Homework Assignment #4

1: Define the Hermite polynomials \( \{ H_m \geq 0 \} \) by \( H_m(x) = (-1)^m e^{x^2} \partial^m e^{-x^2} \). Equivalently, 
\( H_0 = 1 \) and \( H_{m+1}(x) = (x - \partial)H_m(x) \).

(i) Show that \( H_m \) is an \( m \)th order polynomial in which the coefficient of \( x^m \) is 1 and the coefficient of \( x^k \) is 0 if and only if the parity of \( k \) is the same as that of \( m \). Furthermore, show that
\[
e^{\lambda x - \frac{\lambda^2}{2}} = \sum_{m=0}^{n} \frac{\lambda^m}{m!} H_m(x),
\]
and conclude that \( H_{2m}(0) = (-1)^m \frac{(2m)!}{2^m m!} = (-1)^m \prod_{k=1}^{m} (2k - 1) \). Hence,
\[
H_{2m}(x) = \sum_{k=0}^{m} c_{2m,2k} x^{2k} \quad \text{and} \quad H_{2m+1} = \sum_{k=0}^{m} c_{2m+1,2k+1} x^{2k+1},
\]
where \( c_{m,m} = 1 \) and \( c_{2m,0} = (-1) \prod_{k=1}^{m} (2k - 1) \).

(ii) Given \( a \geq 0 \), set
\[
H_{2m}(x,a) = \sum_{k=0}^{m} c_{2m,2k} a^{m-k} x^{2k} \quad \text{and} \quad H_{2m+1}(x,a) = \sum_{k=0}^{m} c_{2m+1,2k+1} a^{m-k} x^{2k+1},
\]
and show that
\[
e^{\lambda x - \frac{\lambda^2}{2a}} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} H_m(x,a) \quad \text{for all} \quad a \geq 0.
\]

2: Let \( I \) be an \( \mathbb{R} \)-valued and \( A \) a bounded, non-negative random variable, and assume that
\[
\mathbb{E}[e^{\xi I - \frac{\xi^2}{2} A}] = 1 \quad \text{for all} \quad \xi \in \mathbb{R}.
\]

(i) Show that
\[
\mathbb{E}[e^{\frac{I^2}{2(1 + \|A\|_u)}}] \leq (1 + \|A\|_u)^\frac{1}{2}.
\]

(ii) Show that, for each \( m \geq 0 \), \( \mathbb{E}[H_m(I,A)] = 0 \) and therefore that
\[
\sum_{k=0}^{m} c_{2m,2k} \mathbb{E}[X^{2k}A^{m-k}] = 0.
\]
In particular, conclude that \( \mathbb{E}[I^2] = \mathbb{E}[A] \).
(iii) Assume that \( m \geq 2 \). Using Hölder’s inequality and the fact that
\[
a^\theta b^{1-\theta} \leq \theta t^{\frac{1}{\theta}} a + (1 - \theta)t^{\frac{1}{1-\theta}} b
\]
for any \( a, b \geq 0, \theta \in [0, 1] \), and \( t > 0 \), show that
\[
\mathbb{E}[I^{2m}] \leq (-1)^{m+1} \frac{(2m)!}{2^m m!} \mathbb{E}[A^m] + f_m(t) \mathbb{E}[I^{2m}] + g_m(t) \mathbb{E}[A^m],
\]
where
\[
f_m(t) = \sum_{k=1}^{m-1} |c_{2m,2k}| t^{\frac{m}{k}} \quad \text{and} \quad g_m(t) = \sum_{k=1}^{m-1} |c_{2m,2k}| t^{\frac{m}{m-k}}.
\]
Conclude that if \( s_m > 0 \) is determined by \( f_m(s_m) = \frac{1}{2} \), then
\[
\mathbb{E}[I^{2m}] \leq 2 \left( (-1)^{m+1} \frac{(2m)!}{2^m m!} + g_m(s_m) \right) \mathbb{E}[A^m].
\]

(iv) Again assume that \( m \geq 2 \), determine \( t_m > 0 \) by \( g_m(t_m) = \frac{(2m)!}{2^{m+1} m!} \), and show that
\[
\mathbb{E}[A^m] \leq \frac{2^{m+1} m!}{(2m)!} \left((-1)^{m+1} + f_m(t_m)\right) \mathbb{E}[I^{2m}].
\]

By combining the preceding, one sees that, for each \( m \geq 1 \) there is a \( \kappa_{2m} \in [1, \infty) \) such that
\[
\kappa_{2m}^{-1} \mathbb{E}[A^m] \leq \mathbb{E}[I^{2m}] \leq \kappa_{2m} \mathbb{E}[A^m].
\]

3: Let \( (B(t), \mathcal{F}_t, \mathbb{P}) \) be an \( \mathbb{R}^M \)-valued Brownian motion, \( \eta \) an \( \mathbb{R}^M \)-valued, \( \{\mathcal{F}_t : t \geq 0\} \)-progressively measurable function for which
\[
\mathbb{E}^\mathbb{P} \left[ \int_0^t |\eta(\tau)|^2 d\tau \right] < \infty,
\]
and
\[
I_\eta(t) = \int_0^t (\eta(\tau), dB(\tau))_{\mathbb{R}^M}.
\]
Show that
\[
\kappa_{2m}^{-1} \mathbb{E}^\mathbb{P} \left[ \left( \int_0^\zeta |\eta(\tau)|^2 d\tau \right)^m \right] \leq \mathbb{E}^\mathbb{P} \left[ I_\eta(\zeta)^{2m} \right] \leq \kappa_{2m} \mathbb{E}^\mathbb{P} \left[ \left( \int_0^\zeta |\eta(\tau)|^2 d\tau \right)^m \right]
\]
for all \( m \geq 1 \) and \( \{\mathcal{F}_t : t \geq 0\} \)-stopping times \( \zeta \).