Homework Assignment #2

Exercise 2.1:

(**i**) Suppose that $\{Y_n : n \geq 1\} \subseteq \mathfrak{G}$ and that $Y_n \longrightarrow Y$ in $L^2(\mathbb{P};\mathbb{R})$. Then $m_n \equiv \mathbb{E}^{\mathbb{P}}[Y_n] \longrightarrow m \equiv \mathbb{E}^P[Y]$ and $V_n \equiv \mathrm{Var}(Y_n) \longrightarrow V \equiv \mathrm{Var}(Y)$, and therefore

$$\mathbb{E}^{\mathbb{P}}\left[e^{i\xi Y}\right] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\left[e^{i\xi Y_n}\right] = \lim_{n\to\infty} e^{i\xi m_n + \frac{\xi^2}{2}V_n} = e^{i\xi m + \frac{\xi^2}{2}V}.$$

Thus Y is Gaussian, and so $\overline{\mathfrak{G}}$ is a Gaussian family. The remaining assertions are trivial.

(**ii**) Clearly $\sigma(L) = \sigma(\tilde{L})$. Next,

$$X - \Pi_{\mathbf{1}\oplus L}X = \tilde{X} + \mathbb{E}^{\mathbb{P}}[X] - \Pi_{\mathbf{1}\oplus L}\tilde{X} - \mathbb{E}^{\mathbb{P}}[X] = \tilde{X} - \Pi_{\mathbf{1}\oplus L}\tilde{X}.$$

Since $\tilde{L} \subseteq \mathbf{1}\oplus L$, we will know that $\Pi_{\mathbf{1}\oplus L}\tilde{X} = \Pi_{\tilde{L}}\tilde{X}$ once we show that $\tilde{X} - \Pi_{\tilde{L}}\tilde{X} \perp \mathbf{1}\oplus L$. But $\mathbf{1} \perp \tilde{L}$. Thus if $Y \in \mathbf{1}\oplus L$ and $\tilde{Y} = Y - \mathbb{E}^{\mathbb{P}}[Y]$, then $\tilde{Y} \in \tilde{L}$ and so

$$\mathbb{E}^{\mathbb{P}}\left[(\tilde{X} - \Pi_{\tilde{L}}\tilde{X})Y\right] = \mathbb{E}^{\mathbb{P}}\left[(\tilde{X} - \Pi_{\tilde{L}}\tilde{X})\tilde{Y}\right] + \mathbb{E}^{\mathbb{P}}[Y]\mathbb{E}^{\mathbb{P}}\left[\tilde{X} - \Pi_{\tilde{L}}\tilde{X}\right] = 0.$$

Finally, by Lemma 2.1.1, $\tilde{X} - \Pi_{\tilde{L}}\tilde{X}$ is independent of $\sigma(\tilde{L}) = \sigma(L)$.

Exercise 2.2: Let $\mathfrak{G}$ be the centered Gaussian family generated by $\{B(t) : t \geq 0\}$. Since

$$\left\{\big(\xi, B(T)\big)_{\mathbb{R}^N} : \xi \in \mathbb{R}^N\right\} \cup \left\{\big(\xi, \theta_T(t)\big)_{\mathbb{R}^N} : t \geq 0 \ \&\ t \geq 0\right\} \subseteq \mathfrak{G},$$

the independence of $B(T)$ from $\sigma\big(\{\theta_T(t) : t \geq 0\}\big)$ follows from

$$\mathbb{E}^P\left[\big(\xi, B(T)\big)_{\mathbb{R}^N}\big(\eta, \theta_T(t)\big)\right] = (\xi,\eta)_{\mathbb{R}^N}\big(T\wedge t - Th_T(t)\big) = 0.$$

Next,

$$\mathbb{E}^P\left[\Phi\circ B(T),\, B(T) \in \Gamma\right] = \mathbb{E}^P\left[\Phi\circ\big(\theta_T + h_T B(T)\big)\right] = \int_\Gamma \mathbb{E}^P\left[\Phi\circ\big(\theta_T + h_T\mathbf{y}\big)\right]\gamma_{\mathbf{0},TI}(d\mathbf{y}).$$

Exercise 2.3: If $0 \leq s < t$, then $B(t) - B(s)$ is independent of $\mathcal{F}_s$ and therefore

$$\mathbb{E}^{\mathbb{P}}\left[E_\xi(t)\,\middle|\,\mathcal{F}_s\right] = E_\xi(s)e^{-\frac{\xi^2}{2}(t-s)}\mathbb{E}^{\mathbb{P}}\left[e^{\xi(B(t)-B(s))}\,\middle|\,\mathcal{F}_s\right] = E_\xi(s).$$

The reasoning given shows that $\mathbb{P}\big(\|B(\,\cdot\,)\|_{[0,t]} \geq R\big) \leq 2\exp\big(-\xi R + \frac{t\xi^2}{2}\big)$ for all $\xi \geq 0$, and so the $\mathbb{P}\big(\|B(\,\cdot\,)\|_{[0,t]} \geq R\big) \leq 2e^{-\frac{R^2}{2t}}$ follows when one takes $\xi = \frac{R}{t}$. When $N \geq 2$, observe that $\|B(\,\cdot\,)\|_{[0,t]} \leq N^{\frac{1}{2}}\max_{1\leq j\leq N}\|B(\,\cdot\,)_j\|_{[0,t]}$, and therefore

$$\begin{aligned}\mathbb{P}\big(\|B(\,\cdot\,)\|_{[0,t]} \geq R\big) &\leq \mathbb{P}\left(\max_{1\leq j\leq N}\|B(\,\cdot\,)_j\|_{[0,t]} \geq N^{\frac{1}{2}}R\right)\\ &\leq N\max_{1\leq j\leq N}\mathbb{P}\big(\|B(\,\cdot\,)_j\|_{[0,t]} \geq N^{\frac{1}{2}}R\big) \leq 2Ne^{-\frac{R^2}{2Nt}}.\end{aligned}$$

Exerise 2.4: Set $\breve{B}(0) = 0$ and $\breve{B}(t) = tB\left(\frac{1}{t}\right)$ for $t > 0$. Then $\big(\breve{B}(t), \mathcal{F}_{\frac{1}{t}}, \mathbb{P}\big)$ is a Brownian motion, and so

$$\lim_{t\to\infty}\frac{B(t)}{t} = \lim_{t\searrow 0}\breve{B}(t) = 0.$$

If A is orthogonal, then $\{T_A w(t) : t \geq 0\}$ generates a centered, Gaussian family with same coveriance as $\{w(t) : t \geq 0\}$ and therefore has the same distribution. If A is not orthogonal, then, as $n \to \infty$,

$$\sum_{m=1}^{2^n} \big(T_A w(m2^{-n}) - T_A w((m-1)2^{-n})\big) \otimes \big(T_A w(m2^{-n}) - T_A w((m-1)2^{-n})\big)$$

tends to $AA^\top \neq \mathbf{I}$ $\mathcal{W}$-almost surely, and therefore $(T_A)_*\mathcal{W} \perp \mathcal{W}$.

Exercise 2.5:

(**i**) If $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is a Brownian motion, then the argument given above in Exercise 2.2 shows that $\big(e^{i(\boldsymbol{\xi}, X(t))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}}, \mathcal{F}_t, \mathbb{P}\big)$ is a martingale. Conversely, if it is a martingale, then

$$\mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, X(t)-X(s))_{\mathbb{R}^N}} \,\big|\, \mathcal{F}_s\big] = e^{-\frac{|\boldsymbol{\xi}|^2}{2}(t-s)},$$

and so $\big(X(t), \mathcal{F}_t, \mathbb{P}\big)$ is a Brownian motion.

(**ii**) By Hunt's stopping time theorem,

$$\mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, B(t+\zeta))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}(t+\zeta)} \,\big|\, \mathcal{F}_{s+\zeta}\big] = e^{i(\boldsymbol{\xi}, B(s+\zeta))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}(s+\zeta)}.$$

Thus, since ζ and $B(\zeta)$ are $\mathcal{F}_{s+\zeta}$-measurable,

$$\mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, B(t+\zeta)-B(\zeta))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}t} \,\big|\, \mathcal{F}_{s+\zeta}\big] = e^{i(\boldsymbol{\xi}, B(s+\zeta)-B(\zeta))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}s},$$

and so, by (**i**), $\big(B(t+\zeta) - B(\zeta), \mathcal{F}_{t+\zeta}, \mathbb{P}\big)$ is a Brownian motion.

(**iii**) Let $A \in \mathcal{F}_s$, and check that $A \cap \{\zeta > s\} \in \mathcal{F}_{s\wedge\zeta} \subseteq \mathcal{F}_s \cap \mathcal{F}_{t\wedge\zeta}$. Using Hunt's stopping time theorem, one has

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, \check{B}(t))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}t}, A\big] &= \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, 2B(s\wedge\zeta)-B(t))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}t}, A \cap \{\zeta \leq s\}\big] \\
&\qquad + \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, 2B(t\wedge\zeta)-B(t))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}t}, A \cap \{\zeta > s\}\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, 2B(s\wedge\zeta)-B(s))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}s}, A \cap \{\zeta \leq s\}\big] \\
&\qquad + \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, B(t\wedge\zeta))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}t\wedge\zeta}, A \cap \{\zeta > s\}\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, \check{B}(s))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}s}, A \cap \{\zeta \leq s\}\big] + \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, B(s))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}s}, A \cap \{\zeta > s\}\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[e^{i(\boldsymbol{\xi}, \check{B}(s))_{\mathbb{R}^N} + \frac{|\boldsymbol{\xi}|^2}{2}s}, A\big].
\end{aligned}$$

Hence, by (**i**), $\big(\check{B}(t), \mathcal{F}_t, \mathbb{P}\big)$ is a Brownian motion.

(**iv**) & (**v**) Just follow the steps outlined.

(**vi**) Since $\big(B(t)_{N+1}, \mathcal{F}_t, \mathbb{P}\big)$ is an $\mathbb{R}$-valued Brownian motion, we know from (**v**) that $\mathbb{P}(\zeta \leq t) = 2\mathbb{P}\big(B(t)_{N+1} \geq a\big)$. In particular, this means that $\zeta < \infty$ (a.s., $\mathbb{P}$). Further, because $\{B(t)_{N+1} : t \geq 0\}$ is independent of $\{B(t)_j : 1 \leq j \leq N \;\&\; 1 \leq j \leq N\}$, the calulation in Exercise 1.4 justifies to

$$\begin{aligned}
\mathbb{P}(X \in \Gamma) &= \int_0^\infty \mathbb{P}\Big(\big(B(t)_1, \dots, B(t)_N\big) \in \Gamma\Big)\, \mathbb{P}(\zeta \in dt) \\
&\frac{2a}{\sqrt{2\pi}} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{1}{2t}} \gamma_{\mathbf{0}, t\mathbf{I}}(\Gamma)\, dt = \frac{2a}{\omega_N} \int_\Gamma \big(a^2 + |\mathbf{y}|^2\big)^{-\frac{N+1}{2}}\, d\mathbf{y}.
\end{aligned}$$

Exercise 2.6: Set $B_{m,n} = B(m2^{-n})$ and $\Delta_{m,n} = B_{m+1,n} - B_{m,n}$. Clearly,

$$B_{m+1,n}^2 - B_{m,n}^2 = \Delta_{m,n}^2 + 2B_{m,n}\Delta_{m,n} = -\Delta_{m,n}^2 + 2B_{m+1,n}\Delta_{m,n}.$$

From these, one has

$$B(1)^2 = \sum_{m=0}^{2^n-1} \Delta_{m,n}^2 + 2\sum_{m=0}^{2^n-1} B_{m,n}\Delta_{m,n} \text{ and } B(1)^2 = -\sum_{m=0}^{2^n-1} \Delta_{m,n}^2 + 2\sum_{m=0}^{2^n-1} B_{m+1,n}\Delta_{m,n},$$

and so, by (2.1.2), the first and third equations are proved. To prove the second equation, note that

$$\begin{aligned} B_{m+1,n}^2 - B_{m,n}^2 - 2B_{2m+1,n+1}\Delta_{m,n} &= \big(B_{m+1,n} + B_{m,n} - 2B_{2m+1,n+1}\big)\Delta_{m,n} \\ &= \Delta_{2m+1,n+1}\Delta_{m,n} - \Delta_{2m,n+1}\Delta_{m,n} = \Delta_{2m+1,n+1}^2 - \Delta_{2m,n+1}^2. \end{aligned}$$

Next, proceeding in exactly the same way as in the derivation of (2.1.2), one sees that

$$\lim_{n\to\infty} \sum_{m=0}^{2^n-1} \Delta_{2m+1,n+1}^2 = \frac{1}{2} = \lim_{n\to\infty} \sum_{m=0}^{2^n-1} \Delta_{2m,n+1}^2,$$

and therefore, after another application of (2.1.2), that

$$B(1)^2 = 2\lim_{n\to\infty} \sum_{m=0}^{2^n-1} B_{2m+1,n+1}\Delta_{m,n}.$$