Homework Assignment #1

1: Given \( m \in \mathbb{R}^N \) and a non-negative definite, symmetric matrix \( C \), define the Borel measure \( \gamma_{m,C} \) on \( \mathbb{R}^N \) to be the distribution of \( y \sim m + C^{\frac{1}{2}}y \) under the standard Gaussian measure

\[
\gamma_{0,1}(y) = (2\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{2}} dy.
\]

Show that

\[
\widehat{\gamma_{0,C}}(\xi) = e^{-\frac{(\xi, C\xi)}{2}}.
\]

In addition, if \( C \) is non-degenerate, show that

\[
\gamma_{0,C}(y) = ((2\pi)^N \det(C))^{-\frac{1}{2}} \exp\left( -\frac{(y, C^{-1}y)_{\mathbb{R}^N}}{2} \right) dy.
\]

2: Show that

\[
\int_{\mathbb{R}^N \setminus \{0\}} \left( e^{i(\xi, y)_{\mathbb{R}^N}} - 1 - B(0, 1)(y)(\xi, y)_{\mathbb{R}^N} \right) \frac{dy}{|y|^{N+1}} = |\xi| \int_{\mathbb{R}^N \setminus \{0\}} \left( \cos(e, y)_{\mathbb{R}^N} - 1 \right) \frac{dy}{|y|^{N+1}}
\]

for any \( e \in S^{N-1} \), and conclude that there is a \( c > 0 \) such the \( \ell(\xi) \) corresponding to \( m = 0, C = 0 \), and the Lévy measure \( M(dy) = c1_{\mathbb{R}^N \setminus \{0\}}(y)|y|^{-N-2} dy \) is equal to \(-|\xi|\). The associated infinitely divisable laws are called Cauchy distributions. To see what they look like, begin by showing that

\[
\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{a^2}{2t} - \frac{b^2}{2t}} dt = \frac{\sqrt{2\pi}e^{-ab}}{b}
\]

and then that

\[
\int_0^\infty t^{-\frac{3}{2}} e^{-\frac{a^2}{2t} - \frac{b^2}{2t}} dt = \frac{\sqrt{2\pi}e^{-ab}}{a}.
\]

To do the first of these, try the change of variable \( \tau = bt^{-\frac{1}{2}} - at^{-\frac{1}{2}} \), and get the second by differentiating the first with respect to \( a \). Now apply the second one to see that

\[
\int_0^\infty \tau^{-\frac{3}{2}} \widehat{\gamma_{0,1}}(\xi) d\tau = e^{-|\xi|}.
\]

and conclude from this that if

\[
P_t(dy) = \left( \frac{t}{(2\pi)^{\frac{N+1}{2}}} \int_0^\infty \tau^{-\frac{N+3}{2}} e^{-\frac{|y|^2}{2\tau} + \frac{t^2}{2\tau}} d\tau \right) dy = \frac{2}{\omega_N} \left( \frac{t}{t^2 + |y|^2} \right)^{\frac{N+1}{2}} dy,
\]

where \( \omega_N = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \) is the surface area of \( S^N \), then

\[
\hat{P}_t(\xi) = e^{-|\xi|}.
\]

Finally, use this to show that the constant \( c \) above is \( \frac{2}{\omega_N} \).