Homework #5

Exercise 4.4:

(**i**) Because $A(t) - A(s)$ is non-negative definite,

$$
\begin{aligned}
|A_{ij}(t) - A_{ij}(s)| &\le \big(A_{ii}(t) - A_{ii}(s)\big)^{\frac12}\big(A_{jj}(t) - A_{jj}(s)\big)^{\frac12} \\
&\le \frac{\big(A_{ii}(t) - A_{ii}(s)\big) + \big(A_{jj}(t) - A_{jj}(s)\big)}{2} \le \mathrm{Trace}\big(A(t) - A(s)\big)
\end{aligned}
$$

for $0 \le s < t$. Hence, $A_{ij}(\,\cdot\,)$ is absolutely continuous, and so, if $a(t) = \big((\dot A_{ij})\big)_{1\le i,j\le N}$, then $A(t) = \int_0^t a(\tau)\,dt$, from which it is clear that a can be chosen so that it is non-negative definite. Finally, since

$$
\int_s^t \mathrm{Trace}\big(a(\tau)\big)\,d\tau = \mathrm{Trace}\big(A(t) - A(s)\big) \le t - s,
$$

it follows that a can be chosen so that $\mathrm{Trace}\big(a(t)\big) \le 1$.

(**ii**)–(**iv**) Just follow the outline.

Exercise 5.1: Set $M(t) = \psi(t) - \mathbf{x} - \int_0^t b\big(\psi(\tau)\big)\,d\tau$. Then $\big(M(t), \mathcal{B}_t, \mathbb{P}_{\mathbf{x}}\big)$ is a continuous, local $\mathbb{R}^N$-valued martingale and $A \equiv \big((\langle M_i, M_j\rangle)\big)_{1\le i,j\le N} = 0$. Hence, since $\big(|M(t)|^2 - \mathrm{Trace}\big(A(t)\big), \mathcal{B}_t, \mathbb{P}_{\mathbf{x}}\big)$ is a martingale, $\mathbb{E}^{\mathbb{P}}\big[|M(t)|^2\big] = 0$.

Exercise 5.2: Note that $dE(t) = -E(t)\big(\beta(t), dB(t)\big)_{\mathbb{R}^N}$, and then apply Itô's formula to see that, for any $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R})$,

$$
d\Big(E(t)\varphi\big(X(t)\big)\Big) = E(t)\Big(\nabla\varphi\big(X(t)\big) - \beta(t), dB(t)\Big)_{\mathbb{R}^N} + \frac12 E(t)\Delta\varphi\big(X(t)\big)\,dt,
$$

and therefore that

$$
\left(E(t)\varphi\big(X(t)\big) - \frac12\int_0^t E(\tau)\Delta\varphi\big(X(\tau)\big)\,d\tau, \mathcal{F}_t, \mathbb{P}\right)
$$

is a martingale. Thus, if $0 \le s < t$ and $A \in \mathcal{F}_s$, then

$$
\begin{aligned}
&\mathbb{E}^{\tilde P}\Big[\varphi\big(X(t)\big) - \varphi\big(X(s)\big)\,A\Big] = \mathbb{E}^{\mathbb{P}}\Big[E(t)\varphi\big(X(t)\big) - E(s)\varphi\big(X(s)\big)\,A\Big] \\
&= \frac12\int_s^t \mathbb{E}^{\mathbb{P}}\Big[E(\tau)\Delta\varphi\big(X(\tau)\big),\, A\Big]\,d\tau = \frac12\int_s^t \mathbb{E}^{\mathbb{P}}\Big[E(t)\Delta\varphi\big(X(\tau)\big),\, A\Big]\,d\tau \\
&= \frac12\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_s^t \Delta\varphi\big(X(\tau)\big)\,d\tau,\, A\right],
\end{aligned}
$$

which means that

$$
\left(\varphi\big(X(t)\big) - \frac12\int_0^t \Delta\varphi\big(X(\tau)\big)\,d\tau, \mathcal{F}_t, \tilde{\mathbb{P}}\right)
$$

is a martingale and therefore that $\big(X(t), \mathcal{F}_t, \tilde{\mathbb{P}}\big)$ is a Brownian motion.

Exercise 5.3: By following the outline, one arrives at the conclusion that

$$
\mathcal{W}(\mathfrak{e} \le 1) \ge \mathcal{W}\big(w(t) \ge -1 \text{ for } t \in [0,1]\big) > 0.
$$

Thus

$$
\mathbb{E}^{\mathcal{W}}\left[\exp\left(\int_0^t w(\tau)^2\,dw(\tau) - \frac12\int_0^2 w(\tau)^4\,d\tau\right)\right] = \mathcal{W}(\mathfrak{e} > t) < 1.
$$

Exercise 5.7:

(**i**) By Theorem 3.5.3, we know that the span of

$$\mathbb{R} \oplus \{\tilde{I}^{(m)}_{g^{\otimes m}}(\infty) : m \geq 1 \ \&\ g \in L^2([0,\infty);\mathbb{R})\}$$

is dense in $L^2(\mathcal{W};\mathbb{R})$, and from this it is clear that the span of

$$\mathbb{R} \oplus \{\tilde{I}^{(m)}_{(\dot{f})^{\otimes m}}(\infty) : m \geq 1 \ \&\ f \in H^1(\mathbb{R}^N)\big((0,\infty);\mathbb{R}\big)\}$$

is also dense there. Futher, by Exercise 3.6, $\tilde{I}^{(m)}_{(\dot{f})^{\otimes m}}(\infty) = H_m\big(I(\dot{f}); \|\dot{f}\|^2_{H^1(\mathbb{R}^N)}\big)$ for $m \geq 1$. Hence $\mathscr{P}\big(\mathbb{W}(\mathbb{R}^N);\mathbb{R}\big)$ is dense in $L^2(\mathcal{W};\mathbb{R})$, and from this it is easy to show that $\mathscr{P}\big(\mathbb{W}(\mathbb{R}^N);H\big)$ is dense in $L^2(\mathcal{W};H)$. Finally, (5.6.1) is a simple application of the chain rule and the fact that $DI(\dot{f}) = f$.

(**ii**) By (3.1.6),

$$\begin{aligned}\big(\Psi, D\Phi\big)_{L^2(\mathcal{W};H^1(\mathbb{R}^N))} &= \mathbb{E}^{\mathcal{W}}\big[D_h\Phi(\Psi,h)_{H^1(\mathbb{R}^N)}\big] = \mathbb{E}^{\mathcal{W}}\big[\Phi\big(I(\dot{h}) - D_h\big)(\Psi,h)_H\big] \\ &= \Big(\big(I(\dot{h}) - D_h\big)(\Psi,h)_H, \Phi\Big)_{L^2(\mathcal{W};\mathbb{R})}.\end{aligned}$$

(**iii**) Suppose that $\Phi \in \mathrm{Dom}(D)$ and $\Phi \in \mathrm{Dom}(D^\top)$, and choose $\{\Phi_n : n \geq 1\} \subseteq \mathscr{P}(\mathbb{W}(\mathbb{R}^N);\mathbb{R})$ so that $\Phi_n \longrightarrow \Phi$ in $L^2(\mathcal{W};\mathbb{R})$ and $D\Phi_n \longrightarrow D\Phi$ in $L^2\big(\mathcal{W};H^1(\mathbb{R}^N)\big)$. Then

$$\begin{aligned}\big(\Phi, D^\top\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} &= \lim_{n\to\infty}\big(\Phi_n, D^\top\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} \\ &= \lim_{n\to\infty}\big(D\Phi_n, \Psi\big)_{L^2(\mathcal{W};H^1(\mathbb{R}^N))} = \big(D\Phi, \Psi\big)_{L^2(\mathcal{W};H^1(\mathbb{R}^N))}.\end{aligned}$$

(**iv**) It suffices to treat the case when $\Psi = G\big(I(\dot{f}_1),\dots,I(\dot{f}_L)\big)h$, where G is a polynomial and $\|h\|_{H^1(\mathbb{R}^N)} = 1$. In that case,

$$\sum_{k=1}^n I(\dot{h}_k)(\Psi,h_k)_{H^1(\mathbb{R}^N)} = G\big(I(\dot{f}_1),\dots,I(\dot{f}_L)\big)I\left(\sum_{k=1}^n (h,h_k)_{H^1(\mathbb{R}^N)}\dot{h}_k\right).$$

Since $\sum_{k=1}^n (h,h_k)_{H^1(\mathbb{R}^N)}\dot{h}_k \longrightarrow h$ in $L^2([0,\infty);\mathbb{R}^N)$, $I\left(\sum_{k=1}^n (h,h_k)_{H^1(\mathbb{R}^N)}\dot{h}_k\right) \longrightarrow I(\dot{h})$ is $L^p(\mathcal{W};\mathbb{R})$ for all $p \in [1,\infty)$, and therefore

$$\sum_{k=1}^n I(\dot{h}_k)(\Psi,h_k)_{H^1(\mathbb{R}^N)} \longrightarrow I(\dot{h})(\Psi,h)_{H^1(\mathbb{R}^N)} \text{ in } L^2(\mathcal{W};\mathbb{R}).$$

Next,

$$\begin{aligned}\sum_{k=1}^n D_{h_k}(\Psi,h_k)_{H^1(\mathbb{R}^N)} &= \sum_{k=1}^n (h,h_k)_{H^1(\mathbb{R}^N)}\left(\sum_{\ell=1}^L (f_\ell,h_k)_{H^1(\mathbb{R}^N)}\partial_{x_\ell}G\big(I(\dot{f}_1),\dots,I(\dot{f}_L)\big)\right) \\ &\longrightarrow \sum_{\ell=1}^L (f_\ell,h)_{H^1(\mathbb{R}^N)}\partial_{x_\ell}G\big(I(\dot{f}_1),\dots,I(\dot{f}_L)\big) = D_h(\Psi,h)_{H^1(\mathbb{R}^N)}\end{aligned}$$

in $L^2(\mathcal{W};\mathbb{R})$.

Exercise 5.8:

(**i**) The only assertion here that needs comment is the final one. Thus, suppose that $\Phi \in \mathrm{Dom}(\mathcal{L})$, and choose $\{\Phi_n : n \geq 1\} \subseteq \mathscr{P}\big(\mathbb{W}(\mathbb{R}^N);\mathbb{R}\big)$ so that $\Phi_n \longrightarrow \Phi$ and $\mathcal{L}\Phi_n \longrightarrow \mathcal{L}\Phi$ in $L^2(\mathcal{W};\mathbb{R})$. Then, for $1 \leq m < n$,

$$\begin{aligned}\big\|D\Phi_n - D\Phi_m\big\|^2_{L^2(\mathcal{W};H^1(\mathbb{R}^N))} &= \big(\Phi_n - \Phi_m, \mathcal{L}(\Phi_n - \Phi_m)\big)_{L^2(\mathcal{W};\mathbb{R})} \\ &\leq \|\Phi_n - \Phi_m\|_{L^2(\mathcal{W};\mathbb{R})}\|\mathcal{L}\Phi_n - \mathcal{L}\Phi_m\|_{L^2(\mathcal{W};\mathbb{R})},\end{aligned}$$

and therefore $\{D\Phi_n : n \geq 1\}$ converges in $L^2(\mathcal{W};H^1(\mathbb{R}^N))$. Hence $\Phi \in \mathrm{Dom}(D)$, $D\Phi_n \longrightarrow D\Phi$ in $L^2\big(\mathcal{W};H^1(\mathbb{R}^N)\big)$, and

$$\begin{aligned}\|D\Phi\|^2_{L^2(\mathcal{W};H^1(\mathbb{R}^N))} &= \lim_{n\to\infty}\|D\Phi_n\|^2_{L^2(\mathcal{W};H^1(\mathbb{R}^N))} \\ &= \lim_{n\to\infty}\big(\Phi_n, \mathcal{L}\Phi_n\big)_{L^2(\mathcal{W};\mathbb{R})} = (\Phi, \mathcal{L}\Phi)_{L^2(\mathcal{W};\mathbb{R})},\end{aligned}$$

from which the asserted result follows by polarization.

(**ii**) This is an immediate consequence of (5.6.2).

(**iii**) By following the outline, one arrives at $H'_m = mH_{m-1}$ for $m \geq 1$. Hence, since $xH_{m-1}(x) - H'_{m-1} = H_m$, it follows that $xH'_m(x) - H''_m(x) = mH_m(x)$. Finally, since $H_m(x,a) = a^{\frac{m}{2}}H_m(a^{-\frac{1}{2}}x)$, this leads to $x\partial_x H_m(x,a) - a\partial_x^2 H_m(x,a) = mH_m(x,a)$.

(**iv**) By Theorem 3.5.3, we know that $\mathbb{R} \oplus \{\tilde{I}^{(m)}_{f^{\otimes m}}(\infty) : f \in H^1(\mathbb{R}^N)\}$ is dense in $Z^{(m)}$, and, by Exercise 3.6, we know that $\tilde{I}^{(m)}_{f^{\otimes m}}(\infty) = H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big)$. Next, set $\Phi = H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big)$, and choose an orthonormal basis $\{h_m : m \geq 1\}$ with $h_1 = \frac{f}{\|f\|_{H^1(\mathbb{R}^N)}}$. Then

$$\begin{aligned}\mathcal{L}\Phi = I(\dot{h}_1)D_{h_1}\Phi - D^2_{h_1}\Phi &= I(\dot{h}_1)(f,h_1)_{H^1(\mathbb{R}^N)}\partial_x H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) \\ &\qquad - (f,h_1)^2_{H^1(\mathbb{R}^N)}\partial_x^2 H_m\big(I(\dot{h}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) \\ &= I(\dot{f})\partial_x H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) - \|f\|^2_{H^1(\mathbb{R}^N)}\partial_x^2 H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) \\ &= mH_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big).\end{aligned}$$

Now suppose that

$$\{\Phi_n : n \geq 1\} \subseteq \mathrm{span}\Big(\mathbb{R} \oplus \{H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) : f \in H^1(\mathbb{R}^N)\}\Big)$$

and that $\Phi_n \longrightarrow \Phi$ in $L^2(\mathcal{W};\mathbb{R})$. Then $\mathcal{L}\Phi_n = m\Phi_n \longrightarrow m\Phi$ in $L^2(\mathcal{W};\mathbb{R})$ and so $\Phi \in \mathrm{Dom}(\mathcal{L})$ and $\mathcal{L}\Phi = m\Phi$. Conversely, suppose that $\Phi \in \mathrm{Dom}(\mathcal{L})$ and that $\mathcal{L}\Phi = m\Phi$. If $\Psi \in Z^{(n)}$, then

$$m(\Phi,\Psi)_{L^2(\mathcal{W};\mathbb{R})} = \big(\mathcal{L}\Phi, \Psi\big)_{L^2(\mathcal{W};\mathbb{R})} = \big(\Phi, \mathcal{L}\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} = n(\Phi,\Psi)_{L^2(\mathcal{W};\mathbb{R})},$$

and so either $m = n$ or $(\Phi,\Psi)_{L^2(\mathcal{W};\mathbb{R})} = 0$. Hence $\Phi \in Z^{(m)}$. Finally, observe that for any $a \geq 0$ and $m \geq 1$, $\partial_x H_m(x,a) = mH_{m-1}(x,a)$ follows from $H_m(x,a) = a^{\frac{m}{2}}H_m(a^{-\frac{1}{2}}\mathbf{x})$ and $H'_m = mH_{m-1}$. Thus, if $\Phi = H_m\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big)$, then $D_h\Phi = m(f,h)_{H^1(\mathbb{R}^N)}H_{m-1}\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big)$, and so

$$\mathcal{L}D_h\Phi = m(f,h)_{H^1(\mathbb{R}^N)}(m-1)H_{m-1}\big(I(\dot{f}), \|f\|^2_{H^1(\mathbb{R}^N)}\big) = (m-1)D_h\Phi.$$

Starting from this, it is an easy matter to check that $\mathcal{L}D_h\Phi = (m-1)D_h\Phi$ for all $\Phi \in Z^{(m)}$.

(**v**) Suppose that $\Phi \in L^2(\mathcal{W};\mathbb{R})$ and that

$$(*) \qquad \sum_{m=0}^{\infty} m^2 \|\Pi_{Z^{(m)}}\Phi\|^2_{L^2(\mathcal{W};\mathbb{R}^N)} < \infty.$$

Set $\Phi_n = \sum_{m=0}^{n} \Pi_{Z^{(m)}}\Phi$. Then $\Phi_n \longrightarrow \Phi$ in $L^2(\mathcal{W};\mathbb{R})$ and $\mathcal{L}\Phi_n = \sum_{m=0}^{n} m\Pi_{Z^{(m)}}\Phi$. Thus, because of $(*)$, we know that $\mathcal{L}\Phi_n \longrightarrow \sum_{m=0}^{\infty} m\Pi_{Z^{(m)}}\Phi$ in $L^2(\mathcal{W};\mathbb{R})$, $\Phi \in \mathrm{Dom}(\mathcal{L})$ and $\mathcal{L}\Phi = \sum_{m=0}^{\infty} m\Pi_{Z^{(m)}}\Phi$.

Next suppose that $\Phi \in \mathrm{Dom}(\mathcal{L})$. Then, for any $\Psi \in L^2(\mathcal{W};\mathbb{R})$,

$$\begin{aligned}\big(\Pi_{Z^{(m)}}\mathcal{L}\Phi, \Psi\big)_{L^2(\mathcal{W};\mathbb{R})} &= \big(\mathcal{L}\Phi, \Pi_{Z^{(m)}}\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} = \big(\Phi, \mathcal{L}\Pi_{Z^{(m)}}\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} \\ &= m\big(\Phi, \Pi_{Z^{(m)}}\Psi\big)_{L^2(\mathcal{W};\mathbb{R})} = m\big(\Pi_{Z^{(m)}}\Phi, \Psi\big)_{L^2(\mathcal{W};\mathbb{R})},\end{aligned}$$

and so $\Pi_{Z^{(m)}}\mathcal{L}\Phi = m\Pi_{Z^{(m)}}\Phi$. Hence,

$$\sum_{m=0}^{\infty} m^2 \|\Pi_{Z^{(m)}}\Phi\|^2_{L^2(\mathcal{W};\mathbb{R}^N)} = \|\mathcal{L}\Phi\|^2_{L^2(\mathcal{W};\mathbb{R})} < \infty.$$