aligned}$$

Since each of the term in the sum over $m \in \mathbb{Z}^+$ is positive, we have that

$$p^{(-1,1)}(t,x,y) > g(t,y-x)\big[1 - 2e^{-\frac{2(1-|x|)(1-|y|)}{t}}\big] \ge \big(1 - \tfrac{2}{e}\big)g(t,y-x)$$

if $t \le 2(1-|x|)(1-|y|)$. Hence, for each $\theta \in (0,1)$, $p^{(-1,1)}(t,x,y) > 0$ for all $(t,x,y) \in [0,2\theta^2]\times[-1+\theta,1-\theta]^2$. Finally, to handle $x,y \in [-1+\theta,1-\theta]$ and $t > 2\theta^2$, apply (7.3.6) with $I = (-1,1)$ to see that

$$p^{(-1,1)}\big((m+1)\theta^2,x,y\big) \ge \int_{|z|\le(1-\theta)} p^{(-1,1)}(\theta^2,x,z)p^{(-1,1)}(m\theta^2,z,y)\,dz,$$

and use this and induction to see that $p^{(-1,1)}(m\theta^2,x,y) > 0$ for all $m \ge 1$. Thus, if $n \in \mathbb{Z}^+$ is chosen so that $n\theta^2 < t \le (n+1)\theta^2$, then another application of (7.3.6) shows that

$$p^{(-1,1)}(t,x,y) \ge \int_{|z|\le(1-\theta)} p^{(-1,1)}(t-n\theta^2,x,z)p^{(-1,1)}(n\theta^2,z,y)\,dz > 0. \quad \square$$

Exercises for § 7.3

EXERCISE 7.3.7. Suppose that G is a non-empty, open subset of $\mathbb{R}^N$, define $\zeta_x^G : C(\mathbb{R}^N) \longrightarrow [0,\infty]$ by

$$\zeta_{\mathbf{x}}^G(\boldsymbol{\psi}) = \inf\{t \ge 0 : \mathbf{x} + \boldsymbol{\psi}(t) \notin G\},$$

and set

$$P^G(t,\mathbf{x},\Gamma) = \mathcal{W}^{(N)}\big(\{\boldsymbol{\psi} : \mathbf{x} + \boldsymbol{\psi}(t) \in \Gamma \ \&\ \zeta_{\mathbf{x}}^G(\boldsymbol{\psi}) > t\}\big)$$

for $(t,\mathbf{x}) \in (0,\infty) \times G$ and $\Gamma \in \mathcal{B}_G$.

(**i**) Show that

$$P^G(s+t,\mathbf{x},\Gamma) = \int_G P^G(t,\mathbf{z},\Gamma)\, P^G(s,\mathbf{x},d\mathbf{y}).$$

(**ii**) As an application of Exercise 7.1.25, show that

$$P^G(t,\mathbf{x},\Gamma) = \gamma_{\mathbf{0},t\mathbf{I}}(\Gamma - \mathbf{x}) - \mathbb{E}^{\mathcal{W}^{(N)}}\big[\gamma_{\mathbf{0},(t-\zeta_{\mathbf{x}}^G)\mathbf{I}}\big(\Gamma - \mathbf{x} - \boldsymbol{\psi}(\zeta_{\mathbf{x}}^G)\big),\ \zeta_{\mathbf{x}}^G \le \Gamma\big].$$

. This is the probabilistic version of Duhamel's formula, which we will see again in § 10.3.1.

(**iii**) As a consequence of (**ii**), show that there is a Borel measurable function $p^G : (0,\infty) \times G^2 \longrightarrow [0,\infty)$ such that $(t,\mathbf{y}) \rightsquigarrow p^G(t,\mathbf{x},\mathbf{y})$ is continuous for each $\mathbf{x} \in G$ and $P^G(t,\mathbf{x},d\mathbf{y}) = p^G(t,\mathbf{x},\mathbf{y})\,d\mathbf{y}$ for each $(t,\mathbf{x}) \in (0,\infty) \times G$. In particular, use this in conjunction with (**i**) to conclude that

$$p^G(s+t,\mathbf{x},\mathbf{y}) = \int_G p^G(t,\mathbf{z},\mathbf{y}) p^G(s,\mathbf{x},\mathbf{z})\, d\mathbf{z}.$$

Hint: Keep in mind that $(\tau,\boldsymbol{\xi}) \rightsquigarrow (2\pi\tau)^{-\frac{N}{2}} e^{-\frac{|\boldsymbol{\xi}|^2}{2\tau}}$ is smooth and bounded as long as $\boldsymbol{\xi}$ stays away from the origin.

(**iv**) Given $\mathbf{c} = (c_1,\dots,c_N) \in \mathbb{R}^N$ and $r > 0$, let $Q(\mathbf{c},r)$ denote the open cube $\prod_{i=1}^N (c_i - r, c_i + r)$, and show that (cf. Corollary 7.3.4)

$$p^{Q(\mathbf{c},r)}(t,\mathbf{x},\mathbf{y}) = \prod_{i=1}^N p^{(c_i - r, c_i + r)}(t, x_i, y_i)$$

for $\mathbf{x} = (x_1,\dots,x_N)$, $\mathbf{y} = (y_1,\dots,y_N) \in Q(\mathbf{c},r)$. In particular, conclude that $p^{Q(\mathbf{c},r)}(t,\mathbf{x},\mathbf{y})$ is uniformly positive on compact subsets of $(0,\infty) \times Q(\mathbf{c},r)^2$.

(**v**) Assume that G is connected, and show that $p^G(t,\mathbf{x},\mathbf{y})$ is uniformly positive on compact subsets of $(0,\infty) \times G^2$.

Hint: If $Q(\mathbf{c},r) \subseteq G$, show that $p^G(t,\mathbf{x},\mathbf{y}) \ge p^{Q(\mathbf{c},r)}(t,\mathbf{x},\mathbf{y})$ on $(0,\infty) \times Q(\mathbf{c},r)^2$.