## Homework #3

**3.2**: The first part is trivial, and the second part follows when one writes $I_\zeta(\infty)$ as

$$\big((1,i),(I_{\boldsymbol{\eta}_1}(\infty),I_{\boldsymbol{\eta}_2}(\infty)\big)_{\mathbb{R}^2}.$$

**Exercise 3.3**: When $u\in C^{1,2}_{\mathrm{b}}(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$, this is simply an application of Itô's formula and Doob's stopping time theorem. To handle the general case, choose a sequence $\{\mathfrak{G}_n : n\ge 1\}$ of bounded, open sets such that $\overline{\mathfrak{G}_n}\subseteq\mathfrak{G}$ and $\mathfrak{G}_n\nearrow\mathfrak{G}$, and choose $\eta_n\in C^\infty\big(\mathbb{R}\times\mathbb{R}^N;[0,1]\big)$ so that $\eta_n=1$ on $\overline{\mathfrak{G}_n}$ and vanishes off of a compact subset of $\mathfrak{G}$. Then, by the preceding applied to $\nu_n u$ and $\mathfrak{G}_n$, the desired result holds with $\zeta_{s,\mathbf{x}}$ replaced by

$$\zeta_n=\inf\{t\ge 0:\ \big(s+w(t),\mathbf{x}+w(t)\big)\notin\mathfrak{G}_n\}.$$

Finally, observe that $t\wedge\zeta_n\nearrow t\wedge\zeta_{s,\mathbf{x}}$.

**Exercises 3.4 & 3.5**: Just follow the outlines.

**Exercise 3.6**: The initial assertions are easy applictions of Itô's formula and Doob's stopping time theorem.

(**i**) Because

$$\begin{aligned}\int_{\mathbb{R}} e^{\xi X-\frac{\xi^2}{2}A}\,\gamma_{0,1}(d\xi)&=(1+A)^{-\frac12}\int_{\mathbb{R}} e^{\xi X}\,\gamma_{0,(1+A)^{-1}}(d\xi)\\&=(1+A)^{-\frac12}e^{\frac{X^2}{2(1+A)}}\ge(1+\|A\|_{\mathrm{u}})^{-\frac12}e^{\frac{X^2}{2(1+\|A\|_{\mathrm{u}})}},\end{aligned}$$

the inequality becomes an application of Fubini's theorem.

(**ii**) Just follow the outline given.

(**iii**) Observe that $H_m(x,a)=a^{\frac m2}H_m(a^{-\frac12}x)$, and therefore

$$\sum_{m=0}^\infty\frac{\xi^m}{m!}H_m(x,a)=\sum_{m=0}^\infty\frac{(a^{\frac12}\xi)^m}{m!}H_m(a^{-\frac12}x)=e^{\xi x-\frac{\xi^2}{2}a}.$$

Hence,

$$\sum_{m=0}^\infty\frac{\xi^m}{m!}\mathbb{E}^{\mathbb{P}}\big[H_m(X,A)\big]=\mathbb{E}^{\mathbb{P}}\big[e^{\xi X-\frac{\xi^2}{2}A}\big]=1,$$

and so $\mathbb{E}^{\mathbb{P}}\big[H_m(X,A)\big]=0$ for $m\ge1$.

Because $\mathbb{E}^{\mathbb{P}}\big[H_{2m}(X,A)\big]=0$ for all $m\ge1$ and $H_2(X,A)=X^2-A$, $\mathbb{E}^{\mathbb{P}}\big[X^2\big]=\mathbb{E}^{\mathbb{P}}[A]$, and, for $m\ge2$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X^{2m}\big]&=-\sum_{k=0}^{m-1}c_{2k,2k}\mathbb{E}^{\mathbb{P}}\big[X^{2k}A^{m-k}\big]\\&\le(-1)^{m+1}\frac{(m)!}{2^m m!}\mathbb{E}^{\mathbb{P}}\big[A^m\big]+\sum_{k=1}^{m-1}|c_{2m,2k}|\Big(\tfrac{k}{m}t^{\frac mk}\mathbb{E}^{\mathbb{P}}\big[X^{2m}\big]+\tfrac{m-k}{m}t^{\frac{m}{m-k}}\mathbb{E}^{\mathbb{P}}\big[A^m\big]\Big).\end{aligned}$$

From this it is an easy step to the estimate of $\mathbb{E}^{\mathbb{P}}[X^{2m}]$ in terms of $\mathbb{E}^{\mathbb{P}}[A^m]$, and the complementary estimate is obtained in essentially the same way.

(**iv**) The first assertion requires no comment. Given it, the second assertion is an application of Fatou's lemma and the monotone convergence, and the final part follows from it and the inequality $|I_\sigma(t)|^{2m}\le N^{m-1}\sum_{j=1}^N\big(I_\sigma(t)_j\big)^{2m}$.

(**v**) When $A(\,\cdot\,)$ is bounded, this inequality is an easy application of the final one in (**iii**), and it follows in general by the monotone convergence theorem.