Homework #4

Exercise 3.7:

(**i**) Because $\sigma\big(\{\beta(t) : t \geq 0\}\big)$ is independent of $\sigma\big(\{B(t) : t \geq 0\}\big)$,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N}\right)^2\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \big(f(t), \sigma\sigma^\top(t) f(t)\big)_{\mathbb{R}^N}\, dt\right] + \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \big(f(t), \beta(t)\big)_{\mathbb{R}^N}\, dt\right)^2\right].$$

Hence,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N}\right)^2\right] \geq \kappa^2 \|f\|^2_{L^2([0,T];\mathbb{R}^N)}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N}\right)\right] \leq \left(\kappa^{-2} + \mathbb{E}^{\mathbb{P}}\left[\int_0^T |\beta(t)|^2\, dt\right]\right) \|f\|^2_{L^2([0,T];\mathbb{R}^N)}.$$

(**ii**) Clearly L_T is a linear subspace. Next, let $\{f_n : n \geq 1\} \subseteq L^2([0,T];\mathbb{R}^N)$, set $X_n = \int_0^T \big(f_n(t), dZ(t)\big)_{\mathbb{R}^N}$, and suppose that $X_n \longrightarrow X$ in $L^2([0,T];\mathbb{R}^N)$. By the lower bound in (**i**), one sees that $\{f_n : n \geq 1\}$ is Cauchy convergent in $L^2([0,T];\mathbb{R}^N)$ and therefore that there exists an $f \in L^2([0,T];\mathbb{R}^N)$ to which it converges. Since this means that $X_n \longrightarrow \int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N}$, it follows that $X \in L_T$ and therefore that L_T is closed. In addition, by taking $f(t) = \mathbf{1}_{[0,t]}\sigma(t)^{-1}\boldsymbol{\xi}$ one sees that $\big(\boldsymbol{\xi}, Z(t)\big)_{\mathbb{R}^N} \in L_T$ for all $t \in [0,T]$ and $\boldsymbol{\xi} \in \mathbb{R}^N$. Finally, suppose that L is a closed, linear subspace that contains $\big(\boldsymbol{\xi}, Z(t)\big)_{\mathbb{R}^N}$ for all $t \in [0,T]$ and $\boldsymbol{\xi} \in \mathbb{R}^N$. If $f : [0,T] \longleftrightarrow \mathbb{R}^N$ has locally bounded variation, then $\int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N}$ is a Riemann-Stieltjes integral and its Riemman sum approximations are in L. Thus $\int_0^T \big(f(t), dZ(t)\big)_{\mathbb{R}^N} \in L$ when f has locally bounded variation, and so it is also in L for every $f \in L^2([0,T];\mathbb{R}^N)$. Hence, $L_T \subseteq L$.

(**iii**) Riemann sum approximations make it obvious that $\big(\boldsymbol{\xi}, Z(t)\big)_{\mathbb{R}^N} \in \mathfrak{G}$ of all $(t, \boldsymbol{\xi}) \in [0,\infty) \times \mathbb{R}^N$ if σ has locally bounded variation and β is continuous as a function of t. Therefore, since $\mathfrak{G}$ is closed, the same is true in general. Hence, by Exercise 2.1, for any $X \in \mathfrak{G}$ and $T > 0$,

$$\mathbb{E}^{\mathbb{P}}\big[X \,\big|\, \sigma\big(\{Z(t) : t \in [0,T]\}\big)\big] = \Pi_{\mathbf{1}\oplus L_T} X,$$

and, by (**ii**)

$$\Pi_{1\oplus L_T} X = m_{T,X} + \int_0^T \big(f_{T,X}(t), dZ(t)\big)_{\mathbb{R}^N}$$

for some $m_{T,X} \in \mathbb{R}$ and $f_{T,X} \in L^2([0,T];\mathbb{R}^N)$.

Exercise 4.1: If $\mathbb{E}^{\mathbb{P}}\big[|M(t\wedge\zeta_k)|^p\big] \leq C(t) < \infty$ for some $p > 1$ and all $t \geq 0$ and $k \geq 1$, then, for each $t \geq 0$, $\{M(t\wedge\zeta_k) : k \geq 1\}$ is uniformly P-integrable and therefore $M(t\wedge\zeta_k) \longrightarrow M(t)$ in $L^1(P;\mathbb{R})$. Hence the martingale property for

$\{M(t) : t \geq 0\}$ is inherited from that for the $\{M(t \wedge \zeta_k) : t \geq 0\}$'s. Next, assume that $M(t) \geq 0$ for all $t \geq 0$. If $0 \leq s < t$ and $A \in \mathcal{F}_s$, then, by Fatou's lemma,

$$\mathbb{E}^{\mathbb{P}}\big[M(t),\, A\big] \leq \varliminf_{k\to\infty} \mathbb{E}^{\mathbb{P}}\big[M(t \wedge \alpha_k),\, A\big] = \lim_{k\to\infty} \mathbb{E}^{\mathbb{P}}\big[M(s \wedge \zeta_k), A\big] = \mathbb{E}^{\mathbb{P}}\big[M(s), A\big],$$

and so $\big(M(t), \mathcal{F}_t, \mathbb{P}\big)$ is a supermartingale. Moreover, $\mathbb{E}^{\mathbb{P}}\big[M(t)\big] = E^{\mathbb{P}}\big[M(0)\big]$ if and only if $\mathbb{E}^{\mathbb{P}}\big[M(t)\big] = E^{\mathbb{P}}\big[M(t \wedge \zeta_k)\big]$ for all $k \geq 1$, and therefore $M(t \wedge \zeta_k) \longrightarrow M(t)$ if $L^1(\mathbb{P}; \mathbb{R})$ if and only if $\mathbb{E}^{\mathbb{P}}\big[M(t)\big] = E^{\mathbb{P}}\big[M(0)\big]$.

Exercise 4.2: Without loss in generality, we will assume that $\big(M_1(t), \mathcal{F}_t, \mathbb{P}\big)$ and $\big(M_1(t), \mathcal{F}_t, \mathbb{P}\big)$ are martingales. In addition, by Corollary 4.3.4, we know that $\langle M_1, M_2\rangle$ does not depend on which filtration M_1 and M_2 are martingales with respect to, and so we can assume that $\mathcal{F}_t = \sigma\big(\big\{\big(M_1(\tau), M_2(\tau)\big) : \tau \in [0,t]\big\}\big)$. Now suppose that $A_i \in \sigma\big(\{M_i(\tau) : \tau \in [0,s]\}\big)$. Then, for $t > s$, when $M_1(\,\cdot\,)$ is independent of $M_2(\,\cdot\,)$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[M_1(t)M_2(t),\, A_1 \times A_2\big] &= \mathbb{E}^{\mathbb{P}}\big[M_1(t),\, A_1\big]\mathbb{E}^{\mathbb{P}}\big[M_2(t),\, A_2\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[M_1(s),\, A_1\big]\mathbb{E}^{\mathbb{P}}\big[M_2(s),\, A_2\big] = \mathbb{E}^{\mathbb{P}}\big[M_1(s)M_2(s),\, A_1 \times A_2\big],\end{aligned}$$

and so, since

$$\mathcal{F}_s = \sigma\big(\{M_1(\tau) : \tau \in [0,s]\}\big) \times \sigma\big(\{M_2(\tau) : \tau \in [0,s]\}\big),$$

$\big(M_1(t)M_2(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale and therefore $\langle M_1, M_2\rangle = 0$.

To produce an example in which $\langle M_1, M_2\rangle = 0$ but $M_1(\,\cdot\,)$ is not independent of $M_2(\,\cdot\,)$, let $\big(B(t), \mathcal{F}_t, \mathbb{P}\big)$ be an $\mathbb{R}^2$-valued Brownian motion, and set $M_1(t) = \int_0^t B_1(\tau)\, dB_1(\tau)$ and $M_2(t) = \int_0^t B_1(\tau)\, dB_2(\tau)$. Then $\langle M_1, M_2\rangle(t) = B_1(t)^2\langle B_1, B_2\rangle(t) = 0$. If $M_1(\,\cdot\,)$ were independent of $M_2(\,\cdot\,)$, then $\mathbb{E}^{\mathbb{P}}\big[M_1(1)M_2(1)\big]$ would be 0. However, $M_1(t) = \frac{B_1(t)^2 - t}{2}$ and

$$M_2(1)^2 = 2\int_0^1 M_2(\tau)B_1(\tau)\, dB_2(\tau) + \int_0^1 B_1(\tau)^2\, d\tau.$$

Thus,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\big[M_1(1)M_2(1)^2\big] = \int_0^1 \mathbb{E}^{\mathbb{P}}\big[M_1(1)B_1(\tau)^2\big]\, d\tau = \int_0^1 \mathbb{E}^{\mathbb{P}}\big[M_1(\tau)B_1(\tau)^2\big]\, d\tau \\ &= \frac{1}{2}\int_0^1 \mathbb{E}^{\mathbb{P}}\big[B_1(\tau)^4\big]\, d\tau - \frac{1}{2}\int_0^1 \mathbb{E}^{\mathbb{P}}\big[B_1(\tau)^2\big]\, d\tau = \frac{3}{2}\int_0^1 \tau^2\, d\tau - \frac{1}{2}\int_0^1 \tau\, d\tau = \frac{1}{4},\end{aligned}$$

whereas $\mathbb{E}^{\mathbb{P}}\big[M_1(1)M_2(1)^2\big]$ would be 0 if $M_1(\,\cdot\,)$ were independent of $M_2(\,\cdot\,)$.

Finally, let M_j be the martingale part of X_j. Then the martingale part of $\varphi \circ X$ is

$$\sum_{j=1}^N \int_0^t \partial_j\varphi \circ X(\tau)\, dM_j(\tau),$$

and so

$$\langle \varphi_1 \circ X, \varphi_2 \circ X\rangle(t) = \sum_{j_1,j_2=1}^N \int_0^t (\partial_{x_{j_1}}\varphi_1) \circ X(\tau)\partial_{x_{j_2}}\varphi_2) \circ X(\tau)\, d\langle X_{j_1}, X_{j_2}\rangle(\tau).$$

Exercise 4.3:

(**i**) To reduce to the case when M and $\langle\langle M\rangle\rangle$ are bounded, set $\zeta_n = \inf\{t \geq 0 : |M(t)| \vee \langle\langle M\rangle\rangle(t) \geq n\}$ and $M_n(t) = M(t\wedge\zeta_n)$. If $\mathbb{E}^{\mathbb{P}}\big[|M(t)|^p\big] < \infty$, then, by Hunt's stopping time theorem, for fixed $t \geq 0$, $\big(M_n(t), \mathcal{F}_{t\wedge\zeta_n}, \mathbb{P}\big)$ is a discrete parameter martingale, and therefore, by Doob's inequality, $\mathbb{E}^{\mathbb{P}}\big[\sup_{n\geq 1} |M_n(t)|^p\big] < \infty$. Hence, by the monotone convergence and Lebesgue's dominated convergence theorems, (4.6.1) for M_n would imply it for M. When $\mathbb{E}^{\mathbb{P}}\big[|M(t)|^p\big] = \infty$, (4.2.6) is trivial.

(**ii**) Just follow the outline.