Homework #1

Exercise 1.1: It is a simple calculus exercise to check that if $\partial_t u = \frac{1}{2}\Delta u$ and $v(t,\mathbf{x}) = u\big(\frac{1-e^{-2t}}{2}, \mathbf{y} - e^{-t}\mathbf{x}\big)$, then $\partial_t v = Lv$. Thus, if $\varphi \in C(\mathbb{R}^N;\mathbb{R})$ and

$$v(t,\mathbf{x}) = \int_{\mathbb{R}^N} \varphi(\mathbf{y}) g\left(\frac{1-e^{-2t}}{2}, \mathbf{y} - e^{-t}\mathbf{x}\right) d\mathbf{y},$$

then $\partial_t v = Lv$ in $(0,\infty)\times\mathbb{R}^N$ and $\lim_{t\searrow 0} v(t,\,\cdot\,) = \varphi$. Therefore,

$$\frac{d}{d\tau}\langle v(t-\tau,\,\cdot\,), P(\tau,\mathbf{x},\,\cdot\,)\rangle = 0 \text{ for } \tau \in (0,t),$$

and so

$$v(t,\mathbf{x}) = \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, P(t,\mathbf{x},d\mathbf{y}).$$

Equivalently $P(t,\mathbf{x},d\mathbf{y}) = g\big(\frac{1-e^{-2t}}{2}, \mathbf{y} - e^{-t}\mathbf{x}\big)$.

Exercise 1.2: To prove $(**)$, suppose that $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. Then $\hat\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$ and

$$\sup_{n\ge 1} n\big|\big(e^{-\frac{i}{n}\ell(\boldsymbol{\xi})} - 1\big)\hat\varphi(\boldsymbol{\xi})\big| \le |\ell(\boldsymbol{\xi})||\hat\varphi(\boldsymbol{\xi})|.$$

Hence, by Parseval's indentity, (i) & (ii), and Lebesgue's dominated convergence theorem,

$$(2\pi)^N n\big(\langle\varphi,\mu_{\frac{1}{n}}\rangle - \varphi(\mathbf{0})\big) = n\int_{\mathbb{R}^N}\big(e^{\frac{i}{n}\ell(-\boldsymbol{\xi})} - 1\big)\hat\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi} \longrightarrow \int_{\mathbb{R}^N}\ell(-\boldsymbol{\xi})\hat\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

Since $\varphi \ge \varphi(\mathbf{0}) \implies \langle\varphi,\mu_{\frac{1}{n}}\rangle \ge \varphi(\mathbf{0})$, it is clear that A satisfied the minimum principle. Next suppose that $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. Then $\widehat{\varphi_R}(\boldsymbol{\xi}) = R^N\hat\varphi(R\boldsymbol{\xi})$, and so, by (ii) and Lebesgue's dominated convergence theorem,

$$(2\pi)^N A\varphi_R = R^N\int_{\mathbb{R}^N}\ell(-\boldsymbol{\xi})\hat\varphi(R\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int_{\mathbb{R}^N}\ell(-R^{-1}\boldsymbol{\xi})\hat\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi} \longrightarrow 0$$

as $R\to\infty$. Hence, by Theorem 1.1.1, there is a Lévy system $(\mathbf{m},C,M)$ for which (1.1.3) holds. Equivalently, for any $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$,

$$\int_{\mathbb{R}^N}\ell(-\boldsymbol{\xi})\hat\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\Big(e^{-i(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + i\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\,M(d\mathbf{y})\right)\hat\varphi(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

and so ℓ is given by the expression in $(*)$.

Exercise 1.3: Because $\hat\mu = (\widehat{\mu}_{\frac{1}{n}})^n$, $\hat\mu = e^{n\ell_{\frac{1}{n}}}$ on $B(\mathbf{0},r)$, and therefore, by uniqueness, $\ell = n\ell_{\frac{1}{n}}$.

Next, using the power series for log, one sees that

$$|\log z| \le \sum_{m=1}^\infty \frac{|1-z|^m}{m} \le |1-z|\sum_{m=0}^\infty |1-z|^m \le 2|1-z| \text{ if } |1-z| \le \frac{1}{2}.$$

Hence $|\ell(\boldsymbol{\xi})| \le 2|1-\hat\mu(\boldsymbol{\xi})| \le 1$ for $|\boldsymbol{\xi}| \le r$, and so, since $\mathfrak{Re}\big(\ell(\boldsymbol{\xi})\big) \le 0$,

$$\big|1 - \widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})\big| = \big|1 - e^{\frac{1}{n}\ell(\boldsymbol{\xi})}\big| \le \frac{|\ell(\boldsymbol{\xi})|}{n} \le \frac{1}{n} \text{ for } |\boldsymbol{\xi}| \le r.$$

Let $\mathbf{e} \in \mathbb{S}^{N-1}$ be given. By averaging the inequality

$$\big|1 - \widehat{\mu_{\frac{1}{n}}}(\rho\mathbf{e})\big| \ge \int_{\mathbb{R}^N}\big(1 - \cos\rho(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}\,\mu_{\frac{1}{n}}(d\mathbf{y})$$

with respect to $\rho \in [0, r]$, one sees that

$$\frac{1}{n} \geq \int_{\mathbb{R}^N} \left(1 - \frac{\sin r(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}}{r(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}}\right) \mu_{\frac{1}{n}}(d\mathbf{y}) \geq s(rR)\mu_{\frac{1}{n}}\big(\{\mathbf{y} : |(\mathbf{e}, \mathbf{y})| \geq R\}\big),$$

where

$$s(T) \equiv \inf_{t \geq T} \left(1 - \frac{\sin t}{t}\right).$$

Since $\sin t = \int_0^t \cos\tau \, d\tau < t$ and $\frac{\sin t}{t} \longrightarrow 0$ as $t \to \infty$, $s(T) > 0$ for all $T > 0$, and therefore we now know that

$$\mu_{\frac{1}{n}}\big(\{\mathbf{y} : |(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}| \geq R\}\big) \leq \frac{1}{ns(rR)}.$$

Hence, since, for any $\rho > 0$,

$$\big|1 - \widehat{\mu}_{\frac{1}{n}}(\rho\mathbf{e})\big| \leq \int_{\mathbb{R}^N} \big|1 - e^{i\rho(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}}\big| \, \mu_{\frac{1}{n}}(d\mathbf{y}) \leq \rho R + 2\mu_{\frac{1}{n}}\big(\{\mathbf{y} : |(\mathbf{e}, \mathbf{y})| \geq R\}\big),$$

by taking $R = \frac{1}{4\rho}$, we have that

$$\sup_{|\boldsymbol{\xi}| \leq \rho} \big|1 - \widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})\big| \leq \frac{1}{4} + \frac{2}{ns\big(\frac{r}{4\rho}\big)}$$

and therefore that $(**)$ holds. In particular, this means that ℓ admits a continuous extention to the whole of $\mathbb{R}^N$ such that $\widehat{\mu_{\frac{1}{n}}} = e^{\frac{1}{n}\ell}$.

Since $|\ell(\boldsymbol{\xi})| \leq 2n\big|1 - \widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})\big|$ if $\big|1 - \widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})\big| \leq \frac{1}{2}$, $(**)$ implies that $|\ell(\boldsymbol{\xi})| \leq n$ if $|\boldsymbol{\xi}| \leq \rho$ and $n \geq \frac{8}{s\left(\frac{r}{4\rho}\right)}$. By using the Taylor series for sin, it is easy to check that $\lim_{t \searrow 0} t^{-2}\big(1 - \frac{\sin t}{t}\big) = \frac{1}{6}$, and therefore there exists a $T_0 \in (0, 1]$ such that $1 - \frac{\sin t}{t} \geq \frac{t^2}{12}$ for $t \in (0, T_0]$. Hence, $s(T) \geq \epsilon(T \wedge 1)^2$ where $\epsilon = \frac{1}{12} \wedge s(T_0)$. Finally, suppose that $|\boldsymbol{\xi}| \geq \frac{r}{4}$ and choose $n \geq 1$ so that $n \geq \frac{16|\boldsymbol{\xi}|^2}{\epsilon r^2} > n - 1$. Then $|\ell(\boldsymbol{\xi})| \leq n$, and so one can find a $C < \infty$ such that $|\ell(\boldsymbol{\xi})| \leq C(1 + |\boldsymbol{\xi}|^2)$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$.

Exercise 1.4: Begin with the observation that, because of the rotation invariance of Lebesgue measure,

$$\boldsymbol{\xi} \rightsquigarrow \int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} \left(e^{i(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1 - i\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}\right) \frac{d\mathbf{y})}{|\mathbf{y}|^{N+1}}$$

is a function of $|\boldsymbol{\xi}|$. Next let $\mathbf{e} \in \mathbb{S}^{N-1}$ and $\rho > 0$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} \left(e^{i\rho(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}} - 1 - i\rho\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}\right) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}} \\
&= \lim_{r \searrow 0} \int_{|\mathbf{y}| \geq r} \big(\cos\rho(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N} - 1\big) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}} \\
&\qquad + i \int_{|\mathbf{y}| \geq r} \big(\sin\rho(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N} - \rho\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}\big) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}} \\
&= \rho \lim_{r \searrow 0} \int_{|\mathbf{y}| \geq r} \big(\cos(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N} - 1\big) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}} = \rho \int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} \big(\cos(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N} - 1\big) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}}.
\end{aligned}$$

Thus, if

$$c = -\left(\int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} \big(\cos(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N} - 1\big) \frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}}\right)^{-1}$$

and $M(d\mathbf{y}) = c\mathbf{1}_{\mathbb{R}^N\setminus\{\mathbf{0}\}}\frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}}$, then the ℓ corresponding to the Lévy system $(\mathbf{0}, 0, M)$ is given by $\ell(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|$.

Turning to the second part, use the quadratic formula to see that $t^{\frac{1}{2}} = \frac{\tau+\sqrt{\tau^2+4ab}}{2b}$ and therefore that

$$t^{-\frac{1}{2}}dt = b^{-1}d\tau + \frac{\tau}{b\sqrt{\tau^2+4ab}}d\tau.$$

Since the coefficient in the second term is odd and $\tau^2 + 2ab = b^2t + \frac{a^2}{t}$, this means that

$$\int_0^\infty t^{-\frac{1}{2}}e^{-\frac{a^2}{2t}-\frac{b^2t}{2}}\,dt = \frac{e^{-ab}}{b}\int_{\mathbb{R}} e^{-\frac{\tau^2}{2}}\,d\tau = \frac{\sqrt{2\pi}e^{-ab}}{b},$$

from which

$$\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{a^2}{2t}-\frac{b^2t}{2}}\,dt = \frac{\sqrt{2\pi}e^{-ab}}{a}$$

follows when one differentiates with respect to a.

Because $\widehat{\gamma_{\mathbf{0},\tau\mathbf{I}}}(\boldsymbol{\xi}) = e^{-\frac{\tau|\boldsymbol{\xi}|^2}{2}}$, it follows that

$$t\int_0^\infty \tau^{-\frac{3}{2}}e^{-\frac{t^2}{2\tau}}\widehat{\gamma_{\mathbf{0},\tau\mathbf{I}}}\,d\tau = \sqrt{2\pi}e^{-t|\boldsymbol{\xi}|}.$$

Hence, if

$$P_t(d\mathbf{y}) = \frac{t}{\sqrt{2\pi}}\int_0^\infty \tau^{-\frac{3}{2}}e^{-\frac{t^2}{2\tau}}\gamma_{\mathbf{0},\tau\mathbf{I}}(d\mathbf{y})\,d\tau,$$

then $\widehat{P}_t(\boldsymbol{\xi}) = e^{-t|\boldsymbol{\xi}|}$, and, after a making the change of variables $\tau \to \frac{1}{\tau}$, one sees that

$$P_t(d\mathbf{y}) = \frac{1}{\omega_N}\frac{t}{(t^2+|\mathbf{y}|^2)^{\frac{N+1}{2}}}\,d\mathbf{y}.$$

Finally, since, on the one hand,

$$\frac{\widehat{P}_t(\mathbf{e})-1}{t} = \frac{2}{\omega_N}\int_{\mathbb{R}^N}\frac{e^{i(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}}-1}{(t^2+|\mathbf{y}|^2)^{\frac{N+1}{2}}}\,d\mathbf{y} = \frac{2}{\omega_N}\int_{\mathbb{R}^N}\frac{\cos(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}-1}{(t^2+|\mathbf{y}|^2)^{\frac{N+1}{2}}}\,d\mathbf{y}$$
$$\longrightarrow \frac{2}{\omega_N}\int_{\mathbb{R}^N}\big(\cos(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}-1\big)\,\frac{d\mathbf{y}}{|\mathbf{y}|^{N+1}},$$

and, on the other hand,

$$\frac{\widehat{P}_t(\mathbf{e})-1}{t} = \frac{e^{-t}-1}{t} \longrightarrow -1$$

as $t \searrow 0$, it follows that $c = \frac{2}{\omega_N}$.