Brownian Motion, the Gaussian Lévy Process

Deconstructing Brownian Motion: My construction of Brownian motion is based on an idea of Lévy's; and in order to explain Lévy's idea, I will begin with the following line of reasoning.

Assume that $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion in $\mathbb{R}^N$. That is, $\{\mathbf{B}(t) : t \geq 0\}$ starts at $\mathbf{0}$, has independent increments, any increment $\mathbf{B}(s+t) - \mathbf{B}(s)$ has distribution $\gamma_{\mathbf{0},t\mathbf{I}}$, and the paths $t \rightsquigarrow \mathbf{B}(t)$ are continuous. Next, given $n \in \mathbb{N}$, let $t \rightsquigarrow \mathbf{B}_n(t)$ be the polygonal path obtained from $t \rightsquigarrow \mathbf{B}(t)$ by linear interpolation during each time interval $[m2^{-n}, (m+1)2^{-n}]$. Thus,

$$\mathbf{B}_n(t) = \mathbf{B}(m2^{-n}) + 2^n\big(t - m2^{-n}\big)\Big(\mathbf{B}\big((m+1)2^{-n}\big) - \mathbf{B}(m2^{-n})\Big)$$

for $m2^{-n} \leq t \leq (m+1)2^{-n}$. The distribution of $\{\mathbf{B}_0(t) : t \geq 0\}$ is very easy to understand. Namely, if $\mathbf{X}_{m,0} = \mathbf{B}(m) - \mathbf{B}(m-1)$ for $m \geq 1$, then the $\mathbf{X}_{m,0}$'s are independent, standard normal $\mathbb{R}^N$-valued random variables, $\mathbf{B}_0(m) = \sum_{1 \leq m \leq n} \mathbf{X}_{m,0}$, and $\mathbf{B}_0(t) = (m-t)\mathbf{B}_0(m-1) + (t-m+1)\mathbf{B}_0(m)$ for $m-1 \leq t \leq m$. To understand the relationship between successive $\mathbf{B}_n$'s, observe that $\mathbf{B}_{n+1}(m2^{-n}) = \mathbf{B}_n(m2^{-n})$ for all $m \in \mathbb{N}$ and that

$$\begin{aligned}
\mathbf{X}_{m,n+1} &\equiv 2^{\frac{n}{2}+1}\Big(\mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) - \mathbf{B}_n\big((2m-1)2^{-n-1}\big)\Big) \\
&= 2^{\frac{n}{2}+1}\left(\mathbf{B}\big((2m-1)2^{-n-1}\big) - \frac{\mathbf{B}\big(m2^{-n}\big) + \mathbf{B}\big((m-1)2^{-n}\big)}{2}\right) \\
&= 2^{\frac{n}{2}}\Big[\Big(\mathbf{B}\big((2m-1)2^{-n-1}\big) - \mathbf{B}\big((m-1)2^{-n}\big)\Big) \\
&\qquad - \Big(\mathbf{B}\big(m2^{-n}\big) - \mathbf{B}\big((2m-1)2^{-n-1}\big)\Big)\Big],
\end{aligned}$$

and therefore $\{\mathbf{X}_{m,n+1} : m \geq 1\}$ is again a sequence of independent standard normal random variables. What is less obvious is that $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ is also a family of independent random variables. In fact, checking this requires us to make essential use of the fact that we are dealing with Gaussian random variables.

In preparation for proving the preceding independence assertion, say that $\mathfrak{G} \subseteq L^2(\mathbb{P};\mathbb{R})$ is a **Gaussian family** if $\mathfrak{G}$ is a linear subspace and each element of $\mathfrak{G}$ is a **centered** (i.e., mean value 0), $\mathbb{R}$-valued Gaussian random variable. My interest in Gaussian families at this point is that the linear span $\mathfrak{G}(\mathbf{B})$ of $\big\{\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N} : t \geq 0 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N\big\}$ is one. To see this, simply note that, for any $0 = t_0 < t_1 < \cdots t_n$ and $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n \in \mathbb{R}^N$,

$$\sum_{m=1}^{n}\big(\boldsymbol{\xi}_m, \mathbf{B}(t_m)\big)_{\mathbb{R}^N} = \sum_{\ell=1}^{n}\left(\sum_{m=\ell}^{n}\big(\boldsymbol{\xi}_m, \mathbf{B}(t_\ell) - \mathbf{B}(t_{\ell-1})\big)_{\mathbb{R}^N}\right)_{\mathbb{R}^N},$$

which, as a linear combination of independent centered Gaussians, is itself a centered Gaussian.

The crucial fact about Gaussian families is the content of the next lemma.

LEMMA 1. *Suppose that $\mathfrak{G} \subseteq L^2(\mathbb{P};\mathbb{R})$ is a Gaussian family. Then the closure of $\mathfrak{G}$ in $L^2(\mathbb{P};\mathbb{R})$ is again a Gaussian family. Moreover, for any $S \subseteq \mathfrak{G}$, S is independent of $S^\perp \cap \mathfrak{G}$, where $S^\perp$ is the orthogonal complement of S in $L^2(\mathbb{P};\mathbb{R})$.*

PROOF: The first assertion is easy since Gaussian random variables are closed under convergence in probability.

Turning to the second part, what I must show is that if $X_1, \dots, X_n \in S$ and $X'_1, \dots, X'_n \in S^\perp \cap \mathfrak{G}$, then

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m} \prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right] = \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m}\right] \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right]$$

for any choice of $\{\xi_m : 1 \le m \le n\} \cup \{\xi'_m : 1 \le m \le n\} \subseteq \mathbb{R}$. But the expectation value on the left is equal to

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \left(\xi_m X_m + \xi'_m X'_m\right)\right)^2\right]\right) \\
&\quad = \exp\left(-\frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \xi_m X_m\right)^2\right] - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=1}^n \xi'_m X'_m\right)^2\right]\right) \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi_m X_m}\right] \mathbb{E}^{\mathbb{P}}\left[\prod_{m=1}^n e^{\sqrt{-1}\,\xi'_m X'_m}\right],
\end{aligned}$$

since $\mathbb{E}^{\mathbb{P}}[X_m X'_{m'}] = 0$ for all $1 \le m, m' \le n$. $\square$

Armed with Lemma 1, we can now check that $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ is independent. Indeed, since, for all $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$ and $\boldsymbol{\xi} \in \mathbb{R}^N$, $\big(\boldsymbol{\xi}, \mathbf{X}_{m,n}\big)_{\mathbb{R}^N}$ a member of the Gaussian family $\mathfrak{G}(\mathbf{B})$, all that we have to do is check that, for each $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$, $\ell \in \mathbb{N}$, and $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in (\mathbb{R}^N)^2$,

$$\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{X}_{m,n+1}\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\big)_{\mathbb{R}^N}\big] = 0.$$

But, since, for $s \le t$, $\mathbf{B}(s)$ is independent of $\mathbf{B}(t) - \mathbf{B}(s)$,

$$\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big] = \mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(s)\big)_{\mathbb{R}^N}\big] = s\big(\boldsymbol{\xi}, \boldsymbol{\eta}\big)_{\mathbb{R}^N}$$

and therefore

$$\begin{aligned}
&2^{-\frac{n}{2}-1}\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}, \mathbf{X}_{m,n+1}\big)_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\big)_{\mathbb{R}^N}\big] \\
&\quad = \mathbb{E}^{\mathbb{P}}\Big[\Big(\boldsymbol{\xi}, \mathbf{B}\big((2m-1)2^{-n-1}\big)\Big)_{\mathbb{R}^N}\Big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\Big)_{\mathbb{R}^N}\Big] \\
&\qquad\qquad - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\Big[\Big(\boldsymbol{\xi}, \mathbf{B}\big(m2^{-n}\big) + \mathbf{B}\big((m-1)2^{-n}\big)\Big)_{\mathbb{R}^N}\Big(\boldsymbol{\eta}, \mathbf{B}(\ell 2^{-n})\Big)_{\mathbb{R}^N}\Big] \\
&\quad = 2^{-n}\big(\boldsymbol{\xi}, \boldsymbol{\eta}\big)_{\mathbb{R}^N}\left[\big(m - \tfrac{1}{2}\big) \wedge \ell - \frac{m \wedge \ell + (m-1) \wedge \ell}{2}\right] = 0.
\end{aligned}$$

Lévy's Construction of Brownian Motion: Lévy's idea was to invert the reasoning just given. That is, start with a family $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ of independent $N(\mathbf{0},\mathbf{I})$-random variables. Next, define $\{\mathbf{B}_n(t) : t \geq 0\}$ inductively so that $t \rightsquigarrow \mathbf{B}_n(t)$ is linear on each interval $[(m-1)2^{-n}, m2^{-n}]$, $\mathbf{B}_0(m) = \sum_{1\leq \ell \leq m} \mathbf{X}_{\ell,0}$, $m \in \mathbb{N}$, $\mathbf{B}_{n+1}(m2^{-n}) = \mathbf{B}_n(m2^{-n})$ for $m \in \mathbb{N}$, and

$$\mathbf{B}_{n+1}\big((2m-1)2^{-n}\big) = \mathbf{B}_n\big((2m-1)2^{-n-1}\big) + 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1} \quad \text{for } m \in \mathbb{Z}^+.$$

If Brownian motion exists, then the distribution of $\{\mathbf{B}_n(t) : t \geq 0\}$ is the distribution of the process obtained by polygonalizing it on each of the intervals $[(m-1)2^{-n}, m2^{-n}]$, and so the limit $\lim_{n\to\infty} \mathbf{B}_n(t)$ should exist uniformly on compacts and should be Brownian motion.

To see that this procedure works, one must first verify that the preceding definition of $\{\mathbf{B}_n(t) : t \geq 0\}$ gives a process with the correct distribution. That is, we need to show that $\big\{\mathbf{B}_n\big((m+1)2^{-n}\big) - \mathbf{B}_n\big(m2^{-n}\big) : m \in \mathbb{N}\big\}$ is a sequence of independent $N(\mathbf{0}, 2^{-n}\mathbf{I})$-random variables. But, since this sequence is contained in the Gaussian family spanned by $\{\mathbf{X}_{m,n} : (m,n) \in \mathbb{Z}^+ \times \mathbb{N}\}$, Lemma 1 says that we need only show that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[&\Big(\boldsymbol{\xi}, \mathbf{B}_n\big((m+1)2^{-n}\big) - \mathbf{B}_n\big(m2^{-n}\big)\Big)_{\mathbb{R}^N} \\ &\times \Big(\boldsymbol{\xi}', \mathbf{B}_n\big((m'+1)2^{-n}\big) - \mathbf{B}_n\big(m'2^{-n}\big)\Big)_{\mathbb{R}^N}\Big] = 2^{-n}\big(\boldsymbol{\xi}, \boldsymbol{\xi}'\big)_{\mathbb{R}^N}\delta_{m,m'}\end{aligned}$$

for $\boldsymbol{\xi}$, $\boldsymbol{\xi}' \in \mathbb{R}^N$ and m, $m' \in \mathbb{N}$. When $n = 0$, this is obvious. Now assume that it is true for n, and observe that

$$\begin{aligned}&\mathbf{B}_{n+1}(m2^{-n}) - \mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) \\ &\qquad = \frac{\mathbf{B}_n(m2^{-n}) - \mathbf{B}_n\big((m-1)2^{-n}\big)}{2} - 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1}\end{aligned}$$

and

$$\begin{aligned}&\mathbf{B}_{n+1}\big((2m-1)2^{-n-1}\big) - \mathbf{B}_{n+1}\big((m-1)2^{-n}\big) \\ &\qquad = \frac{\mathbf{B}_n(m2^{-n}) - \mathbf{B}_n\big((m-1)2^{-n}\big)}{2} + 2^{-\frac{n}{2}-1}\mathbf{X}_{m,n+1}.\end{aligned}$$

Using these expressions and the induction hypothesis, it is easy to check the required equation.

Second, and more challenging, we must show that, $\mathbb{P}$-almost surely, these processes are converging uniformly on compact time intervals. For this purpose, consider the difference $t \rightsquigarrow \mathbf{B}_{n+1}(t) - \mathbf{B}_n(t)$. Since this path is linear on each interval $[m2^{-n-1}, (m+1)2^{-n-1}]$,

$$\begin{aligned}\max_{t\in[0,2^L]}\big|\mathbf{B}_{n+1}(t) - \mathbf{B}_n(t)\big| &= \max_{1\leq m \leq 2^{L+n+1}}\big|\mathbf{B}_{n+1}(m2^{-n-1}) - \mathbf{B}_n(m2^{-n-1})\big| \\ &= 2^{-\frac{n}{2}-1}\max_{1\leq m\leq 2^{L+n}}|\mathbf{X}_{m,n+1}| \leq 2^{-\frac{n}{2}-1}\left(\sum_{m=1}^{2^{L+n}}|\mathbf{X}_{m,n+1}|^4\right)^{\frac{1}{4}}.\end{aligned}$$

Thus, by Jensen's Inequality,

$$\mathbb{E}^{\mathbb{P}}\big[\|\mathbf{B}_{n+1}-\mathbf{B}_n\|_{[0,2^L]}\big] \le 2^{-\frac{n}{2}-1}\left(\sum_{m=1}^{2^{L+n}} \mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_{m,n+1}|^4\big]\right)^{\frac14} = 2^{-\frac{n-L-4}{4}} C_N,$$

where $C_N \equiv \mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_{1,0}|^4\big]^{\frac14} < \infty$.

Starting from the preceding, it is an easy matter to show that there is a measurable $\mathbf{B} : [0,\infty)\times\Omega \longrightarrow \mathbb{R}^N$ such that $\mathbf{B}(0) = \mathbf{0}$, $\mathbf{B}(\,\cdot\,,\omega) \in C\big([0,\infty);\mathbb{R}^N\big)$ for each $\omega \in \Omega$, and $\|\mathbf{B}_n - \mathbf{B}\|_{[0,t]} \longrightarrow 0$ both $\mathbb{P}$-almost surely and in $L^1(\mathbb{P};\mathbb{R})$ for every $t \in [0,\infty)$. Furthermore, since $\mathbf{B}(m2^{-n}) = \mathbf{B}_n(m2^{-n})$ $\mathbb{P}$-almost surely for all $(m,n) \in \mathbb{N}^2$, it is clear that $\big\{\mathbf{B}\big((m+1)2^{-n}\big) - \mathbf{B}(m2^{-n}) : m \ge 0\big\}$ is a sequence of independent $N(\mathbf{0}, 2^{-n}\mathbf{I})$-random variables for all $n \in \mathbb{N}$. Hence, by continuity, it follows that $\{\mathbf{B}(t) : t \ge 0\}$ is a Brownian motion.

We have now completed the Lévy's construction, but, before moving on, it is only proper to recognize that, clever as his method is, Lévy was not the first to construct a Brownian motion. Instead, it was N. Wiener who was the first. In fact, his famous[1] 1923 article "Differential Space" in *J. Math. Phys.* #**2** contains three different approaches.

Lévy's Construction in Context: There are elements of Lévy's construction that admit interesting generalizations, perhaps the most important of which is **Kolmogorov's Continuity Criterion**.

THEOREM 2. *Suppose that $\{X(\mathbf{x}) : \mathbf{x} \in [0,R]^N\}$ is a family of random variables taking values in a Banach space B, and assume that, for some $p \in [1,\infty)$, $C < \infty$, and $r \in (0,1]$,*

$$\mathbb{E}\big[\|X(\mathbf{y}) - X(\mathbf{x})\|_B^p\big]^{\frac1p} \le C|\mathbf{y}-\mathbf{x}|^{\frac{N}{p}+r} \quad \text{for all } \mathbf{x},\,\mathbf{y} \in [0,R]^N.$$

Then there exists a family $\{\tilde X(\mathbf{x}) : \mathbf{x} \in [0,R]^N\}$ of random variables such that $X(\mathbf{x}) = \tilde X(\mathbf{x})$ $\mathbb{P}$-almost surely for each $\mathbf{x} \in [0,R]^N$ and $\mathbf{x} \in [0,R]^N \longmapsto \tilde X(\mathbf{x},\omega) \in B$ is continuous for all $\omega \in \Omega$. In fact, for each $\alpha \in [0,r)$, there is a $K < \infty$, depending only on N, p, r, and α, such that

$$\mathbb{E}\left[\sup_{\substack{\mathbf{x},\mathbf{y}\in[0,R]^N\\ \mathbf{x}\ne\mathbf{y}}} \left(\frac{\|\tilde X(\mathbf{y}) - \tilde X(\mathbf{x})\|_B}{|\mathbf{y}-\mathbf{x}|^\alpha}\right)^p\right]^{\frac1p} \le K R^{\frac{N}{p}+r-\alpha}.$$

[1] Wiener's article is remarkable, but I must admit that I have never been convinced that it is complete. Undoubtedly, my skepticism are more a consequence of my own ineptitude than of his.

Proof: First note that, by sm easy rescaling argument, it suffices to treat the case when $R=1$.

Given $n\ge 0$, set[2]

$$M_n=\max_{\substack{\mathbf{k},\mathbf{m}\in\mathbb{N}^N\cap[0,2^n]^N\\ \|\mathbf{m}-\mathbf{k}\|_\infty=1}}\big\|X(\mathbf{m}2^{-n})-X\big(\mathbf{k}2^{-n}\big)\big\|_B,$$

and observe that

$$\begin{aligned}\mathbb{E}\big[M_n^p\big]^{\frac1p}&\le\mathbb{E}\left[\left(\sum_{\substack{\mathbf{k},\mathbf{m}\in\mathbb{N}^N\cap[0,2^n]^N\\ \|\mathbf{m}-\mathbf{k}\|_\infty=1}}\big\|X(\mathbf{m}2^{-n})-X\big(\mathbf{k}2^{-n}\big)\big\|_B^p\right)^{\frac1p}\right]\\ &\le\left(\sum_{\substack{\mathbf{k},\mathbf{m}\in\mathbb{N}^N\cap[0,2^n]^N\\ \|\mathbf{m}-\mathbf{k}\|_\infty=1}}\mathbb{E}\big[\big\|X(\mathbf{m}2^{-n})-X\big(\mathbf{k}2^{-n}\big)\big\|_B^p\big]\right)^{\frac1p}\le K2^{-nr},\end{aligned}$$

where $K=C(2N)^{\frac1p}$.

Let $n\ge1$ be given. Because $X_n(\mathbf{x})-X_{n-1}(\mathbf{x})$ is a multilinear function on each cude $\mathbf{m}2^{-n}+[0,2^{-n}]^N$,

$$\sup_{\mathbf{x},\mathbf{y}\in[0,1]^N}\|X_n(\mathbf{y})-X_{n-1}(\mathbf{x})\|_B=\max_{\mathbf{m}\in\mathbb{N}^N\cap[0,2^n]^N}\big\{\|X_n(\mathbf{m}2^{-n})-X_{n-1}(\mathbf{m}2^{-n})\|_B\big\}.$$

Since $X_n(\mathbf{m}2^{-n})=X(\mathbf{m}2^{-n})$ and either $X_{n-1}(\mathbf{m}2^{-n})=X(\mathbf{m}2^{-n})$ or

$$X_{n-1}(\mathbf{m}2^{-n})=\sum_{\substack{\mathbf{k}\in\mathbb{N}^N\cap[0,2^n]\\ \|\mathbf{k}-\mathbf{m}\|_\infty=1}}\theta_{\mathbf{m}.\mathbf{k}}X(\mathbf{k}2^{-n}),$$

where the $\theta_{\mathbf{m},\mathbf{k}}$'s are non-negative and sum to 1, it follows that

$$\sup_{\mathbf{x},\mathbf{y}\in[0,1]^N}\|X_n(\mathbf{y})-X_{n-1}(\mathbf{x})\|_B\le M_n$$

and therefore that

$$\mathbb{E}\left[\sup_{\mathbf{x},\mathbf{y}\in[0,1]^N}\|X_n(\mathbf{y})-X_{n-1}(\mathbf{x})\|_B^p\right]^{\frac1p}\le K2^{-nr}.$$

[2] Given $\mathbf{x}\in\mathbb{R}^N$, $\|\mathbf{x}\|_\infty=\max_{1\le j\le N}|x_j|$.

Hence

$$\mathbb{E}\left[\sup_{n'>n}\sup_{\mathbf{x},\mathbf{y}\in[0,1]^N}\|X_{n'}(\mathbf{y})-X_n(\mathbf{x})\|_B^p\right]^{\frac{1}{p}}\le\frac{K}{1-2^{-r}}2^{-nr},$$

and so there exists a measurable map $\tilde{X}:[0,1]^N\times\Omega\longrightarrow B$ such that $\mathbf{x}\rightsquigarrow\tilde{X}(\mathbf{x},\omega)$ is continuous for each $\omega\in\Omega$ and

$$\mathbb{E}\left[\sup_{\mathbf{x}\in[0,1]^N}\|\tilde{X}(\mathbf{x})-X_n(\mathbf{x})\|_B^p\right]^{\frac{1}{p}}\le\frac{K}{1-2^{-r}}2^{-nr}.$$

Furthermore, $\tilde{X}(\mathbf{x})=X(\mathbf{x})$ a.s. if $\mathbf{x}=\mathbf{m}2^{-n}$ for some $n\ge 0$ and $\mathbf{m}\in\mathbb{N}^N\cap[0,2^n]^N$. Hence, since $\mathbf{x}\rightsquigarrow\tilde{X}(\mathbf{x})$ is continuous and

$$\mathbb{E}\big[\|X([2^n\mathbf{x}]2^{-n})-X(\mathbf{x})\|_B^p\big]^{\frac{1}{p}}\le C2^{-n(\frac{N}{p}+r)},$$

it follows that $X(\mathbf{x})=\tilde{X}(\mathbf{x})$ a.s. for all $\mathbf{x}\in[0,1]^N$.

Finally, to prove the final estimate, suppose that $2^{-n-1}<|\mathbf{y}-\mathbf{x}|\le 2^{-n}$. Then

$$\|X_n(\mathbf{y})-X_n(\mathbf{x})\|_B\le N^{\frac{1}{2}}2^n|\mathbf{x}-\mathbf{y}|M_n,$$

and so

$$\|\tilde{X}(\mathbf{y})-\tilde{X}(\mathbf{x})\|_B\le 2\sup_{\boldsymbol{\xi}\in[0,1]^N}\|\tilde{X}(\boldsymbol{\xi})-X_n(\boldsymbol{\xi})\|_B+N^{\frac{1}{2}}2^n|\mathbf{x}-\mathbf{y}|M_n.$$

Hence, by the preceding,

$$\mathbb{E}\left[\sup_{\substack{\mathbf{x},\mathbf{y}\in[0,1]^N\\2^{-n-1}<|\mathbf{y}-\mathbf{x}|\le 2^{-n}}}\left(\frac{\|\tilde{X}(\mathbf{y})-\tilde{X}(\mathbf{x})\|_B}{|\mathbf{y}-\mathbf{x}|^\alpha}\right)^p\right]^{\frac{1}{p}}\le K'2^{-n(r-\alpha)},$$

where $K'=\frac{4K}{1-2^{-r}}+2N^{\frac{1}{2}}K$, and therefore

$$\mathbb{E}\left[\left(\sup_{\substack{\mathbf{x},\mathbf{y}\in[0,1]^N\\\mathbf{y}\ne\mathbf{x}}}\frac{\|\tilde{X}(\mathbf{y})-\tilde{X}(\mathbf{x})\|_B}{|\mathbf{y}-\mathbf{x}|^\alpha}\right)^p\right]^{\frac{1}{p}}\le\frac{K'}{1-2^{-(r-\alpha)}}.\quad\square$$

Corollary 3. *Assume that there is a* $p \in [1,\infty)$, $\beta > \frac{N}{p}$, *and* $C < \infty$ *such that*

$$\mathbb{E}\big[\|\tilde{X}(\mathbf{y}) - \tilde{X}(\mathbf{x})\|_B^p\big]^{\frac{1}{p}} \le C|\mathbf{y}-\mathbf{x}|^\beta \quad \text{for all } \mathbf{x},\, \mathbf{y} \in [0,\infty)^N.$$

Then, for each for each $\gamma > \beta$,

$$\lim_{|\mathbf{x}|\to\infty} \frac{X(\mathbf{x})}{|\mathbf{x}|^\gamma} = 0 \text{ a.s.}.$$

Proof: Take $\alpha = 0$ in Theorem 2. Then

$$\begin{aligned}\mathbb{E}&\left[\left(\sup_{\mathbf{x}\in[2^{n-1},2^n]^N} \frac{\|\tilde{X}(\mathbf{x}) - \tilde{X}(\mathbf{0})\|_B}{|\mathbf{x}|^\gamma}\right)^p\right]^{\frac{1}{p}} \\ &\le 2^{-\gamma(n-1)}\mathbb{E}\left[\sup_{\mathbf{x}\in[0,2^n n]^N} \|\tilde{X}(\mathbf{x}) - \tilde{X}(\mathbf{0})\|_B^p\right]^{\frac{1}{p}} \le 2^\gamma K 2^{(\beta-\gamma)n},\end{aligned}$$

and so

$$\mathbb{E}\left[\left(\sup_{\mathbf{x}\in[2^{m-1},\infty)^N} \frac{\|\tilde{X}(\mathbf{x}) - \tilde{X}(\mathbf{0})\|_B}{|\mathbf{x}|^\gamma}\right)^p\right]^{\frac{1}{p}} \le \frac{2^\gamma K}{1-2^{\beta-\gamma}} 2^{(\beta-\gamma)m}. \quad \square$$

Wiener Measure and Brownian Motion: Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a non-decreasing family $\{\mathcal{F}_t : t \ge 0\}$ of sub-σ-algebras, and a family $\{B(t) : t \ge 0\}$ of $\mathbb{R}^N$-valued random variables, one says that $\big(B(t),\mathcal{F}_t,\mathbb{P}\big)$ is an $\mathbb{R}^N$-valued **Brownian motion** if

(i) $\mathbb{P}$-almost surely, $B(0) = \mathbf{0}$ and $t \rightsquigarrow B(t)$ is continuous.
(ii) For each $s \ge 0$, $B(s)$ is $\mathcal{F}_s$-measurable, and, for $t > s$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$ and has distribution $\gamma_{\mathbf{0},(t-s)\mathbf{I}}$.

Endow $C\big([0,\infty);\mathbb{R}^N\big)$ with the topology of uniform convergence on compacts. Equivalently, if the metrice ρ on $C\big([0,\infty);\mathbb{R}^N\big)$ is defined by

$$\rho(w_1,\, w_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|w_2 - w_1\|_{[0,m]}}{1 + \|w_2 - w_1\|_{[0,m]}}$$

for $w_1, w_2 \in C\big([0,\infty);\mathbb{R}^N\big)$, then we are giving $C\big([0,\infty);\mathbb{R}^N\big)$ the topology determined by ρ. It is an elementary exercise to show that this metric space is separable and complete. Furtheremore, the associated Borel field $\mathcal{B}$ is the smallest σ-algebra $\sigma\big(\{w(t) : t \ge 0\}\big)$ for which all the maps $w \rightsquigarrow w(t)$ are measurable. Indeed, since $w \rightsquigarrow w(t)$ is continuous, it is Borel measurable, and therefore $\sigma\big(\{w(t) : t \geq 0\}\big) \subseteq \mathcal{B}$. To prove the opposite inclusion, begin by observing that every open subset of $C\big([0,\infty);\mathbb{R}^N\big)$ can be written as the countable union of sets of the form $B_T(w) = \{w' : \|w' - w\|_{[0,T]} \leq r\}$. Since

$$B_T(w) = \bigcap_{t \in \mathbb{Q} \cap [0,T]} \{w' : |w'(t) - w(t)| \leq r\} \in \sigma\big(\{w(t) : t \geq 0\}\big),$$

where $\mathbb{Q}$ denotes the set of rational numbers, there is nothing more to do.

In view of the preceding, we know that two Borel, probability measures μ_1 and μ_2 on $C\big([0,\infty);\mathbb{R}^N\big)$ are equal if, for all $n \geq 1$ and $0 \leq t_1 < \cdots < t_n$, the distribution of $w \rightsquigarrow \big(w(t_1), \dots, w(t_n)\big)$ is the same under μ_1 and μ_2. In particular, the measure induced on $C\big([0,\infty);\mathbb{R}^N\big)$ by one Brownian motion is the same as that induced by any other Brownian motion. Namely, if μ is such a measure, then $\mu\big(\{w : w(0) = \mathbf{0}\}\big) = 1$ and, for each $0 \leq s < t$, $w(t) - w(s)$ is independent of $\mathcal{B}_s \equiv \sigma\big(\{w(\tau) : \tau \in [0,s]\}\big)$ and has distribution $\gamma_{\mathbf{0},(t-s)\mathbf{I}}$. Hence, for any $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$, $w(t_1) - w(t_0), \dots, w(t_n) - w(t_{n-1})$ are mutually independent and the mth one has distribution $\gamma_{\mathbf{0},(t_m - t_{m-1})\mathbf{I}}$. This unique measure is called **Wiener measure**, and I will use $\mathcal{W}$ is denote it.