The Lévy-Khinchine Formula

Given the characterization of linear functionals that satisfy the minimum principle and are quasi-local, it is quite easy to derive the Lévy-Khinchine formula.

Let $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ be infinitely devisable in the sence that, for each $n \geq 1$, there is a $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$ such that $\mu = \mu_{\frac{1}{n}}^{*n}$. To prove that there is a Lévy system $(\mathbf{m}, \mathbf{C}, M)$ such that $\hat{\mu} = e^{\ell}$ where

$$\ell(\boldsymbol{\xi}) = i(\mathbf{m}, \boldsymbol{\xi})_{\mathbb{R}^N} - \tfrac{1}{2}(\boldsymbol{\xi}, \mathbf{C}\boldsymbol{\xi})_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \Big(e^{i(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - i\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}\Big) M(d\mathbf{y}),$$

it suffices to show that

(i) $\hat{\mu}$ never vanishes

(ii) If $\ell \in C(\mathbb{R}^N; \mathbb{C})$ is determined by $\ell(\mathbf{0}) = 1$ and $\hat{\mu} = e^{\ell}$, then $|\ell(\boldsymbol{\xi})| \leq C\big(1 + |\boldsymbol{\xi}|^2\big)$ for some $C < \infty$.

Indeed, (i) guarantees that the ℓ described in (ii) exists and is unique. Further, because $\hat{\mu} = (\widehat{\mu_{\frac{1}{n}}})^n$, (i) guarantees that $\widehat{\mu_{\frac{1}{n}}}$ never vanishes, and clearly $\frac{1}{n}\ell$ is the unique choice of $\ell_{\frac{1}{n}} \in C(\mathbb{R}^N; \mathbb{C})$ such that $\ell_{\frac{1}{n}}(\mathbf{0}) = 0$ and $\widehat{\mu_{\frac{1}{n}}} = e^{\ell_{\frac{1}{n}}}$. Hence, using (ii) and Parseval's Identity, one sees, that

$$A\varphi \equiv \lim_{n\to\infty} n\big(\langle \varphi, \mu_{\frac{1}{n}}\rangle - \varphi(\mathbf{0})\big) = (2\pi)^{-N} \int_{\mathbb{R}^N} \ell(-\boldsymbol{\xi})\hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

for $\mathscr{S}(\mathbb{R}^N; \mathbb{C})$. Obviously A satisfies the minimum principle. In addition, if $\varphi \in \mathscr{S}(\mathbb{R}^N; \mathbb{C})$, then

$$(2\pi)^N A\varphi_R = R^N \int_{\mathbb{R}^N} \ell(-\boldsymbol{\xi})\hat{\varphi}(R\boldsymbol{\xi})\, d\boldsymbol{\xi} = \int_{\mathbb{R}^N} \ell(-R^{-1}\boldsymbol{\xi})\hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Thus A is quasi-local. As a consequence, there exists a $(\mathbf{m}, \mathbf{C}, M)$ such that

$$\begin{aligned} A\varphi = (2\pi)^{-N} \int_{\mathbb{R}^N} \Bigg(& i(\mathbf{m}, \boldsymbol{\xi})_{\mathbb{R}^N} - \tfrac{1}{2}(\boldsymbol{\xi}, \mathbf{C}\boldsymbol{\xi})_{\mathbb{R}^N} \\ & + \int_{\mathbb{R}^N} \Big(e^{i(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - i\mathbf{1}_{B(\mathbf{0},1)}(\mathbf{y})(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}\Big) M(\mathbf{y}) \Bigg) \hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \end{aligned}$$

To verify that (i) and (ii) hold, begin by observing that

$$|1 - \hat{\mu}(\boldsymbol{\xi})| \geq \mathfrak{Re}\big(1 - \hat{\mu}(\boldsymbol{\xi})\big) = \int_{\mathbb{R}^N} \big(1 - \cos(\boldsymbol{\xi}, \mathbf{y})\big)\, \mu(d\mathbf{y}),$$

and therefore that, for any $\mathbf{e} \in \mathbb{S}^{N-1}$,

$$\sup_{|\boldsymbol{\xi}|\leq r} |1 - \hat{\mu}(\boldsymbol{\xi})| \geq \frac{1}{r} \int_{\mathbb{R}^N} \left(\int_0^r \big(1 - \cos t(\boldsymbol{\xi}, \mathbf{y})\big)\, dt \right) \mu(d\mathbf{y}) = \int_{\mathbb{R}^N} \left(1 - \frac{\sin r(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}}{r(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}} \right) \mu(d\mathbf{y}).$$

Hence, if $s(T) \equiv \inf_{|t|\geq T}\Big(1-\frac{\sin r(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}}{r(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}}\Big)$ for $T>0$, then, $s>0$ and, for any $R>0$,

$$(*)\qquad \sup_{|\boldsymbol{\xi}|\leq r}|1-\hat{\mu}(\boldsymbol{\xi})|\geq s(rR)\mu\big(\{\mathbf{y}:\,|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\geq R\}\big).$$

Now choose $r>0$ so that $\sup_{|\boldsymbol{\xi}|\leq r}|1-\hat{\mu}(\boldsymbol{\xi})|\leq\frac{1}{2}$. Then, since $|1-z|\leq\frac{1}{2}\implies|\log z|\leq 2|1-z|$, $|\ell(\boldsymbol{\xi})|\leq 1$, and so

$$|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})|=\big|1-e^{\frac{\ell(\boldsymbol{\xi})}{n}}\big|\leq\frac{1}{n}$$

for $|\boldsymbol{\xi}|\leq r$. Applying (*) with μ replaced by $\mu_{\frac{1}{n}}$ and using this estimate, we have that, for any $R>0$,

$$\frac{1}{n}\geq s(rR)\mu_{\frac{1}{n}}\big(\{\mathbf{y}:\,|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\geq R\}\big),$$

which, because

$$|1-\widehat{\mu_{\frac{1}{n}}}(\rho\mathbf{e})|\leq\rho R+2\mu_{\frac{1}{n}}\big(\{\mathbf{y}:\,|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\geq R\}\big),$$

means that

$$\sup_{|\boldsymbol{\xi}|\leq\rho}|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})|\leq\rho R+\frac{2}{ns(rR)}\quad\text{for all }(\rho,R)\in(0,\infty).$$

In particularly, by taking $R=\frac{1}{4\rho}$, we get

$$\sup_{|\boldsymbol{\xi}|\leq\rho}|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})|\leq\frac{1}{4}+\frac{2}{ns\big(\frac{r}{4\rho}\big)},$$

and so, for all $\rho>0$,

$$(**)\qquad \sup_{|\boldsymbol{\xi}|\leq\rho}|1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})|\leq\frac{1}{2}\quad\text{if }n\geq\frac{8}{s\big(\frac{r}{4\rho}\big)}.$$

From (**), it is clear that, for each $\rho>0$, there is an n such that $|\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})|\geq\frac{1}{2}$ and therefore $|\hat{\mu}(\boldsymbol{\xi})|\geq 2^{-n}$ for $|\boldsymbol{\xi}|\leq\rho$, and this completes the proof of (i). Finally, from (**), $|\ell(\boldsymbol{\xi})|\leq n$ if $|\boldsymbol{\xi}|\leq\rho$ and $n\geq\frac{8}{s\big(\frac{r}{4\rho}\big)}$. Since there is an ϵ such that $s(T)\geq\epsilon T^2$ for $T\in(0,1]$, it follows that

$$|\ell(\boldsymbol{\xi})|\leq 1+\frac{128|\boldsymbol{\xi}|^2}{\epsilon r^2}.$$