Chapter IV

Non-Gaussian Lévy Processes

Although analysis was the engine which drove the proofs in Chapter III, probability theory can do a lot to explain the meaning of the results there. Specifically, in this and the next chapters we will give an intuitively appealing way of thinking about a random variable $\mathbf{X}$ whose distribution is infinitely divisible. That is,

$$\mathbb{E}^{\mathbb{P}}\bigl[e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{X})_{\mathbb{R}^N}}\bigr] = \exp\biggl(\sqrt{-1}\bigl(\boldsymbol{\xi},\mathbf{m}\bigr) + \bigl(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\bigr)_{\mathbb{R}^N} \\ + \int_{\mathbb{R}^N}\Bigl[e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\,\mathbf{1}_{[0,1]}\bigl(|\mathbf{y}|\bigr)\Bigr]\,M(d\mathbf{y})\biggr)$$

for some $\mathbf{m} \in \mathbb{R}^N$, some symmetric, non-negative definite $\mathbf{C} \in \operatorname{Hom}(\mathbb{R}^N;\mathbb{R}^N)$, and Lévy measure $M \in \mathfrak{M}_2(\mathbb{R}^N)$. In the present chapter we will deal with the case when there is no Gaussian component. That is, we will be assuming that $\mathbf{C} = 0$. Because it is destinctly different, we will treat the Gaussian component separately in the next chapter. However, we begin with some general comments which apply to the considerations in both this and the next chapter.

The key idea, which seems to have been Lévy's, is to develop a dynamical picture of $\mathbf{X}$. To understand the origin of this idea, denote by $\mu \in \mathcal{I}(\mathbb{R}^N)$ the distribution of $\mathbf{X}$, and, for $n \in \mathbb{Z}^+$, choose a sequence $\{\mathbf{X}_{k,\frac{1}{n}} : k \in \mathbb{Z}^+\}$ of independent randoms variables with distribution $\mu_{\frac{1}{n}}$. Then, we can realize $\mathbf{X}$ as $\mathbf{Z}_{\frac{1}{n}}(1)$, where $\mathbf{Z}_{\frac{1}{n}}(0) = \mathbf{0}$, $\mathbf{Z}_{\frac{1}{n}}\bigl(\frac{m}{n}\bigr) = \sum_{k=1}^{m}\mathbf{X}_{k,\frac{1}{n}}$ for $m \in \mathbb{Z}^+$, and $\mathbf{Z}_{\frac{1}{n}}(t) = \mathbf{Z}_{\frac{1}{n}}\bigl(\frac{m-1}{n}\bigr)$ when $\frac{m-1}{n} < t \le \frac{m}{n}$. That is, $\mathbf{X}$ can be thought of the place at which a random evolution starting from $\mathbf{0}$ arrives at time 1. Of course, the evolution $t \rightsquigarrow \mathbf{Z}_{\frac{1}{n}}(t)$ looks a little contrived and depends on which n one chooses. On the other, there is an inherent consistency between these evolutions. Namely, if $n_1,\, n_2 \in \mathbb{Z}^+$ and $\frac{n_2}{n_1} \in \mathbb{Z}^+$, then the joint distribution of $\bigl\{\mathbf{Z}_{\frac{1}{n_1}}\bigl(\frac{m}{n_1}\bigr) : m \in \mathbb{N}\bigr\}$ is the same as the joint distribution of $\bigl\{\mathbf{Z}_{\frac{1}{n_2}}\bigl(\frac{m}{n_1}\bigr) : m \in \mathbb{N}\bigr\}$. Thus, there is reason to hope that it is possible to let $n \to \infty$ and thereby arrive at a family $\bigl\{\mathbf{Z}(t) : t \in [0,\infty)\bigr\}$ of random variables such that $\bigl\{\mathbf{Z}\bigl(\frac{m}{n}\bigr) : m \in \mathbb{N}\bigr\}$ has the same joint distribution as $\bigl\{\mathbf{Z}_{\frac{1}{n}}\bigl(\frac{m}{n}\bigr) : m \in \mathbb{N}\bigr\}$ for all $n \in \mathbb{Z}^+$. Further, one should suspect that the better one understands the mechanism by which $t \rightsquigarrow \mathbf{Z}(t)$ evolves, the better one will understand $\mathbf{X}$.

Assuming that the family $\{\mathbf{Z}(t) : t \in [0,\infty)\}$ exists, notice that we already know what the joint distribution of $\{\mathbf{Z}(t_k) : k \in \mathbb{N}\}$ must be for any choice of $0 = t_0 < \cdots < t_k < \cdots$. Indeed, if we define $\mu_t \in \mathbf{M}_1(\mathbb{R}^N)$ for $t \in [0,\infty)$ so that (cf. Theorem 3.2.7) $\widehat{\mu_t} = e^{t\ell_\mu}$, then it should be clear that $\mathbf{Z}(0) = \mathbf{0}$ and $\{\mathbf{Z}(t_k) - \mathbf{Z}(t_{k-1}) : k \in \mathbb{Z}^+\}$ is a sequence of independent random variables, the kth one of which has distribution $\mu_{t_k - t_{k-1}}$. This is obvious when $nt_k \in \mathbb{N}$ for some $n \in \mathbb{Z}^+$ and follows in general when one replaces each t_k by[*] $n^{-1}[nt_k]$ and lets $n \to \infty$. For this reason, $\{\mathbf{Z}(t) : t \in [0,\infty)\}$ used to be called a **process with independent, homogeneous increments**, the term "process" being the common one for continuous families of random variables and the adjective "homogeneous" referring to the fact that the distribution of the increment $\mathbf{Z}(t) - \mathbf{Z}(s)$ for $0 \le s < t$ depends only on the length $t - s$ of the time interval over which it is taken. In more recent times, a process with independent, homogeneous increments is said to a **Lévy process**, and so we will adopt this terminology.

Unfortunately, before we can carry out this program, we need to deal with a few technical, book keeping matters.

§4.1 Stochastic Processes, Some Generalities

Given an index $\mathcal{A}$ with some nice structure and a family $\{X(\alpha) : \alpha \in \mathcal{A}\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in some measurable space $(E, \mathcal{B})$, it is often helpful to think about $\{X(\alpha) : \alpha \in A\}$ in terms of the map $\omega \in \Omega \longmapsto \mathbf{X}(\,\cdot\,,\omega) \in E^{\mathcal{A}}$. For instance, if $\mathcal{A}$ is linearly ordered, then $\omega \rightsquigarrow X(\,\cdot\,,\omega)$ can be thought of as random evolution. More generally, when probabilists want to indicate that they are thinking about $\{X(\alpha) : \alpha \in \mathcal{A}\}$ as the map $\omega \rightsquigarrow X(\,\cdot\,,\omega)$, they call $\{X(\alpha) : \alpha \in \mathcal{A}\}$ a **stochastic process** on $\mathcal{A}$ with **state space** $(E, \mathcal{B})$.

The **distribution of a stochastic process** is the probability measure $X_*\mathbb{P}$ on[†] $(E^{\mathcal{A}}, \mathcal{B}^{\mathcal{A}})$ obtained by pushing $\mathbb{P}$ forward under the map $\omega \rightsquigarrow X(\,\cdot\,,\omega)$. Hence two stochastic processes $\{X(\alpha) : \alpha \in \mathcal{A}\}$ and $\{Y(\alpha) : \alpha \in \mathcal{A}\}$ on $(E, \mathcal{B})$ have the same distribution if and only if

$$\mathbb{P}\big(X(\alpha_k) \in \Gamma_k,\, 0 \le k \le K\big) = \mathbb{P}\big(Y(\alpha_k) \in \Gamma_k,\, 0 \le k \le K\big)$$

for all $K \in \mathbb{Z}^+$, $\{\alpha_0, \dots, \alpha_K\} \subseteq \mathcal{A}$, and $\Gamma_0, \dots, \Gamma_K \in \mathcal{B}$.

As long as $\mathcal{A}$ is countable, there are no problems because $E^{\mathcal{A}}$ is a reasonably tame object and $\mathcal{B}^{\mathcal{A}}$ contains lots of sets. However, when $\mathcal{A}$ is uncountable, $E^{\mathcal{A}}$ is a hideously large space and $\mathcal{B}^{\mathcal{A}}$ will be too meager to contain many of the

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[*] We use $[t]$ to denote the integer part of $t \in \mathbb{R}$. That is, $[t]$ is the largest integer dominated by t.

[†] Recall that $\mathcal{B}^{\mathcal{A}}$ is the σ-algebra over $E^{\mathcal{A}}$ which generated by all the maps $\psi \in E^{\mathcal{A}} \longmapsto \psi(\alpha) \in E$ as α runs over $\mathcal{A}$.

subsets in which one is interested. The point is that for $B \in \mathcal{B}^{\mathcal{A}}$, there must (cf. Exercise 4.1.8) be a countable subset $\{\alpha_k : k \in \mathbb{N}\}$ of $\mathcal{A}$ such that one can determine whether or not $\psi \in B$ by knowing $\{\psi(\alpha_k) : k \in \mathbb{N}\}$. Thus, for instance, $C\big([0,\infty);\mathbb{R}\big) \notin \mathcal{B}_{\mathbb{R}}^{[0,\infty)}$.

Probabilists expended a great deal of effort to overcome the problem raised in the preceding paragraph. For instance, using a remarkable piece of measure theoretic reasoning, J.L. Doob* proved that in the important case when $\mathcal{A} = [0,\infty)$ and $E = \mathbb{R}$, one can always make a modification, what he called the "separable modification," so that sets like $C\big([0,\infty);\mathbb{R}\big)$ become measurable. However, in recent times, probabilists have tried to simplify their lives by constructing their processes in such a way that these unpleasant measurability questions never arise. That is, if they suspect that the process should have some property which is not measurable with respect to $\mathcal{B}^{\mathcal{A}}$, they shun constructions based on general principles, like Kolmogorov's Extension Theorem (cf. part (**iii**) of Exercise 5.1.8 below), and instead adopt a construction procedure which produces the process with the desired properties already present.

The rest of this chapter contains important examples of this approach, and the rest of this section contains a few technical preparations.

§4.1.1. The Space $D(\mathbb{R}^N)$. Unless its Lévy measure is zero, a Lévy process for $\mu \in \mathcal{I}(\mathbb{R}^N)$ cannot be constructed so that it has continuous paths. That is, $t \rightsquigarrow \mathbf{Z}_\mu(t,\omega)$ will not be continuous. On the other hand, it can be constructed so that its paths are reasonably nice. Specifically, its paths can be made to be right continuous everywhere and have no oscillatory discontinuities. For this reason, we introduce the space $D(\mathbb{R}^N)$ of paths $\boldsymbol{\psi} : [0,\infty) \longrightarrow \mathbb{R}^N$ such that $\boldsymbol{\psi}(t) = \boldsymbol{\psi}(t+) \equiv \lim_{\tau \searrow t} \boldsymbol{\psi}(\tau)$ for each $t \in [0,\infty)$ and $\psi(t-) \equiv \lim_{\tau \nearrow t} \boldsymbol{\psi}(\tau)$ exists in $\mathbb{R}^N$ for each $t \in (0,\infty)$. Equivalently, for each $t \in (0,\infty)$ and $\epsilon > 0$, there is a $\delta \in (0,t)$ such that $\sup\{|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(\tau)| : \tau \in (t, t+\delta)\} < \epsilon$ and $\sup\{|\boldsymbol{\psi}(t-) - \boldsymbol{\psi}(\tau)| : \tau \in (t-\delta, t)\} < \epsilon$.

The following lemma presents a few basic properties possessed by elements of $D(\mathbb{R}^N)$. In its statement, for $n \in \mathbb{N}$ and $\tau \in (0,\infty)$, $[\tau]_n^+ = \min\{m2^{-n} : m \in \mathbb{Z}^+ \text{ and } m \geq 2^n\tau\}$ and $[\tau]_n^- = [\tau]_n^+ - 2^{-n} = \max\{m2^{-n} : m \in \mathbb{N} \text{ and } m < 2^n\tau\}$. In addition, for $0 \leq a < b$,

$$\|\boldsymbol{\psi}\|_{[a,b]} \equiv \sup_{t \in [a,b]} |\boldsymbol{\psi}(t)|$$

is the uniform norm of $\boldsymbol{\psi} \restriction [a,b]$, and

$$\operatorname{var}_{[a,b]}(\boldsymbol{\psi}) = \sup\left\{\sum_{k=1}^{K} |\boldsymbol{\psi}(t_k) - \boldsymbol{\psi}(t_{k-1})| : K \in \mathbb{Z}^+ \right.$$
$$\left. \text{and } a = t_0 < t_1 < \cdots < t_K = b\right\}$$

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[*] See Chapter II of Doob's *Stochastic Processes*, published by J. Wiley

is the total variation of $\boldsymbol{\psi} \restriction [a,b]$.

Lemma 4.1.1. *If $\boldsymbol{\psi} \in D(\mathbb{R}^N)$, then, for each $t > 0$ and $r > 0$, $\|\boldsymbol{\psi}\|_{[0,t]} < \infty$ and the set*

$$J(t,r,\boldsymbol{\psi}) \equiv \{\tau \in (0,t] : |\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)| \geq r\}$$

is finite subset of $(0,t]$. In addition, there exists an $n(t,r,\boldsymbol{\psi}) \in \mathbb{N}$ such that for every $n \geq n(t,r,\boldsymbol{\psi})$ and $m \in \mathbb{Z}^+ \cap (0,2^n]$

$$\big|\boldsymbol{\psi}\big(m2^{-n}t\big) - \boldsymbol{\psi}\big((m-1)2^{-n}t\big)\big| \geq r \implies m2^{-n} = \left[\tfrac{\tau}{t}\right]_n^+ \text{ for some } \tau \in J(t,r,\boldsymbol{\psi}).$$

Finally,

$$\|\boldsymbol{\psi}\|_{[0,t]} = \lim_{n\to\infty} \max\big\{|\boldsymbol{\psi}(m2^{-n}t)| : m \in \mathbb{N} \cap [0,2^n]\big\}$$

and

$$\mathrm{var}_{[0,t]}(\boldsymbol{\psi}) = \lim_{n\to\infty} \sum_{m\in\mathbb{Z}^+\cap[0,2^n]} \big|\boldsymbol{\psi}\big(m2^{-n}t\big) - \boldsymbol{\psi}\big((m-1)2^{-n}t\big)\big|.$$

Proof: Begin by noting that it suffices to treat the case when $t = 1$, since one can always reduce to this case by replacing $\boldsymbol{\psi}$ by $\tau \rightsquigarrow \boldsymbol{\psi}(t\tau)$.

If $\|\boldsymbol{\psi}\|_{[0,1]}$ were infinite, then we could find a sequence $\{\tau_n : n \geq 1\} \subseteq [0,1]$ such that $|\boldsymbol{\psi}(\tau_n)| \longrightarrow \infty$, and clearly, without loss in generality, we could choose this sequence so that $\tau_n \longrightarrow \tau \in [0,1]$ and either $\{\tau_n : n \geq 1\}$ is strictly decreasing or strictly increasing. But, in the first case this would contradict right continuity, and in the second it would contradict the existence of left limits. Thus, $\|\boldsymbol{\psi}\|_{[0,1]}$ must be finite.

Essentially the same reasoning shows that $J(1,r,\boldsymbol{\psi})$ is finite. If it were not, then we could find a sequence $\{\tau_n : n \geq 0\}$ of distinct points in $(0,1]$ such that $|\boldsymbol{\psi}(\tau_n) - \boldsymbol{\psi}(\tau_n-)| \geq r$, and again we could choose them so that either they were strictly increasing or strictly decreasing. If they were strictly increasing, then $\tau_n \nearrow \tau$ for some $\tau \in (0,1]$ and, for each $n \in \mathbb{Z}^+$, there would exist a $\tau_n' \in (\tau_{n-1},\tau_n)$ such that $|\boldsymbol{\psi}(\tau_n) - \boldsymbol{\psi}(\tau_n')| \geq \frac{r}{2}$, which would contradict the existence of a left limit at τ. Similarly, right continuity would be contradicted if the τ_n were decreasing.

Although it has the same flavor, the proof of the existence of $n(1,r,\boldsymbol{\psi})$ is a little trickier. Let $0 \leq \tau_1 < \cdots \tau_K \leq 1$ be the elements of $J(1,r,\boldsymbol{\psi})$. If $n(1,r,\boldsymbol{\psi})$ failed to exist, then we could choose a subsequence $\{(m_j,n_j) : j \geq 1\}$ from $\mathbb{Z}^+ \times \mathbb{N}$ so that $\{n_j : j \geq 1\}$ is strictly increasing, $t_j \equiv m_j 2^{-n_j} \in (0,1]$ satisfies $\big|\boldsymbol{\psi}\big(t_j\big) - \boldsymbol{\psi}\big(t_j - 2^{-n_j}\big)\big| \geq r$ for all $j \in \mathbb{Z}^+$, but $t_j \neq [\tau_k]_{n_j}^+$ for any $j \in \mathbb{Z}^+$ or $1 \leq k \leq K$. If $t_j = t$ infinitely often for some t, then we would have the contradiction that $t \notin J(1,r,\boldsymbol{\psi})$ and yet $|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| \geq r$. Hence, we will assume that the t_j's are distinct. Further, without loss in generality, we assume that $\{t_j : j \geq 1\}$ is a subset of one of the intervals $(0,\tau_1)$, (τ_{k-1},τ_k) for some $2 \leq k \leq K$, or $(\tau_K,1]$. Finally, we may and will assume that either $t_j \nearrow t \in (0,1]$ or that $t_j \searrow t \in [0,1)$. But, since $|\boldsymbol{\psi}(t_j) - \boldsymbol{\psi}(t_j - 2^{-n_j})| \geq r$, $t_j \nearrow t$ contradicts the existence of $\boldsymbol{\psi}(t-)$. Similarly, if $t_j \searrow t$ and $t_j - 2^{-n_j} \geq t$ for infinitely many $j's$, then we get a contradiction with right continuity at t. Thus, the only remaining case is when $t_j \searrow t$ and $t_j - 2^{-n_j} < t \leq t_j$ for all but a finite number of j's, in which case we get the contradiction that $t \notin J(1, r, \boldsymbol{\psi})$ and yet

$$|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| = \varliminf_{j\to\infty} \bigl|\boldsymbol{\psi}(t_j) - \boldsymbol{\psi}\bigl(t_j - 2^{-n_j}\bigr)\bigr| \geq r.$$

To prove the assertion about $\|\boldsymbol{\psi}\|_{[0,1]}$, simply observe that, by monotonicity, the limit exists and that, for any $t \in [0,1]$,

$$|\boldsymbol{\psi}(t)| = \lim_{n\to\infty} \bigl|\boldsymbol{\psi}\bigl([t]_n^+\bigr)\bigr| \leq \lim_{n\to\infty} \max_{0\leq m\leq 2^n} \bigl|\boldsymbol{\psi}(m2^{-n})\bigr| \leq \|\boldsymbol{\psi}\|_{[0,1]}.$$

The assertion about $\mathrm{var}_{[0,1]}(\boldsymbol{\psi})$ is proved in essentially the same, although now the monotonicity comes from the triangle inequality and the first step in the preceding must be replaced by $|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-)| = \lim_{n\to\infty} |\boldsymbol{\psi}([t]_n^+) - \boldsymbol{\psi}([t]_n^-)|$. $\square$

We next give $D(\mathbb{R}^N)$ the topological structure corresponding to uniform convergence on compacts. Equivalently, the topological structure for which

$$\rho(\boldsymbol{\psi}, \boldsymbol{\psi}') \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{\|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_{[0,n]}}{1 + \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_{[0,n]}}$$

is a metric. Because it is not separable (cf. Exercise 4.1.7), this topological structure is less than ideal. Nonetheless, the metric ρ is complete. To see that it is, first observe that $|\boldsymbol{\psi}(\tau-)| \leq \|\boldsymbol{\psi}\|_{[0,t]}$ for all $0 < \tau \leq t$. Thus, if $\sup_{\ell>k} \rho(\boldsymbol{\psi}_\ell, \boldsymbol{\psi}_k) \longrightarrow 0$ as $k \to \infty$, then there exist paths $\boldsymbol{\psi} : [0,\infty) \longrightarrow \mathbb{R}^N$ and $\tilde{\boldsymbol{\psi}} : (0,\infty) \longrightarrow \mathbb{R}^N$ such that

$$\sup_{\tau\in[0,t]} |\boldsymbol{\psi}_k(\tau) - \boldsymbol{\psi}(\tau)| \longrightarrow 0 \quad \text{and} \quad \sup_{\tau\in(0,t]} |\boldsymbol{\psi}_k(\tau-) - \tilde{\boldsymbol{\psi}}(\tau)| \longrightarrow 0$$

for each $t > 0$. Therefore, if $t \geq \sigma_n \searrow \tau$, then

$$\varlimsup_{n\to\infty} |\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\sigma_n)| \leq 2\|\boldsymbol{\psi} - \boldsymbol{\psi}_k\|_{[0,t]} + \varlimsup_{n\to\infty} |\boldsymbol{\psi}_k(\tau) - \boldsymbol{\psi}_k(\sigma_n)| \leq 2\|\boldsymbol{\psi} - \boldsymbol{\psi}_k\|_{[0,t]}$$

for all $k \in \mathbb{Z}^+$, and so $\boldsymbol{\psi}$ is right continuous. Essentially the same argument shows that $\boldsymbol{\psi}(\tau-) = \tilde{\boldsymbol{\psi}}(\tau)$ for $\tau > 0$, which means, of course, that $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ and that $\sup_{\tau\in(0,t]} |\boldsymbol{\psi}_k(\tau-) - \boldsymbol{\psi}(\tau-)| \longrightarrow 0$ for each $t > 0$.

One might think that we would take the measurable structure on $D(\mathbb{R}^N)$ to be the one given by the Borel field $\mathcal{B}_{D(\mathbb{R}^N)}$ determined by uniform convergence on compacts. However, this is not the choice we will make. Instead, the measurable structure we choose for $D(\mathbb{R}^N)$ is the one that $D(\mathbb{R}^N)$ inherits as a subset of $(\mathbb{R}^N)^{[0,\infty)}$. That is, the we take for $D(\mathbb{R}^N)$ the measurable structure given by the σ-algebra $\mathcal{F}_{D(\mathbb{R}^N)} = \sigma\big(\{\boldsymbol{\psi}(t) : t \in [0,\infty)\}\big)$, the σ-algebra generated by the maps $\boldsymbol{\psi} \in D(\mathbb{R}^N) \longmapsto \boldsymbol{\psi}(t) \in \mathbb{R}^N$ as t runs over $[0,\infty)$. The reason for our insisting on this choice is that we want two $D(\mathbb{R}^N)$-valued stochastic processes $\{\mathbf{X}(t) : t \geq 0\}$ and $\{\mathbf{Y}(t) : t \geq 0\}$ to induce the same measure on $D(\mathbb{R}^N)$ if they have the same distribution. Seeing as (cf. Exercise 4.1.8) $\mathcal{F}_{D(\mathbb{R}^N)} \subsetneqq \mathcal{B}_{D(\mathbb{R}^N)}$, this would not be true were we to choose the Borel structure.

Because $\mathcal{F}_{D(\mathbb{R}^N)} \neq \mathcal{B}_{D(\mathbb{R}^N)}$, a continuous function on $D(\mathbb{R}^N)$ need not be $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. Nonetheless many of them will be. For example, the last part of Lemma 4.1.1 proves that both $\boldsymbol{\psi} \rightsquigarrow \|\boldsymbol{\psi}\|_{[0,t]}$ and $\boldsymbol{\psi} \rightsquigarrow \operatorname{var}_{[0,t]}(\boldsymbol{\psi})$ will be $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable for all $t \in [0,\infty)$. In the next subsection, we will examine other important functions on $D(\mathbb{R}^N)$ and show that they too are $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable.

§4.1.2. Jump functions. Let $\mathfrak{M}_\infty(\mathbb{R}^N)$ be the space of non-negative, Borel measures M on $\mathbb{R}^N$ with the properties that $M(\{\mathbf{0}\}) = 0$ and $M\big(B(\mathbf{0},r)\complement\big) < \infty$ for all $r > 0$. A **jump function** is a map $t \in [0,\infty) \longmapsto j(t,\,\cdot\,) \in \mathfrak{M}_\infty(\mathbb{R}^N)$ with the property that, for each $\Delta \in \mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0} \notin \bar{\Delta}$, $j(0,\Delta) = 0$, $t \rightsquigarrow j(t,\Delta)$ is a non-decreasing element of $D(\mathbb{R}^N)$ such that $j(t,\Delta) - j(t-,\Delta) \in \{0,1\}$ for each $t > 0$.

Lemma 4.1.2. *A map $t \rightsquigarrow j(t,\,\cdot\,)$ is a non-zero jump function if and only if there exists a set $\emptyset \neq J \subset (0,\infty)$ which is finite or countable and a map $\tau \in J \longmapsto \mathbf{y}_\tau \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ such that $\{\tau \in J \cap (0,t] : |\mathbf{y}_\tau| \geq r\}$ is finite for each $(t,r) \in (0,\infty)^2$ and*

$$j(t,\,\cdot\,) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)} \delta_{\mathbf{y}_\tau}. \tag{4.1.3}$$

In particular, if $t \rightsquigarrow j(t,\,\cdot\,)$ is a jump function and $t > 0$, then, either $j(t,\,\cdot\,) = j(t-,\,\cdot\,)$ or $j(t,\,\cdot\,) - j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$.

Proof: It should be obvious that if $\{t_j : j \geq 1\}$ and $\{\mathbf{y}_j : j \geq 1\}$ satisfy the stated conditions, then the $t \rightsquigarrow j(t,\,\cdot\,)$ given by (4.1.3) is a jump function. To go the other direction, suppose that $t \rightsquigarrow j(t,\,\cdot\,)$ is a jump function, and, for each $r > 0$, set $f_r(t) = j\big(t, \mathbb{R}^N \setminus B(\mathbf{0},r)\big)$. Because $t \rightsquigarrow f_r(t)$ is a non-decreasing, right-continuous function satisfying $f_r(0) = 0$ and $f_r(t) - f_r(t-) \in \{0,1\}$ for each $t > 0$, it has at most a countable number of discontinuities and at most $f_r(t)$ of them can occur in the interval $(0,t]$. Furthermore, if f_r has a discontinuity at τ, then $j\big(\tau, B(\mathbf{0},r)\big) - j\big(\tau-, B(\mathbf{0},r)\big) = 0$, and so the measure $\nu_\tau = j(\tau,\,\cdot\,) - j(\tau-,\,\cdot\,)$ is a $\{0,1\}$-valued probability measure on $\mathbb{R}^N$ with assigns mass 0 to $B(\mathbf{0},r)$. Hence (cf. Exercise 4.1.12) $\nu_\tau = \delta_{\mathbf{y}}$ for some $\mathbf{y}_\tau \in \mathbb{R}^N \setminus B(\mathbf{0},r)$. From these considerations, it follows easily that if $J(r) = \{\tau \in (0,\infty) : f_r(\tau) \neq f_r(\tau-)\}$ and if, for each $\tau \in J(r)$, $\mathbf{y}_\tau \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ is chosen so that $j(\tau,\,\cdot\,) - j(\tau-,\,\cdot\,) = \delta_{\mathbf{y}_\tau}$, then $J(r) \cap (0,t]$ is finite for all $t > 0$ and

$$j(t,\,\cdot\,) \restriction B(\mathbf{0},r)\complement = \sum_{\tau \in J(r)} \mathbf{1}_{[\tau,\infty)}(t) \delta_{\mathbf{y}_\tau}.$$

Thus, if $J = \bigcup_{r>0} J(r)$, then J is at most countable, $\{(\tau, \mathbf{y}_\tau) : \tau \in J\}$ has the required finiteness property, and (4.1.3) holds. $\square$

The reason for our introducing jump functions is that every element $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ determines a jump function $t \rightsquigarrow j(t, \cdot\,, \boldsymbol{\psi})$ by the prescription

$$
\begin{gathered}
j(t, \Gamma, \boldsymbol{\psi}) = \sum_{\tau \in J(t,\boldsymbol{\psi})} \mathbf{1}_\Gamma\big(\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)\big), \\
\text{where } J(t, \boldsymbol{\psi}) \equiv \{\tau \in (0,t] : \boldsymbol{\psi}(\tau) \neq \boldsymbol{\psi}(\tau-)\},
\end{gathered}
\tag{4.1.4}
$$

for $\Gamma \subseteq \mathbb{R}^N \setminus \{\mathbf{0}\}$. To check that $j(t, \cdot\,, \boldsymbol{\psi})$ is well-defined and is a jump function, take $J(\boldsymbol{\psi}) = \bigcup_{t>0} J(t, \boldsymbol{\psi})$ and $\mathbf{y}_\tau = \boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)$ when $\tau \in J(\boldsymbol{\psi})$, note that, by Lemma 4.1.1, $J(\boldsymbol{\psi})$ is at most countable and that $\{(\tau, \mathbf{y}_\tau) : \tau \in J(\boldsymbol{\psi})\}$ has the finiteness required in Lemma 4.1.2, and observe that (4.1.3) holds when $j(t, \cdot\,) = j(t, \cdot\,, \boldsymbol{\psi})$ and $J = J(\boldsymbol{\psi})$.

Because it will be important for us to know that the the distribution of a $D(\mathbb{R}^N)$-valued stochastic process determines the distribution of the jump functions for by its paths, we will make frequent reference to the following lemma.

Lemma 4.1.5. *If $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a $\mathcal{B}_{\mathbb{R}^N}$-measurable function which vanishes in a neighborhood of $\mathbf{0}$, then φ is $j(t, \cdot\,, \boldsymbol{\psi})$-integrable for all $(t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N)$, and*

$$
(t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, j(t, d\mathbf{y}, \boldsymbol{\psi}) \in \mathbb{R}
$$

is a $\mathcal{B}_{[0,\infty)} \times \mathcal{F}_{D(\mathbb{R}^N)}$-measurable function which, for each $\boldsymbol{\psi}$, is right-continuous and piecewise constant as a function of t. Finally, for all Borel measurable $\varphi : \mathbb{R}^N \longrightarrow [0,\infty)$, $(t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi(\mathbf{y}) j(t, d\mathbf{y}, \boldsymbol{\psi}) \in [0,\infty]$, is $\mathcal{B}_{[0,\infty)} \times \mathcal{F}_{D(\mathbb{R}^N)}$-measurable.

Proof: The final assertion is an immediate consequence of the earlier one plus the Monotone Convergence Theorem.

Let $r > 0$ be given. If φ is a Borel measurable function which vanishes on $B(\mathbf{0}, r)$, then it is immediate from the first part of Lemma 4.1.1 that φ is $j(t, \cdot\,, \boldsymbol{\psi})$-integrable for all $(t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N)$ and, for each $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ $t \rightsquigarrow \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, j(t, d\mathbf{y}, \boldsymbol{\psi})$ is right-continuous and piecewise constant. Thus, it suffices to show that, for each $t \in (0,\infty)$, $\boldsymbol{\psi} \rightsquigarrow \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, j(t, d\mathbf{y}, \boldsymbol{\psi})$ is $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. Moreover, it suffices to do this when $t = 1$ and φ is continuous, since the set of φ's for which it is true is closed under pointwise convergence. But, by the second part of Lemma 4.1.1, we know that

$$
\sum_{m=1}^{2^n} \varphi\Big(\boldsymbol{\psi}\big(m2^{-n}\big) - \boldsymbol{\psi}\big((m-1)2^{-n}\big)\Big) = \sum_{\tau \in J(1,r,\boldsymbol{\psi})} \varphi\Big(\boldsymbol{\psi}\big([\tau]_n^+\big) - \boldsymbol{\psi}\big([\tau]_n^-\big)\Big)
$$

for $n \geq n(1, r, \boldsymbol{\psi})$, and therefore

$$\int_{\mathbb{R}^N} \varphi(\mathbf{y})\, j(1, d\mathbf{y}, \boldsymbol{\psi}) = \lim_{n\to\infty} \sum_{m=1}^{2^n} \varphi\Big(\boldsymbol{\psi}\big(m2^{-n}\big) - \boldsymbol{\psi}\big((m-1)2^{-n}\big)\Big). \quad \square$$

We now describe some properties of a path $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ which are determined by its relationship to its jump function. First, it should be obvious that $\boldsymbol{\psi} \in C(\mathbb{R}^N) \equiv C\big([0,\infty);\mathbb{R}^N\big)$ if and only if $j(t, \cdot\,, \boldsymbol{\psi}) = 0$ for all $t > 0$. At the opposite extreme, we will say that a $\boldsymbol{\psi}$ is **absolutely pure jump** if and only if $j(t, \cdot\,, \boldsymbol{\psi}) \in \mathfrak{M}_1(\mathbb{R}^N)$ and $\boldsymbol{\psi}(t) = \int \mathbf{y}\, j(t, d\mathbf{y}, \boldsymbol{\psi})$ for all $t > 0$. Among the absolutely pure jump paths are those which are the piecewise constant $\boldsymbol{\psi}$'s: those absolutely pure jump $\boldsymbol{\psi}$'s for which $j(t, \cdot\,, \boldsymbol{\psi}) \in \mathfrak{M}_0(\mathbb{R}^N)$, $t > 0$. Because of Lemma 4.1.5, each of these properties is $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. In particular, if $\{\mathbf{Z}(t) : t \geq 0\}$ is a $D(\mathbb{R}^N)$-valued stochatic process whose paths $\mathbb{P}$-almost surely have any one of these properties, then the paths of every $D(\mathbb{R}^N)$-valued stochastic process with the same distribution as $\{\mathbf{Z}(t) : t \geq 0\}$ will almost surely possess that property.

Finally, we need to address the question of when a jump function is the jump function for some $\boldsymbol{\psi} \in D(\mathbb{R}^N)$.

Theorem 4.1.6. *Let $t \rightsquigarrow j(t, \cdot\,)$ be a non-zero jump function, and set $j^\Gamma(t, d\mathbf{y}) = \mathbf{1}_\Gamma(\mathbf{y}) j(t, d\mathbf{y})$ for $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$. If $\Delta \in \mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0} \notin \bar{\Delta}$ and if $\boldsymbol{\psi}^\Delta(t) = \int_\Delta \mathbf{y}\, j(t, d\mathbf{y})$, then $\boldsymbol{\psi}^\Delta$ is a piecewise constant element of $D(\mathbb{R}^N)$, $j(t, \cdot\,, \boldsymbol{\psi}^\Delta) = j^\Delta(t, \cdot\,)$, and $j(t, \cdot\,, \boldsymbol{\psi} - \boldsymbol{\psi}^\Delta) = j^{\mathbb{R}^N\setminus\Delta}(t, \cdot\,) = j(t, \cdot\,) - j^\Delta(t, \cdot\,)$ for any $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ whose jump function is $t \rightsquigarrow j(t, \cdot\,)$. Finally, suppose that $\{\boldsymbol{\psi}_m : m \geq 0\} \subseteq D(\mathbb{R}^N)$ and satisfy the conditions that $\mathbb{R}^N \setminus \{\mathbf{0}\} = \bigcup_{m=0}^\infty \Delta_m$ and, for each $m \in \mathbb{N}$, $\mathbf{0} \notin \Delta_m$ and $j(t, \cdot\,, \boldsymbol{\psi}_m) = j^{\Delta_m}(t, \cdot\,)$, $t \geq 0$. If $\boldsymbol{\psi}_m \longrightarrow \boldsymbol{\psi}$ uniformly on compacts, then $j(t, \cdot\,, \boldsymbol{\psi}) = j(t, \cdot\,)$, $t \geq 0$.*

Proof: Throughout the proof, we use the notation introduced in Lemma 4.1.2.

Assuming that $\mathbf{0} \notin \bar{\Delta}$, we know that

$$j^\Delta(t, \cdot\,) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t) \mathbf{1}_\Delta(\mathbf{y}_\tau) \delta_{\mathbf{y}_\tau},$$

where, for each $t > 0$, there are only finitely many non-vanishing terms. At the same time,

$$\boldsymbol{\psi}^\Delta(t) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t) \mathbf{1}_\Delta(\mathbf{y}_\tau) \mathbf{y}_\tau$$

and

$$j(t, \cdot\,, \boldsymbol{\psi} - \boldsymbol{\psi}^\Delta) = \sum_{\tau \in J} \mathbf{1}_{[\tau,\infty)}(t) \mathbf{1}_{\mathbb{R}^N\setminus\Delta}(\mathbf{y}_\tau) \delta_{\mathbf{y}_\tau}$$

if $j(t,\,\cdot\,,\boldsymbol{\psi}) = j(t,\,\cdot\,)$. Thus, all that remains is to prove the final assertion. To this end, suppose that $j(t,\,\cdot\,,\boldsymbol{\psi}) \neq j(t-,\,\cdot\,,\boldsymbol{\psi})$. Since $\|\boldsymbol{\psi} - \boldsymbol{\psi}_m\|_{[0,t]} \longrightarrow 0$, there exists an m such that $\boldsymbol{\psi}_m(t) \neq \boldsymbol{\psi}_m(t-)$ and therefore that $j(t,\,\cdot\,) - j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $y \in \Delta_m$. Since this means that $\boldsymbol{\psi}_n(t) - \boldsymbol{\psi}_n(t-) = \mathbf{y}$ for all $n \geq m$, it follows that $\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-) = y$ and therefore that $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = \delta_{\mathbf{y}} = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$. Conversely, suppose that $j(t,\,\cdot\,) \neq j(t-,\,\cdot\,)$ and choose m so that $j(t,\,\cdot\,) - j(t-,\,\cdot\,) = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \Delta_m$. Then $\boldsymbol{\psi}_n(t) - \boldsymbol{\psi}_n(t-) = \mathbf{y}$ for all $n \geq m$. Thus, since this means that $\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t-) = \mathbf{y}$, $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = \delta_{\mathbf{y}} = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$. After combining these, we see that $j(t,\,\cdot\,,\boldsymbol{\psi}) - j(t-,\,\cdot\,,\boldsymbol{\psi}) = j(t,\,\cdot\,) - j(t-,\,\cdot\,)$ for all $t > 0$, from which it is an easy step to $j(t,\,\cdot\,) = j(t,\,\cdot\,,\boldsymbol{\psi})$ for all $t \geq 0$. □

Exercises for § 4.1

Exercise 4.1.7. Let $\mathbf{e} \in \mathbb{S}^{N-1}$, set $\boldsymbol{\psi}_t(\tau) = \mathbf{1}_{[t,\infty)}(\tau)\mathbf{e}$ for $t \in [0,1]$, and show that $\|\boldsymbol{\psi}_t - \boldsymbol{\psi}_s\|_{[0,1]} = 1$ for all $s \neq t$ from $[0,1]$. Conclude from this that $D(\mathbb{R}^N)$ is not separable in the topology of uniform convergence on compacts.

Exercise 4.1.8. Show that a function $\varphi : D(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable if and only if there exists an $(\mathbb{R}^N)^{\mathbb{N}}$-measurable function $\Phi : (\mathbb{R}^N)^{\mathbb{N}} \longrightarrow \mathbb{R}$ and a sequence $\{t_k : k \in \mathbb{N}\} \subseteq [0,\infty)$ such that

$$\varphi(\boldsymbol{\psi}) = \Phi\big(\boldsymbol{\psi}(t_0), \dots, \boldsymbol{\psi}(t_k), \dots\big), \quad \boldsymbol{\psi} \in D(\mathbb{R}^N).$$

Next, define $\boldsymbol{\psi}_t$ as in Exercise 4.1.7, and use that exercise together with the preceding to show that the open set $\{\boldsymbol{\psi} \in D(\mathbb{R}^N) : \exists\, t \in [0,1]\ \|\boldsymbol{\psi} - \boldsymbol{\psi}_t\|_{[0,1]} < 1\}$ is not $\mathcal{F}_{D(\mathbb{R}^N)}$-measurable. Conclude that $\mathcal{B}_{D(\mathbb{R}^N)} \subsetneqq \mathcal{F}_{D(\mathbb{R}^N)}$. Similarly, conclude that neither $D(\mathbb{R}^N)$ nor $C(\mathbb{R}^N)$ is a measurable subset of $(\mathbb{R}^N)^{[0,\infty)}$. Finally, show that $C(\mathbb{R}^N) \in \mathcal{F}_{D(\mathbb{R}^N)}$.

Exercise 4.1.9. Show that

$$\text{var}_{[0,t]}(\boldsymbol{\psi}) \geq \int_{\mathbb{R}^N} |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}), \quad (t, \boldsymbol{\psi}) \in [0,\infty) \times D(\mathbb{R}^N). \tag{4.1.10}$$

Hint: This is most easily seen from the representation of $j(t,\,\cdot\,,\boldsymbol{\psi})$ in terms of point masses at the discontinuities of $\boldsymbol{\psi}$. One can use this representation to see that, for each $r > 0$,

$$\text{var}_{[0,t]}(\boldsymbol{\psi}) \geq \sum_{\tau \in J(t,r,\boldsymbol{\psi})} \big|\boldsymbol{\psi}(\tau) - \boldsymbol{\psi}(\tau-)\big| = \int_{|\mathbf{y}| \geq r} |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}), \quad (t, \boldsymbol{\psi}) \in [0,\infty).$$

Exercise 4.1.11. If $\boldsymbol{\psi}$ is an absolutely pure jump path, show that $\text{var}_{[0,t]}(\boldsymbol{\psi}) = \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi})$ and therefore that $\boldsymbol{\psi}$ has locally bounded variation. Conversely, if $\boldsymbol{\psi} \in C(\mathbb{R}^N)$ has locally bounded variation, show that $\boldsymbol{\psi}$ is an absolutely pure jump path if and only if $\mathrm{var}_{[0,t]}(\boldsymbol{\psi}) = \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi})$. Finally, if $\boldsymbol{\psi} \in D(\mathbb{R}^N)$ and $j(t, \cdot\,, \boldsymbol{\psi}) \in \mathfrak{M}_1(\mathbb{R}^N)$ for all $t \geq 0$, set $\boldsymbol{\psi}_{\mathrm{c}}(t) \equiv \boldsymbol{\psi}(t) - \int \mathbf{y}\, j(t, dy, \boldsymbol{\psi})$ and show that $\boldsymbol{\psi}_{\mathrm{c}} \in C(\mathbb{R}^N)$ and

$$\mathrm{var}_{[0,t]}(\boldsymbol{\psi}) = \mathrm{var}_{[0,t]}(\boldsymbol{\psi}_{\mathrm{c}}) + \int |\mathbf{y}|\, j(t, d\mathbf{y}, \boldsymbol{\psi}).$$

Exercise 4.1.12. If $\nu \in \mathbf{M}_1(\mathbb{R}^N)$, show that $\nu(\Gamma) \in \{0,1\}$ for all $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$ if and only if $\nu = \delta_{\mathbf{y}}$ for some $\mathbf{y} \in \mathbb{R}^N$.

Hint: Begin by showing that it suffices to handle the case when $N = 1$. Next, assuming that $N = 1$, show that ν is compactly supported, let m be its mean value, and show that $\nu = \delta_m$.

§ 4.2 Discontinuous Lévy Processes

In this section we will construct the Lévy processes corresponding to those $\mu \in \mathcal{I}(\mathbb{R}^N)$ with no Gaussian component. That is,

(4.2.1)
$$\hat{\mu}(\boldsymbol{\xi}) = \exp\Bigg(\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{m}_\mu\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})} - 1 - \sqrt{-1}\,\mathbf{1}_{[0,1]}(|\mathbf{y}|)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M_\mu(d\mathbf{y})\Bigg).$$

Because they are the building blocks out of which all such processes are made, we treat separately the case when μ is a Poisson measure π_M for some $M \in \mathfrak{M}_0(\mathbb{R}^N)$ and will call the corresponding Lévy process the **Poisson process** associated with M.

§ 4.2.1. The Simple Poisson Process. We begin with the case when $N = 1$ and $M = \delta_1$, for which π_M is the **simple Poisson measure** $e^{-1}\sum_{m=0}^{\infty} \frac{1}{m!}\delta_m$ whose Fourier transform is $\exp\Big(e^{\sqrt{-1}\xi} - 1\Big)$.

To construct the Poisson process associated with M, we start with a sequence $\{\tau_m : m \geq 1\}$ of independent, **unit exponential** random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is,

$$\mathbb{P}\big(\{\omega : \tau_1(\omega) > t_1, \dots, \tau_n(\omega) > t_n\}\big) = \exp\left(-\sum_{m=1}^{n} t_m^+\right)$$

for all $n \in \mathbb{Z}^+$ and $(t_1, \dots, t_n) \in (\mathbb{R}^N)^n$. Without loss in generality, we may and will assume that $\tau_m(\omega) > 0$ for all $m \in \mathbb{Z}^+$ and $\omega \in \Omega$. In addition, by the Strong Law of Large Numbers, we may and will assume that $\sum_{m=1}^{\infty} \tau_m(\omega) = \infty$ for all $\omega \in \Omega$. Next, set $T_0(\omega) = 0$ and $T_n(\omega) = \sum_{m=1}^{n} \tau_m(\omega)$, and define

$$N(t, \omega) = \max\{n \in \mathbb{N} : T_n(\omega) \leq t\} = \sum_{n=1}^{\infty} \mathbf{1}_{[T_n(\omega),\infty)}(t) \quad \text{for } t \in [0, \infty). \tag{4.2.2}$$

Clearly $t \rightsquigarrow N(t,\omega)$ is a non-decreasing, right-continuous, piecewise constant path, $\mathbb{N}$-valued path which starts at 0 and, whenever it jumps, jumps by $+1$. In particular, $N(\,\cdot\,,\omega) \in D(\mathbb{R}^N)$, $N(t,\omega) - N(t-,\omega) \in \{0,1\}$ for all $t \in (0,\infty)$, and so $j\big(t,\,\cdot\,,N(\,\cdot\,,\omega)\big) = N(t,\omega)\delta_1$.

Clearly, $\mathbb{P}\big(N(t)=n\big) = \mathbb{P}\big(T_n \le t < T_{n+1}\big)$. Thus, $\mathbb{P}\big(N(t)=0\big) = \mathbb{P}(\tau_1 > t) = e^{-t}$, and, when $n \ge 1$,

$$\mathbb{P}\big(T_n \le t < T_{n+1}\big) = \underset{A}{\int\cdots\int} e^{-\sum_{m=1}^{n+1}\tau_m}\,d\tau_1\cdots d\tau_{n+1} = e^{-t}|B|,$$

where $A = \big\{(\tau_1,\dots,\tau_{n+1}) \in (0,\infty)^{n+1} : \sum_{m=1}^{n}\tau_m \le t < \sum_{m=1}^{n+1}\tau_m\big\}$ and $B = \big\{(\tau_1,\dots,\tau_n) \in (0,\infty)^n : \sum_{m=1}^{n}\tau_m \le t\big\}$. By making the change of variables $s_m = \sum_{j=1}^{m}\tau_j$ and remarking that the associated Jacobian is 1, one sees that $|B| = |C|$, where $C = \big\{(s_1,\dots,s_n) \in \mathbb{R}^n : 0 < s_1 < \cdots < s_n \le t\big\}$. Since $|C| = \frac{t^n}{n!}$, we have shown that the $\mathbb{P}$-distribution of $N(t)$ is the Poisson measure $\pi_{t\delta_1}$. In particular, π_{δ_1} is the $\mathbb{P}$-distribution of $N(1)$.

We now want to use the same sort of calculation to show that $\{N(t) : t \in [0,\infty)\}$ is a **simple Poisson process**, that is, a Lévy for π_{δ_1}. See Exercise 4.2.18 for another, perhaps preferable, approach.

Lemma 4.2.3. *For any $(s,t) \in [0,\infty)$, the $\mathbb{P}$-distribution of the increment $N(s+t)-N(s)$ is $\pi_{t\delta_1}$. In addition, for any $K \in \mathbb{Z}^+$ and $0 = t_0 < t_1 < \cdots < t_K$, the increments $\{N(t_k) - N(t_{k-1}) : 1 \le k \le K\}$ are independent.*

Proof: What we have to show is that for all $K \in \mathbb{Z}^+$, $0 = n_0 \le \cdots \le n_K$, and $0 = t_0 < t_1 < \cdots < t_K$,

$$\begin{aligned}\mathbb{P}\big(N(t_k) - N(t_{k-1}) = n_k - n_{k-1},\ 1 \le k \le K\big)&\\ = \prod_{k=1}^{K} \frac{e^{-(t_k-t_{k-1})}(t_k - t_{k-1})^{n_k-n_{k-1}}}{(n_k - n_{k-1})!}&,\end{aligned}$$

which is equivalently to checking that

$$\mathbb{P}\big(N(t_k) = n_k,\ 1 \le k \le K\big) = \prod_{k=1}^{K} \frac{e^{-(t_k-t_{k-1})}(t_k - t_{k-1})^{n_k-n_{k-1}}}{(n_k - n_{k-1})!},$$

and, since the case when $n_K = 0$ is trivial, we will assume that $n_K \ge 1$. To this end, note that

$$\begin{aligned}\mathbb{P}\big(N(t_k) = n_k,\ 0 \le k \le K\big) &= \mathbb{P}\big(T_{n_k} \le t_k < T_{n_{k+1}},\ 1 \le k \le K\big)\\ &= \underset{A}{\int\cdots\int} e^{-\sum_{m=1}^{n_K+1}\tau_m}\,d\tau_1\cdots d\tau_{n_K+1} = e^{-t_K}|B|,\end{aligned}$$

where

$$A = \left\{ (\tau_1, \dots, \tau_{n_K+1}) \in (0,\infty)^{n_K+1} : \sum_{m=1}^{n_k} \tau_m \le t_k < \sum_{m=1}^{n_k+1} \tau_m,\ 1 \le k \le K \right\}$$

and

$$B = \left\{ (\tau_1, \dots, \tau_{n_K}) \in (0,\infty)^{n_K} : t_{k-1} < \sum_{m=1}^{n_k} \tau_m \le t_k : 1 \le k \le K \right\}.$$

To compute $|B|$, set $S = \{1 \le k \le K : n_{k-1} < n_k\}$, and make the change of variables $s_m = \sum_{j=1}^{m} \tau_j$ to see that $|B| = |C|$, where

$$C = \bigl\{ (s_1, \dots, s_{n_K}) \in \mathbb{R}^{n_K} : t_{k-1} < s_{n_{k-1}+1} < \cdots < s_{n_k} \le t_k \text{ for } k \in S \bigr\}.$$

Finally, for $k \in S$, set

$$C_k = \bigl\{ (s_{n_{k-1}+1}, \dots, s_{n_k}) \in \mathbb{R}^{n_k - n_{k-1}} : t_{k-1} < s_{n_{k-1}+1} < \cdots < s_{n_k} \le t_k \bigr\},$$

and check that

$$\begin{aligned} e^{-t_K}|C| = e^{-t_K} \prod_{k \in S} |C_k| &= e^{-t_K} \prod_{k \in S} \frac{(t_k - t_{k-1})^{n_k - n_{k-1}}}{(n_k - n_{k-1})!} \\ &= \prod_{k=1}^{K} \frac{e^{-(t_k - t_{k-1})}(t_k - t_{k-1})^{n_k - n_{k-1}}}{(n_k - n_{k-1})!}. \quad \square \end{aligned}$$

The simple Poisson process $\{N(t) : t \ge 0\}$ is aptly named. It starts at 0, waits a unit exponential holding time before jumping to 1, sits at 1 for another, independent, unit exponential holding time before jumping to 2, etc. Thus, since π_{δ_1} is the distribution of this process at time 1, we now have an appealing picture of the way in which simple Poisson random variables arise.

Given $\alpha \in [0,\infty)$, we will say that a $D(\mathbb{R})$-valued process whose distribution is the same as $\{N(\alpha t) : t \ge 0\}$ is a **simple Poisoon process run at rate** α.

§ 4.2.2. Compound Poisson Processes. We next want to build a Poisson process associated with a general $M \in \mathfrak{M}_0(\mathbb{R}^N)$. If $M = 0$, there is nothing to do, since the corresponding process will simply sit at $\mathbf{0}$ for all time. If $M \ne 0$, we write it as $\alpha\nu$, where $\alpha = M(\mathbb{R}^N)$ and $\nu = \frac{M}{\alpha}$. After augmenting our probability space if necessary, we introduce a sequence $\{\mathbf{X}_n : n \ge 1\}$ of independent, ν-distributed, random variables which are independent of the unit exponential random variables $\{\tau_m : m \ge 1\}$ out of which we built the simple Poisson process

$\{N(t) : t \geq 0\}$ in the preceding subsection. Further, since $M(\{\mathbf{0}\}) = 0$, we may and will assume that none of the $\mathbf{X}_n$'s is ever $\mathbf{0}$. Finally, we set

$$\mathbf{Z}_M(t,\omega) = \sum_{1 \leq n \leq N(\alpha t,\omega)} \mathbf{X}_n(\omega), \tag{4.2.4}$$

with the understanding that a sum over the empty set is $\mathbf{0}$.

Clearly, the process $\{\mathbf{Z}_M(t) : t \geq 0\}$ is nearly as easily understood as is the simple Poisson process. Like the simple Poisson process, its paths are right-continuous, start at $\mathbf{0}$, and are piecewise constant. Further, its holding times and jumps are all independent of one another. The difference is that its holding times are now α-exponential random variable (i.e., exponential with mean value $\frac{1}{\alpha}$) and its jumps are random variables with distribution ν. In particular,

$$j\big(t,\,\cdot\,,\mathbf{Z}_M(\,\cdot\,,\omega)\big) = \sum_{1 \leq n \leq N(\alpha t,\omega)} \delta_{\mathbf{X}_n(\omega)} = \sum_{n=1}^{\infty} \mathbf{1}_{[T_n(\omega),\infty)}(t)\delta_{\mathbf{X}_n(\omega)}. \tag{4.2.5}$$

We now want to check that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Lévy corresponding for π_M, and, as such, deserves to be called a Poisson process associated with M: the one with **rate** $M(\mathbb{R}^N)$ and **jump distribution** $\frac{M}{M(\mathbb{R}^N)}$. That is, we must show that, for each $0 = t_0 < t_1 < \cdots t_K$, the random variables $\mathbf{Z}_M(t_k) - \mathbf{Z}_M(t_{k-1})$, $1 \leq k \leq K$, are independent and that the kth one has distribution $\pi_{(t_k - t_{k-1})M}$. Equivalently, we need to check that, for any $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_K \in \mathbb{R}^N$,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k, \mathbf{Z}_M(t_k) - \mathbf{Z}_M(t_{k-1})\big)_{\mathbb{R}^N}\right)\right] = \prod_{k=1}^{K} \widehat{\pi_{\tau_k M}}(\boldsymbol{\xi}_k),$$

where $\tau_k = t_k - t_{k-1}$. But, because of our independence assumptions, the above expectation is equal to

$$\begin{aligned}
&\sum_{n_K \geq \cdots \geq n_1 \geq 0} \mathbb{P}\big(N(\alpha t_k) - N(\alpha t_{k-1}) = n_k - n_{k-1},\ 1 \leq k \leq K\big) \\
&\qquad\qquad \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^{K}\ \sum_{n_{k-1}+1 < m \leq n_k} \big(\boldsymbol{\xi}_k, \mathbf{X}_m\big)_{\mathbb{R}^N}\right)\right] \\
&= \sum_{n_K \geq \cdots \geq n_1 \geq 0} \prod_{k=1}^{K} \frac{e^{-\alpha\tau_k}\tau_k^{n_k - n_{k-1}}}{(n_k - n_{k-1})!}\left(\int_{\mathbb{R}^N}\left(e^{\sqrt{-1}(\boldsymbol{\xi}_k,\mathbf{y})_{\mathbb{R}^N}} - 1\right)\nu(d\mathbf{y})\right)^{n_k - n_{k-1}} \\
&= \prod_{k=1}^{K} \widehat{\pi_{\tau_k M}}(\boldsymbol{\xi}_k).
\end{aligned}$$

Any stochastic process $\{\mathbf{Z}(t) : t \geq 0\}$ with right-continuous, piecewise constant paths and the same distribution as the process $\{\mathbf{Z}_M(t) : t \geq 0\}$ just constructed is called a **Poisson process** associated with M.

Here is a beautiful and important procedure for transforming one Poisson process into another.

Lemma 4.2.6. *Suppose that $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ is a Borel measurable function which takes the origin in $\mathbb{R}^N$ into the origin in $\mathbb{R}^{N'}$, and, for $M \in \mathfrak{M}_0(\mathbb{R}^N)$, define $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$ by*

$$M^F(\Gamma) = M\Big(F^{-1}\big(\Gamma \setminus \{\mathbf{0}\}\big)\Big) \quad \text{for } \Gamma \in \mathcal{B}_{\mathbb{R}^{N'}}.$$

If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Poisson process associated with π_M and

$$\mathbf{Z}^F(t,\omega) = \int_{\mathbb{R}^N} F(\mathbf{y})\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big) \quad \text{for } (t,\omega) \in [0,\infty) \times \Omega, \tag{4.2.7}$$

then $\{\mathbf{Z}^F(t) : t \geq 0\}$ is a Poisson associated with π_{M^F}. Moreover, if, for each i in an index set $\mathcal{I}$, $F_i : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_i}$ is a Borel measurable satisfying $F_i(\mathbf{0}) = \mathbf{0}$ and, for each $\mathbf{y} \in \mathbb{R}^N$, there is at most one $i \in \mathcal{I}$ for which $F_i(\mathbf{y}) \neq \mathbf{0}$, then the processes $\big\{\{\mathbf{Z}^{F_i}(t) : t \geq 0\} : i \in \mathcal{I}\big\}$ are independent.

Proof: In proving the first part, we will, without loss in generality, assume that (cf. (4.2.4)) $\mathbf{Z} = \mathbf{Z}_M$. But then, by (4.2.5),

$$\mathbf{Z}^F(t,\omega) = \sum_{1 \leq n \leq N(\alpha t,\omega)} F\big(\mathbf{X}_n(\omega)\big),$$

from which the first assertion is immediate.

To prove the second assertion, we begin by observing that it suffices to treat the case when $\mathcal{I} = \{1,2\}$. To see this, suppose that we know the result in this case, and let $n > 2$ and a set $\{i_1,\dots,i_n\}$ of distinct elements from $\mathcal{I}$ be given. By taking $F_1 = (F_{i_1},\dots,F_{i_{n-1}})$ and $F_2 = F_{i_n}$ and applying the assumed result, we would have that $\{\mathbf{Z}^{F_{i_n}}(t) : t \geq 0\}$ is independent of $\big\{\big(\mathbf{Z}^{F_{i_1}}(t),\dots,\mathbf{Z}^{F_{i_{n-1}}}(t)\big) : t \geq 0\big\}$. Hence, proceeding by induction, we would be able to show that the processes $\big\{\{\mathbf{Z}^{F_{i_m}}(t) : t \geq 0\} : 1 \leq m \leq n\big\}$ are independent.

Now assume that $\mathcal{I} = \{1,2\}$. What we have to check is that, for any $K \in \mathbb{Z}^+$, $0 = t_0 < t_1 < \cdots < t_K$ and $\{(\boldsymbol{\xi}_k^1, \boldsymbol{\xi}_k^2) : 1 \leq k \leq K\} \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Bigg[\exp\Bigg(\sqrt{-1}\sum_{k=1}^K \Big[\big(\boldsymbol{\xi}_k^1, \mathbf{Z}^{F_1}(t_k) - \mathbf{Z}^{F_1}(t_{k-1})\big)_{\mathbb{R}^{N_1}} \\
&\qquad\qquad + \big(\boldsymbol{\xi}_k^2, \mathbf{Z}^{F_2}(t_k) - \mathbf{Z}^{F_2}(t_{k-1})\big)_{\mathbb{R}^{N_2}}\Big]\Bigg)\Bigg] \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^K \big(\boldsymbol{\xi}_k^1, \mathbf{Z}^{F_1}(t_k) - \mathbf{Z}^{F_1}(t_{k-1})\big)_{\mathbb{R}^{N_1}}\right)\right] \\
&\qquad\qquad \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sqrt{-1}\sum_{k=1}^K \big(\boldsymbol{\xi}_k^2, \mathbf{Z}^{F_2}(t_k) - \mathbf{Z}^{F_2}(t_{k-1})\big)_{\mathbb{R}^{N_2}}\right)\right].
\end{aligned}$$

For this purpose, take $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_1+N_2}$ to be given by $F(\mathbf{y}) = \big(F_1(\mathbf{y}), F_2(\mathbf{y})\big)$, and set $\boldsymbol{\xi}_k = (\boldsymbol{\xi}_k^1, \boldsymbol{\xi}_k^2)$. Then the first expression in the preceding equals

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\exp\Big(\sqrt{-1}\sum_{k=1}^{K}\big(\boldsymbol{\xi}_k, \mathbf{Z}^F(t_k) - \mathbf{Z}^F(t_{k-1}\big)_{\mathbb{R}^{N_1+N_2}}\Big)\right] \\
&\qquad = \prod_{k=1}^{K}\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\sqrt{-1}\big(\boldsymbol{\xi}_k, \mathbf{Z}^F(t_k - t_{k-1})\big)_{\mathbb{R}^{N_1+N_2}}\Big)\Big],
\end{aligned}$$

since $\{\mathbf{Z}^F(t) : t \geq 0\}$ has independent, homogeneous increments. Hence, it suffices for us to observe that, for any $t > 0$ and $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2)$,

$$\begin{aligned}
\mathbb{E}^{P}\Big[\exp\Big(\big(\boldsymbol{\xi}, \mathbf{Z}^F(t)\big)_{\mathbb{R}^{N_1+N_2}}\Big)\Big] &= \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}, F(\mathbf{y}))_{\mathbb{R}^{N_1+N_2}}} - 1\Big)\, M(d\mathbf{y})\right) \\
= \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}^1, F_1(\mathbf{y}))_{\mathbb{R}^{N_1}}} - 1\Big)\, M(d\mathbf{y})\right)& \\
\times \exp\left(t\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi}^2, F_2(\mathbf{y}))_{\mathbb{R}^{N_2}}} - 1\Big)\, M(d\mathbf{y})\right)& \\
= \mathbb{E}^{P}\Big[\exp\Big(\big(\boldsymbol{\xi}^1, \mathbf{Z}^{F_1}(t)\big)_{\mathbb{R}^{N_1}}\Big)\Big]\mathbb{E}^{P}\Big[\exp\Big(\big(\boldsymbol{\xi}^2, \mathbf{Z}^{F_2}(t)\big)_{\mathbb{R}^{N_2}}\Big)\Big].& \quad \square
\end{aligned}$$

As an essentially immediate consequence of Lemma 4.2.6 and Theorem 4.1.6, we have the following important conclusion.

THEOREM 4.2.8. *If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Poisson process associated with π_M, then, for each $\Delta \in \mathcal{B}_{\mathbb{R}^N \setminus \{\mathbf{0}\}}$, $\big\{j\big(t, \Delta, Z(\,\cdot\,)\big) : t \geq 0\big\}$ is a simple Poisson process run at rate $M(\Delta)$. Moreover, if*

$$\mathbf{Z}^{\Delta}(t) = \int_{\Delta}\mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}\big),$$

then $j\big(t, \Gamma, \mathbf{Z}^{\Delta}\big) = j\big(t, \Gamma \cap \Delta, \mathbf{Z}\big)$ for all $(t, \Gamma) \in [0, \infty) \times \mathcal{B}_{\mathbb{R}^N}$. Finally, if $\{\Delta_i : i \in \mathcal{I}\}$ is a family of mutually disjoint Borel subsets of $\mathbb{R}^N \setminus \{\mathbf{0}\}$, then the processes $\big\{\{j\big(t, \Delta_i, \mathbf{Z}\big) : t \geq 0\} : i \in \mathcal{I}\big\}$ are mutually independent.

The result in Theorem 4.2.8 says that the jumps of Poisson process can be decomposed into a family of mutually independent, simple Poisson process run at rates determined by the measure under of the jump sizes. The next result can be thought of as a re-assembly procedure which complements this decomposition result.

THEOREM 4.2.9. *If $\big\{\{\mathbf{Z}_k(t) : t \geq 0\} : 1 \leq k \leq K\big\}$ are independent Poisson processes associated with $\{M_k : 1 \leq k \leq K\} \subseteq \mathfrak{M}_0(\mathbb{R}^N)$, then*

$$\left\{\mathbf{Z}(t) \equiv \sum_{k=1}^{K}\mathbf{Z}_k(t) : t \geq 0\right\} \text{ is a Poisson process associated with } M \equiv \sum_{k=1}^{K} M_k.$$

Next, suppose that the M_k*'s are mutually singular in the sense that, for each* k*, there exists a* $\Delta_k \in \mathcal{B}_{\mathbb{R}^N \setminus \{\mathbf{0}\}}$ *such that the* Δ_k*'s are mutually disjoint and* $M_k(\Delta_k \complement) = 0 = M_\ell(\Delta_k)$ *for* $\ell \neq k$*. Then, for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*,*

$$j\big(t, \cdot\,, \mathbf{Z}(\,\cdot\,, \omega)\big) = \sum_{k=1}^{K} j\big(t, \cdot\,, \mathbf{Z}_k(\,\cdot\,, \omega)\big), \quad t \in [0, \infty).$$

Equivalently, for $\mathbb{P}$*-almost every* $\omega \in \Omega$ *and all* $t \geq 0$*, there is at most one* k *such that* $\mathbf{Z}_k(t, \omega) \neq \mathbf{Z}_k(t-, \omega)$*.*

PROOF: Clearly, $\{\mathbf{Z}(t) : t \geq 0\}$ starts at $\mathbf{0}$ and has independent increments. In addition, for any s, $t \in [0, \infty)$ and $\boldsymbol{\xi} \in \mathbb{R}^N$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}(s+t) - \mathbf{Z}(s))_{\mathbb{R}^N}}\Big] &= \prod_{k=1}^{K} \mathbb{E}^{\mathbb{P}}\Big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}_k(s+t) - \mathbf{Z}_k(s))_{\mathbb{R}^N}}\Big] \\ &= \prod_{k=1}^{K} \exp\left(t \int_{\mathbb{R}^N} \left(e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1\right) M_k(d\mathbf{y})\right) \\ &= \exp\left(t \int_{\mathbb{R}^N} \left(e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1\right) M(d\mathbf{y})\right).\end{aligned}$$

Now assume that the M_k's are as in the final part of the statement, and choose Δ_k's accordingly. Without loss in generality, we will assume that $\mathbb{R}^N \setminus \{\mathbf{0}\} = \bigcup_{k=1}^{K} \Delta_k$. Also, because the assertion depends only on the joint distribution of the processes involved, we may and will assume that

$$\mathbf{Z}_k(t) = \int_{\Delta_k} \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}\big) \quad \text{for } 1 \leq k \leq K,$$

since then $\mathbf{Z}(t) = \sum_{k=1}^{K} \mathbf{Z}_k(t)$, and, by Theorem 4.2.8, the $\mathbf{Z}_k$'s are independent and the kth one is a Poisson process associated with M_k. But with this choice, another application of Theorem 4.2.8 shows that $j\big(t, \Gamma, \mathbf{Z}_k\big) = j\big(t, \Gamma \cap \Delta_k, \mathbf{Z}\big)$, and therefore

$$j\big(t, \Gamma, \mathbf{Z}\big) = \sum_{k=1}^{K} j\big(t, \Gamma, \mathbf{Z}_k\big), \quad t \in [0, \infty). \quad \square$$

Because the paths of a Poisson process are piecewise constant, they certainly have finite variation on each compact time interval. The first part of next lemma allows us to estimate that variation.

LEMMA 4.2.10. *If* $\{\mathbf{Z}(t) : t \geq 0\}$ *is a Poisson process associated with* $M \in \mathfrak{M}_0(\mathbb{R}^N)$*, then*

$$\mathbb{E}^{\mathbb{P}}\big[\mathrm{var}_{[0,t]}(\mathbf{Z})\big] = \int_{\mathbb{R}^N} |\mathbf{y}|\, M(dy).$$

In addition, if $\int_{\mathbb{R}^N} |\mathbf{y}|\, M(d\mathbf{y}) < \infty$ *and* $\bar{\mathbf{Z}}(t) = \mathbf{Z}(t) - \int_{\mathbb{R}^N} \mathbf{y}\, M(d\mathbf{y})$, *then*

$$\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} \geq R\big) \leq \frac{Nt}{R^2}\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}(t)|^2\big] = \frac{Nt}{R^2}\int_{\mathbb{R}^N} |\mathbf{y}|^2\, M(d\mathbf{y}).$$

Proof: Again, we will assume that $\mathbf{Z} = \mathbf{Z}_M$, in which case

$$\operatorname{var}_{[0,t]}(\mathbf{Z}) = \sum_{1 \leq m \leq N(\alpha t)} |\mathbf{X}_m|.$$

Hence (cf. the notation used in § 4.1.1)

$$\mathbb{E}^{\mathbb{P}}\big[\operatorname{var}_{[0,t]}(\mathbf{Z})\big] = \mathbb{E}^{\mathbb{P}}\big[N(\alpha t)\big]\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|\big] = \alpha t \int_{\mathbb{R}^N} |\mathbf{y}|\, \nu(d\mathbf{y}) = t\int_{\mathbb{R}^N} |\mathbf{y}|\, M(d\mathbf{y}).$$

Turning to the second part, begin by observing that

$$\begin{aligned}\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} > R\big) &= \lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq m\leq 2^n} \big|\bar{\mathbf{Z}}\big(m2^{-n}t\big)\big| > R\right)\\ &\leq N \lim_{n\to\infty} \sup_{\mathbf{e}\in\mathbb{S}^{N-1}} \mathbb{P}\left(\max_{1\leq m\leq 2^n} \big|\big(\mathbf{e}, \bar{\mathbf{Z}}\big(m2^{-n}t\big)\big)_{\mathbb{R}^N}\big| > R\right).\end{aligned}$$

Next, given $\mathbf{e} \in \mathbb{S}^{N-1}$ and $n \geq 1$, write

$$\big(\mathbf{e}, \bar{\mathbf{Z}}(m2^{-n}t)\big)_{\mathbb{R}^N} = \sum_{1\leq \ell\leq m} \big(\mathbf{e}, \bar{\mathbf{Z}}(\ell 2^{-n}t) - \bar{\mathbf{Z}}((\ell-1)2^{-n}t)\big)_{\mathbb{R}^N},$$

and apply Kolmogorov's Inequality to conclude that

$$\mathbb{P}\left(\max_{1\leq m\leq 2^n} \big|\big(\mathbf{e}, \bar{\mathbf{Z}}\big(m2^{-n}t\big)\big)_{\mathbb{R}^N}\big| > R\right) \leq R^{-2}\mathbb{E}^{\mathbb{P}}\big[\big(\mathbf{e}, \bar{\mathbf{Z}}(t)\big)^2_{\mathbb{R}^N}\big].$$

Thus, we will be done once we check that $\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}_M(t)|^2\big] = t\int_{\mathbb{R}^N} |\mathbf{y}|^2\, M(d\mathbf{y})$. To this end, first note that $\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{Z}}(t)|\big] = \mathbb{E}^{\mathbb{P}}\big[|\mathbf{Z}(t)|^2\big] - \alpha^2 t^2 |\mathbf{m}|^2$, where $\mathbf{m} = \int_{\mathbb{R}^N} \mathbf{y}\, \nu(d\mathbf{y})$. At the same time, if $\bar{\mathbf{X}}_m = \mathbf{X}_m - \mathbf{m}$, then $\mathbb{E}^{\mathbb{P}}\big[|\mathbf{Z}(t)|^2\big]$ equals

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left|\sum_{1\leq m\leq N(\alpha t)} \mathbf{X}_m\right|^2\right] &= \mathbb{E}^{\mathbb{P}}\left[\left|\sum_{1\leq m\leq N(\alpha t)} \bar{\mathbf{X}}_m\right|^2\right] + |\mathbf{m}|^2\mathbb{E}^{\mathbb{P}}\big[N(\alpha t)^2\big]\\ &= \alpha t\mathbb{E}^{\mathbb{P}}\big[|\bar{\mathbf{X}}_1|^2\big] + |\mathbf{m}|^2\big(\alpha^2 t^2 + \alpha t\big) = \alpha t\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|^2\big] + \alpha^2 t^2 |\mathbf{m}|^2.\end{aligned}$$

Thus, since $\alpha\mathbb{E}^{\mathbb{P}}\big[|\mathbf{X}_1|^2\big] = \int_{\mathbb{R}^N} |\mathbf{y}|^2\, M(d\mathbf{y})$, the desired equality follows. □

§ 4.2.3. Poisson Jump Processes. Rather than attempting to construct more general Lévy processes directly, we will first construct their jump processes and then construct them out of their jumps. With this idea in mind, we say that $(t,\omega) \rightsquigarrow j(t,\,\cdot\,,\omega)$ is a **Poisson jump process** associated with $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$ if, for each $\omega \in \Omega$, $t \rightsquigarrow j(t,\,\cdot\,,\omega)$ is a jump function, and for each $n \in \mathbb{Z}^+$ and collection $\{\Delta_1,\dots,\Delta_n\} \subseteq \mathcal{B}_{\mathbb{R}^N}$ satisfying $\mathbf{0} \notin \bigcup_{i=1}^n \overline{\Delta_i}$, $\big\{\{j(t,\Delta_i) : t \geq 0\} : 1 \leq i \leq n\big\}$ are independent, simple Poisson processes, the ith of which is run at rate $M(\Delta_i)$ for each $1 \leq i \leq n$. By starting with simple functions and passing to limits, one can easily check that

$$(t,\omega) \in [0,\infty) \times \Omega \longmapsto \int \varphi(\mathbf{y}),\, j(t,d\mathbf{y},\omega) \in [0,\infty]$$

is measurable for every Borel measurable function $\varphi : \mathbb{R}^N \longrightarrow [0,\infty]$. Therefore, if $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ is a Borel measurable function, and, for $T > 0$,

$$\Omega(T) = \left\{\omega : \int |F(\mathbf{y})|\, j(T,d\mathbf{y},\omega) < \infty\right\},$$

then both $\Omega(T)$

$$(t,\omega) \in [0,T] \times \Omega(T) \rightsquigarrow \int F(\mathbf{y})\, j(t,\mathbf{y},\omega) \in \mathbb{R}^{N'}$$

are measurable. Note that if $|F(\mathbf{y})|$ vanishes for $\mathbf{y}$'s in a neighborhood of $\mathbf{0}$, then $\Omega(T) = \Omega$ for all $T > 0$.

Our goal in this subsection is to prove the following existence result.

Theorem 4.2.11. *For each $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$ there exist an associated Poisson jump process.*

Proof: Set $A_0 = \mathbb{R}^N \backslash \overline{B(\mathbf{0},1)}$ and $A_k = \overline{B(\mathbf{0},2^{-k+1})} \backslash \overline{B(\mathbf{0},2^{-k})}$ for $k \in \mathbb{Z}^+$, and define $M_k(d\mathbf{y}) = \mathbf{1}_{A_k}(\mathbf{y})\, M(d\mathbf{y})$. Next, choose independent Poisson processes $\big\{\{\mathbf{Z}_k(t) : t \geq 0\} : k \in \mathbb{N}\big\}$ so that the kth one is associated with M_k, and set $j_k(t,\,\cdot\,,\omega) = j\big(t,\,\cdot\,,\mathbf{Z}_k(\,\cdot\,,\omega)\big)$. Without loss in generality, we may and will assume that $j_k(t,A_k\complement,\omega) = 0$ for all $(t,\omega) \in [0,\infty) \times \Omega$ and $k \in \mathbb{N}$. In addition, by Theorem 4.2.9, if $\mathbf{Z}^{(m)}(t) = \sum_{k=0}^m \mathbf{Z}_k(t)$, then we know that, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$j^{(m)}(t,\,\cdot\,,\omega) \equiv j\big(t,\,\cdot\,,\mathbf{Z}^{(m)}(\,\cdot\,,\omega)\big) = \sum_{k=0}^m j_k(t,\,\cdot\,,\omega), \quad t \geq 0.$$

Hence, we may and will assume that

$$t \rightsquigarrow j(t,\,\cdot\,,\omega) \equiv \sum_{k=1}^\infty j_k(t,\,\cdot\,,\omega)$$

is a jump function for all $\omega \in \Omega$. Finally, suppose that $\{\Delta_i : 1 \le i \le n\} \subseteq \mathcal{B}_{\mathbb{R}^N}$ are disjoint and that $\mathbf{0} \notin \bigcup_{i=1}^n \overline{\Delta_i}$. Choose $m \in \mathbb{N}$ so that $\Delta_m \cap \overline{B(\mathbf{0}, 2^{-m})} = \emptyset$, and note that, $\mathbb{P}$-almost surely, $j(, \Delta_i, \omega) = j^{(m)}(t, \Delta_i, \omega)$ for all $t \ge 0$ and $1 \le i \le n$. Hence, the required property is a consequence of the last part of Theorem 4.2.8. $\square$

In preparation for the next section, we prove the following.

Lemma 4.2.12. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ be a Borel measurable function such that $F(\mathbf{0}) = \mathbf{0}$ and $\mathbf{0} \notin \overline{F^{-1}\big(\mathbb{R}^{N'} \setminus B(\mathbf{0}, r)\big)}$ for any $r > 0$. For any $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$, $M^F \in \mathfrak{M}_\infty(\mathbb{R}^{N'})$. Moreover, if $\{j(t, \cdot\,) : t \ge 0\}$ is a Poisson jump process associated with M, then $\{j^F(t, \cdot\,) : t \ge 0\}$ is a Poisson jump process associated with M^F. Finally, if $\mathbf{0} \notin \overline{F^{-1}(\mathbb{R}^{N'} \setminus \{\mathbf{0}\})}$ and*

$$\mathbf{Z}^F(t, \omega) \equiv \int \mathbf{y}\, j^F(t, d\mathbf{y}, \omega) = \int F(\mathbf{y})\, j(t, d\mathbf{y}, \omega),$$

then $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$, $\{\mathbf{Z}^F(t) : t \ge 0\}$ is a Poisson process associated with M^F, and $j\big(t, \cdot\,, \mathbf{Z}^F(\,\cdot\,, \omega)\big) = j^F(t, \cdot\,, \omega)$.

Proof: To prove the first assertion, suppose that $\{\Delta_1, \dots, \Delta_n\}$ are disjoint, Borel subsets of $\mathbb{R}^{N'}$ such that $\mathbf{0} \notin \bigcup_{i=1}^n \overline{\Delta_i}$. Then $\{F^{-1}(\Delta_1), \dots, F^{-1}(\Delta_n)\}$ satisfy the same conditions as subsets of $\mathbb{R}^N$, and therefore, since $j^F(t, \Delta_i, \omega) = j\big(t, F^{-1}(\Delta_i), \omega\big)$, $\big\{\{j^F(t, \Delta_i) : t \ge 0\} : 1 \le i \le n\big\}$ has the required properties.

Turning to the second assertion, first note that $M^F \in \mathfrak{M}_0(\mathbb{R}^{N'})$ is a immediate consequence of $\mathbf{0} \notin \overline{F^{-1}(\mathbb{R}^{N'} \setminus \{\mathbf{0}\})}$ and that the equality $j\big(t, \cdot\,, \mathbf{Z}^F(\,\cdot\,, \omega)\big) = j^F(t, \cdot\,, \omega)$ is a trivial application the final part of Theorem 4.1.6. To prove that $\{\mathbf{Z}^F(t) : t \ge 0\}$ is a Poisson process associated with M^F, use Theorem 4.2.8 to see that $\{j^F(t, \cdot\,) : t \ge 0\}$ has the same distribution as the jump process for a Poisson process $\{\mathbf{Z}(t) : t \ge 0\}$ associated with M^F. Hence, since $\mathbf{Z}(t) = \int \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z})$, $\{\mathbf{Z}^F(t) : t \ge 0\}$ has the same distribution as $\{\mathbf{Z}(t) : t \ge 0\}$. $\square$

§ 4.2.4. Lévy Processes with Bounded Variation. Although the contents of the previous section provide the machinery to construct a Lévy process for any μ with Fourier transform given by (4.2.1), for reasons made clear in the next lemma, we will treat the special case when $M \in \mathfrak{M}_1(\mathbb{R}^N)$ here and deal with $M \in \mathfrak{M}_2(\mathbb{R}^N) \setminus \mathfrak{M}_1(\mathbb{R}^N)$ in the following subsection.

Lemma 4.2.13. *Let $\{j(t, \cdot\,) : t \ge 0\}$ be a Poisson jump process associated with $M \in \mathfrak{M}_2(\mathbb{R}^N)$, and set $V(t) = \int |\mathbf{y}|\, j(t, d\mathbf{y})$. Then $V(t) < \infty$ almost surely or $V(t) = \infty$ almost surely for all $t > 0$ depending on whether M is or is not in $\mathfrak{M}_1(\mathbb{R}^N)$. (See Exercise 5.1.20 to see that the same conclusion holds for any $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$.)*

Proof: Since, $\int_{|\mathbf{y}|>1} |\mathbf{y}|\, j(t, d\mathbf{y}, \omega) < \infty$ for all $(t, \omega) \in [0, \infty) \times \Omega$, the question is entirely about the finiteness of $V_0(t, \omega) \equiv \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, j(t, d\mathbf{y}, \omega)$. To study this question, set $A_k = \overline{B(\mathbf{0},2^{-k+1})} \setminus \overline{B(\mathbf{0},2^{-k})}$, $F_k(\mathbf{y}) = |\mathbf{y}|\mathbf{1}_{A_k}(\mathbf{y})$, and $V_k(t,\omega) = \int_{A_k} |\mathbf{y}|\, j(t,d\mathbf{y},\omega)$ for $k \ge 1$. Clearly, the processes $\big\{\{V_k(t) : t \ge 0\} : k \in \mathbb{Z}^+\big\}$ are independent. In addition, $t \rightsquigarrow V_k(t)$ is non-decreasing and, by the second part of Lemma 4.2.12, $\{V_k(t) : t \ge 0\}$ is a Poisson process associated with M^{F_k}. Thus, by Lemma 4.2.10,

$$(*)\quad a_k \equiv \mathbb{E}^{\mathbb{P}}\big[V_k(t)\big] = t\int_{A_k} |\mathbf{y}|\, M(d\mathbf{y}) \text{ and } b_k \equiv \operatorname{Var}\big(V_k(t)\big) = t\int |\mathbf{y}|^2\, M(d\mathbf{y}).$$

From the first of these, it follows that

$$\mathbb{E}^{\mathbb{P}}\left[\int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, j(t,d\mathbf{y})\right] = \sum_{k=1}^{\infty} \mathbb{E}^{\mathbb{P}}\big[V_k(t)\big] = \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|\, M(dy),$$

which finishes the case when $M \in \mathfrak{M}_1(\mathbb{R}^N)$. When $M \in \mathfrak{M}_2(\mathbb{R}^N) \setminus \mathfrak{M}_1(\mathbb{R}^N)$, set $\bar{V}_k(t) = V_k(t) - ta_k$. Then, for each $t > 0$, $\{\bar{V}_k(t) : k \in \mathbb{Z}^+\}$ is a sequence of independent random with mean value 0. Furthermore, by the second part of (*),

$$\sum_{k=1}^{\infty} \operatorname{Var}\big(\bar{V}_k(t)\big) = t\sum_{k=1}^{\infty} b_k = t\int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|^2\, M(d\mathbf{y}) < \infty.$$

Hence, by Theorem 1.4.2, $\sum_{k=1}^{\infty} \bar{V}_k(t)$ converges $\mathbb{P}$-almost surely. But, because $M \notin \mathfrak{M}_1(\mathbb{R}^N)$, $\sum_{k=1}^{\infty} a_k = \infty$, and so, for each $t > 0$, $\sum_{k=1}^{\infty} V_k(t)$ must diverge $\mathbb{P}$-almost surely. □

Before stating the main result of the subsection, we introduce the notion of a **generalized Poisson measure**. Namely, if $M \in \mathfrak{M}_1(\mathbb{R}^N) \setminus \mathfrak{M}_0(\mathbb{R}^N)$ and π_M is the element of $\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by

$$\exp\left(\int \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\right) M(d\mathbf{y})\right),$$

or, equivalently, by (4.2.1) with $\mathbf{m} = \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(d\mathbf{y})$, then we call π_M the generalized Poisson measure for M. Similarly, if $\{\mathbf{Z}(t) : t \ge 0\}$ is a Lévy process for a generalized Poisson measure π_M, we will say that it is a **generalized Poisson process** associated with M.

Theorem 4.2.14. *Suppose that $M \in \mathfrak{M}_1(\mathbb{R}^N)$ and that $\{j(t,\cdot\,) : t \ge 0\}$ is a Poisson jump process associated with M. Set $\mathcal{N} = \{\omega : \exists t > 0\ j(t,\cdot\,,\omega) \notin \mathfrak{M}_1(\mathbb{R}^N)\}$, and define $(t,\omega) \rightsquigarrow \mathbf{Z}_M(t,\omega)$ so that*

$$\mathbf{Z}_M(t,\omega) = \begin{cases} \int \mathbf{y}\, j(t,d\mathbf{y},\omega) & \text{if } \omega \notin \mathcal{N} \\ \mathbf{0} & \text{if } \omega \in \mathcal{N}. \end{cases}$$

Then $\mathbb{P}(\mathcal{N}) = 0$ *and* $\{\mathbf{Z}_M(t) : t \geq 0\}$ *is a (possibly generalized) Poisson process associated with* M. *In particular,* $t \rightsquigarrow \mathbf{Z}_M(t, \omega)$ *is absolutely pure jump for all* $\omega \in \Omega$ *and* $\{j(t, \cdot\,, \mathbf{Z}) : t \geq 0\}$ *is a Poisson jump process associated with* M. *Finally, if* $\mu \in \mathcal{I}(\mathbb{R}^N)$ *has Fourier transform given by (4.2.1), then*

$$\left\{ t\left(\mathbf{m} - \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(d\mathbf{y}) \right) + \mathbf{Z}_M(t) : t \geq 0 \right\}$$

is a Lévy process for μ.

Proof: That $\mathbb{P}(\mathcal{N}) = 0$ follows from Lemma 4.2.13. To prove that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Lévy process for π_M, set

$$\mathbf{Z}^{(r)}(t, \omega) = \int_{|\mathbf{y}|>r} \mathbf{y}\, j(t, d\mathbf{y}, \omega)$$

for $r > 0$. By Lemma 4.2.12, $\{\mathbf{Z}^{(r)}(t) : t \geq 0\}$ is a Poisson process associated with $M^{(r)}(d\mathbf{y}) \equiv \mathbf{1}_{(r,\infty)}(\mathbf{y})\, M(d\mathbf{y})$. In addition, if $\omega \notin \mathcal{N}$, then $\mathbf{Z}^{(r)}(\,\cdot\,, \omega) \longrightarrow \mathbf{Z}_M(\,\cdot\,, \omega)$ uniformly on compacts, from which it is easy to check that $\{\mathbf{Z}_M(t) : t \geq 0\}$ is a Poisson process associated with M and that the process in the last assertion is a Lévy process for any μ whose Fourier transform is given by (4.2.1). Finally, by Theorem 4.1.6, $j\big(t, \cdot\,, \mathbf{Z}_M(\,\cdot\,, \omega)\big) = j(t, \cdot\,, \omega)$ when $\omega \notin \mathcal{N}$, from which it is clear that $\{j(t, \cdot\,, \mathbf{Z}) : t \geq 0\}$ is a Poisson jump process associated with M. □

§ 4.2.5. General, Non-Gaussian Lévy Processes. In this subsection we will complete the construction of non-Gaussian Lévy processes.

Theorem 4.2.15. *For each* $\mathbf{m} \in \mathbb{R}^N$ *and* $M \in \mathfrak{M}_2(\mathbb{R}^N)$ *there is a Lévy process for the* $\mu \in \mathcal{I}(\mathbb{R}^N)$ *whose Fourier transform is given by (4.2.1). Moreover, if* $\{\mathbf{Z}(t) : t \geq 0\}$ *is such a process, then* $\{j(t, \cdot\,, \mathbf{Z}) : t \geq 0\}$ *is a Poisson jump process associated with* M. *Finally, if, for* $r \in (0, 1]$,

$$\mathbf{Z}^{(r)}(t) = \int_{|\mathbf{y}|>r} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}) - t \int_{r<|\mathbf{y}|\leq 1} \mathbf{y}\, M(d\mathbf{y}),$$

then

$$\mathbb{P}\left(\sup_{\tau \in [0,t]} \left| \mathbf{Z}(\tau) - \tau\mathbf{m} - \mathbf{Z}^{(r)}(\tau) \right| \geq \epsilon \right) \leq \frac{Nt}{\epsilon^2} \int_{\overline{B(\mathbf{0},r)}} |\mathbf{y}|^2\, M(d\mathbf{y}).$$

Proof: Without loss in generality, we will assume that $\mathbf{m} = \mathbf{0}$.

By Theorem 4.2.11, we know that there is a Poisson jump process $\{j(t, \cdot\,) : t \geq 0\}$ associated with M. Take

$$\bar{j}(t, d\mathbf{y}, \omega) = j(t, d\mathbf{y}, \omega) - t\mathbf{1}_{\overline{B(\mathbf{0},1)}}(\mathbf{y}) M(d\mathbf{y}),$$

and define

$$\mathbf{Z}^{(r)}(t,\omega)=\int_{|\mathbf{y}|>r}\mathbf{y}\,\bar{j}(t,d\mathbf{y},\omega),\quad (t,\omega)\in[0,\infty)\times\Omega,$$

for $r\in(0,1]$. By Lemma 4.2.12, we know that $\{\mathbf{Z}^{(r)}(t):\,t\ge 0\}$ is a Lévy process for $\mu^{(r)}$, where

$$\widehat{\mu^{(r)}}(\boldsymbol{\xi})=\exp\left(\int_{|\mathbf{y}|>r}\left[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}}-1-\sqrt{-1}\,\mathbf{1}_{[0,1]}(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\right]M(d\mathbf{y})\right).$$

Furthermore, by the second part of Lemma 4.2.10, we know that, for $0<r<r'\le 1$,

$$(*)\qquad \mathbb{P}\big(\|\mathbf{Z}^{(r')}-\mathbf{Z}^{(r)}\|_{[0,t]}\ge\epsilon\big)\le\frac{Nt}{\epsilon^2}\int_{r<|\mathbf{y}|\le r'}|\mathbf{y}|^2\,M(d\mathbf{y}).$$

Hence, if $1\ge r_m\searrow 0$ is chosen so that

$$\int_{\overline{B(\mathbf{0},r_m)}}|\mathbf{y}|^2\,M(d\mathbf{y})\le 2^{-m},$$

then

$$\begin{aligned}\mathbb{P}\left(\sup_{n>m}\|\mathbf{Z}^{(r_n)}-\mathbf{Z}^{(r_m)}\|_{[0,t]}\ge\tfrac{1}{m}\right)&\le\sum_{n\ge m}\mathbb{P}\big(\|\mathbf{Z}^{(r_{n+1})}-\mathbf{Z}^{(r_n)}\|_{[0,t]}\ge(m+1)^{-2}\big)\\&\le Nt\sum_{n=m}^{\infty}(n+1)^4 2^{-n},\end{aligned}$$

and therefore, by the first part of the Borel–Cantelli Lemma,

$$\mathbb{P}\left(\exists m\,\forall n\ge m\ \|\mathbf{Z}^{(r_n)}-\mathbf{Z}^{(r_m)}\|_{[0,t]}\le\tfrac{1}{m+1}\right)=1.$$

We now know that there is a $\mathbb{P}$-null set $\mathcal{N}$ such that, for any $\omega\notin\mathcal{N}$, there exists a $\mathbf{Z}(\,\cdot\,,\omega)\in D(\mathbb{R}^N)$ to which $\{\mathbf{Z}^{(r_m)}(\,\cdot\,,\omega):\,n\ge 0\}$ converges uniformly on compacts. Thus, if we take $\mathbf{Z}(t,\omega)=\mathbf{0}$ for $(t,\omega)\in[0,\infty)\times\mathcal{N}$, then is an easy matter to check that $\{\mathbf{Z}(t):\,t\ge 0\}$ is a Lévy process for the $\mu\in\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is given by (4.2.1) with $\mathbf{m}=\mathbf{0}$. In addition, since, by Theorem 4.1.6, we know that $t\rightsquigarrow j(t,\,\cdot\,,\omega)$ is the jump function for $t\rightsquigarrow\mathbf{Z}(t,\omega)$ when $\omega\notin\mathcal{N}$, it is clear that $\{j(t,\,\cdot\,,\mathbf{Z}):\,t\ge 0\}$ is a Poisson jump process associated with M Finally, to prove the final estimate, observe that, for $\omega\notin\mathcal{N}$, the path $t\rightsquigarrow\mathbf{Z}^{(r)}(t,\omega)$ used in our construction coincides with the path described in the statement. Thus, the desired estimate is an easy consequence of the one in (*) above. $\square$

Corollary 4.2.16. *Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ with Fourier transform given by (4.2.1), and suppose that $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ. Then, depending on whether or not $M \in \mathfrak{M}_1(\mathbb{R}^N)$, either $\mathbb{P}$-almost all or $\mathbb{P}$-almost none of the paths $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation. Moreover, if $M \in \mathfrak{M}_1(\mathbb{R}^N)$, then, $\mathbb{P}$-almost surely, $t \rightsquigarrow \mathbf{Z}(t) - t\mathbf{m}$ is an absolutely pure jump path.*

Proof: From Theorem 4.2.14, we already know that $t \rightsquigarrow \mathbf{Z}(t) - t\mathbf{m}$ is almost surely an absolutely pure jump path if $M \in \mathfrak{M}_1(\mathbb{R}^N)$, and so $t \rightsquigarrow \mathbf{Z}(t)$ is almost surely of locally bounded variation. Conversely, if $t \rightsquigarrow \mathbf{Z}(t)$ has locally bounded variation with positive probability, then, by (4.1.10), $j\big(t, \cdot\,, \mathbf{Z}\big) \in \mathfrak{M}_1(\mathbb{R}^N)$ with positive probability. But then, since $\{j\big(t, \cdot\,, \mathbf{Z}\big) : t \geq 0\}$ is a Poisson jump process associated with M, it follows from Lemma 4.2.13 that $M \in \mathfrak{M}_1(\mathbb{R}^N)$. $\square$

Corollary 4.2.17. *Let μ and $\{\mathbf{Z}(t) : t \geq 0\}$ be as in Corollary 4.2.16. Given $\Delta \in \mathcal{B}_{\mathbb{R}^N}$ with $\mathbf{0} \notin \bar{\Delta}$, set*

$$\mathbf{Z}^{\Delta}(t) = \int_{\Delta} \mathbf{y}\, j(t, d\mathbf{y}, \mathbf{Z}),\ M^{\Delta}(d\mathbf{y}) = \mathbf{1}_{\Delta}(\mathbf{y}) M(d\mathbf{y}),\ \text{and } \mathbf{m}^{\Delta} = \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M^{\Delta}(d\mathbf{y}).$$

Then $\{\mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is a Poisson process associated with M^{Δ}, $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is a Lévy process for the element of $\mathcal{I}(\mathbb{R}^N)$ whose Fourier transform is

$$\exp\bigg(\sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{m} - \mathbf{m}^{\Delta}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N \setminus \Delta} \Big[e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y})\bigg),$$

and $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is independent of $\{j\big(t, \cdot\,, \mathbf{Z}^{\Delta}\big) : t \geq 0\}$, and therefore of $\{\mathbf{Z}^{\Delta}(t) : t \geq 0\}$.

Proof: That $\{\mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is a Poisson process associated with M^{Δ} is an immediate consequence of Lemma 4.2.12. Next, define $\mathbf{Z}^{(r)}(t)$ as in Theorem 4.2.15, and set $\bar{j}(t, d\mathbf{y}) = j(t, d\mathbf{y}, \mathbf{Z}) - t\mathbf{1}_{[0,1]}(|\mathbf{y}|)\, M(d\mathbf{y})$. Then, for all $r \in (0, 1]$,

$$\mathbf{Z}^{(r)}(t) - \mathbf{Z}^{\Delta}(t) = \int_{\overline{B(\mathbf{0},r)}\complement} \mathbf{1}_{\mathbb{R}^N \setminus \Delta}(\mathbf{y})\mathbf{y}\, \bar{j}(t, d\mathbf{y}) - t \int_{r < |\mathbf{y}| \leq 1} \mathbf{1}_{\Delta}(\mathbf{y})\mathbf{y}\, M(d\mathbf{y}).$$

In particular, this means that $\{\mathbf{Z}^{(r)}(t) - \mathbf{Z}^{\Delta}(t) : t \geq 0\}$ has independent, homogeneous increments and (cf. Theorem 4.1.6) is independent of $\{j\big(t, \cdot\,, \mathbf{Z}^{\Delta}\big) : t \geq 0\}$. In particular, since, as $r \searrow 0$,

$\mathbf{Z}^{(r)}(t) \longrightarrow \mathbf{Z}(t) - t\mathbf{m}$ in probability, it follows that $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is independent of $\{j(t, \cdot, \mathbf{Z}^{\Delta}) : t \geq 0\}$. In addition,

$$e^{-\sqrt{-1}(\boldsymbol{\xi}, \mathbf{m}-\mathbf{m}^{\Delta})_{\mathbb{R}^N}} \mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}(t)-\mathbf{Z}^{\Delta}(t))_{\mathbb{R}^N}}\big] = \lim_{r \searrow 0} \mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{Z}^{(r)}(t)-\mathbf{Z}^{\Delta}(t)+\mathbf{m}^{\Delta})_{\mathbb{R}^N}}\big]$$

$$= \lim_{r \searrow 0} \exp\left(\int\limits_{(\Delta \cup \overline{B(\mathbf{0},r)})\complement} \Big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y}) \right)$$

$$= \exp\left(\int_{\mathbb{R}^N \setminus \Delta} \Big[e^{\sqrt{-1}(\boldsymbol{\xi}, \mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N}\Big]\, M(d\mathbf{y}) \right).$$

Hence, it follows that $\{\mathbf{Z}(t) - \mathbf{Z}^{\Delta}(t) : t \geq 0\}$ is a Lévy process for the specified element of $\mathcal{I}(\mathbb{R}^N)$. $\square$

Exercises for § 4.2

EXERCISE 4.2.18. Here is another proof that the process $\{N(t) : t \geq 0\}$ in § 4.2.1 has independent, homogeneous increments. Refer to the notation used there.

(**i**) Given $n \in \mathbb{Z}^+$ and measurable functions $f : [0, \infty)^{n+1} \longmapsto [0, \infty)$ and $g : [0, \infty)^n \longrightarrow \mathbb{R}$, show that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\big[f(\tau_1, \dots, \tau_{n+1}),\, \tau_{n+1} > g(\tau_1, \dots, \tau_n)\big] \\ &\quad = \mathbb{E}^{\mathbb{P}}\big[e^{-g(\tau_1,\dots,\tau_n)^+} f\big(\tau_1, \dots, \tau_n, \tau_{n+1} + g(\tau_1, \dots, \tau_n)^+\big)\big].\end{aligned}$$

(**ii**) Let $K \in \mathbb{Z}^+$, $0 = n_0 \leq n_1 \leq \cdots \leq n_K$, and $0 = t_0 \leq t_1 < \cdots < t_K = s$ be given, and set $A = \{N(t_k) = n_k,\, 1 \leq k \leq K\}$. Show that $A = B \cap \{\tau_{n_K+1} > s - T_{n_K}\}$, where $B \in \sigma\big(\{\tau_1, \dots, \tau_{n_K}\}\big)$, and apply (**i**) to see that $\mathbb{P}(A) = \mathbb{E}^{\mathbb{P}}\big[e^{(s-T_{n_K})},\, B\big]$.

(**iii**) Let $n \in \mathbb{Z}^+$ and $t > 0$ be given, and set $\psi(\xi) = \mathbb{P}(T_{n-1} > \xi)$. Again using (**i**), show that

$$\begin{aligned}&\mathbb{P}\big(A \cap \{N(s+t) - N(s) < n\}\big) \\ &\quad = \mathbb{E}^{\mathbb{P}}\big[\psi(t + s - T_{n_K+1}),\, B \cap \{\tau_{n_K+1} > s - T_{n_K}\}\big] \\ &\quad = \mathbb{E}^{\mathbb{P}}\big[e^{-(s-T_{n_K})}\psi(t - \tau_{n_K+1}),\, B\big] = \mathbb{E}^{\mathbb{P}}\big[\psi(t - \tau_{n_K+1})\big]\mathbb{E}^{\mathbb{P}}\big[e^{-(s-T_{n_K})},\, B\big] \\ &\quad = \mathbb{P}\big(N(t) < n\big)\mathbb{P}(A).\end{aligned}$$

EXERCISE 4.2.19. Assume that $\mu \in \mathcal{I}(\mathbb{R})$ has its Fourier transform given by (4.2.1), and let $\{Z(t) : t \geq 0\}$ be a Lévy process for μ. Using Exercise 3.2.23, show that $t \rightsquigarrow Z(t)$ is non-decreasing if and only if $M \in \mathfrak{M}_1(\mathbb{R})$, $M\big((-\infty, 0)\big) = 0$, and $m \geq \int_{[-1,1]} y\, M(dy)$.

EXERCISE 4.2.20. Let $\{N(t) : t \geq 0\}$ be a simple Poisson process, and show that $\lim_{t\to\infty} \frac{N(t)}{t} = 1$ $\mathbb{P}$-almost surely.

Hint: First use the Strong Law of Large Numbers to show that $\lim_{n\to\infty} \frac{N(n)}{n} = 1$ $\mathbb{P}$-almost surely. Second, use

$$\mathbb{P}\left(\sup_{n\leq t\leq n+1} \frac{N(t)-N(n)}{t} \geq \epsilon\right) \leq \mathbb{P}\big(N(1) \geq n\epsilon\big) \leq \frac{2}{\epsilon^2 n^2},$$

and therefore that

$$\lim_{t\to\infty} \left| \frac{N(t)}{t} - \frac{N([t])}{[t]} \right| = 0 \quad \mathbb{P}\text{-almost surely.}$$

EXERCISE 4.2.21. Let $\{j(t,\,\cdot\,) : t \geq 0\}$ be a Poisson jump process associated with some $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$, and suppose that $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Borel measurable, M-integrable function which vanishes at $\mathbf{0}$.

(**i**) Let $\mathcal{N}$ be the set of $\omega \in \Omega$ for which there is a $t > 0$ such that F is not integrable $j\big(t,\,\cdot\,,\omega\big)$-integrable, and show that $\mathbb{P}(\mathcal{N}) = 0$.

(**ii**) Show that $M^F \in \mathfrak{M}_1(\mathbb{R})$ and that, in fact,

$$\int |y|\, M^F(dy) = \int |F(\mathbf{y})|\, M(d\mathbf{y}) < \infty.$$

Next, define

$$Z^F(t,\omega) = \begin{cases} \int \varphi(\mathbf{y})\, j(t, d\mathbf{y}, \omega) & \text{if } \omega \notin \mathcal{N} \\ 0 & \text{if } \omega \in \mathcal{N}. \end{cases}$$

Show that $\{Z^F(t) : t \geq 0\}$ is a (possibly generalized) Poisson process associated with M^F.

(**iii**) Show that

$$\lim_{t\to\infty} \frac{Z^F(t)}{t} = \int F(\mathbf{y})\, M(d\mathbf{y}) \quad \mathbb{P}\text{-almost surely.}$$

Hint: Begin by using Lemma 4.2.10 to show that it suffices to handle F's which vanish in a neighborhood of $\mathbf{0}$. When F vanishes in a neighborhood of $\mathbf{0}$, use Lemma 4.2.12 to see that $\{Z^F(t) : t \geq 0\}$ is a Poisson process associated with M^F. Finally, use the representation of a Poisson process in terms of a simple Poisson process and independent random variables, and apply the Strong Law of Large Numbers together with the result in Exercise 4.2.20.

Exercise 4.2.22. Let $\{\mathbf{Z}(t) : t \geq 0\}$ be a Lévy process for the $\mu \in \mathcal{I}(\mathbb{R}^N)$ with Fourier transform given by (4.2.1), and set $\bar{\mathbf{Z}}(t) = \mathbf{Z}(t) - t\mathbf{m}$. Show that for all $R \in [1, \infty)$ and $t \in (0, \infty)$, $\mathbb{P}\big(\|\bar{\mathbf{Z}}\|_{[0,t]} \geq R\big)$ is dominated by t times

$$\frac{4N}{R^2} \int_{\overline{B(\mathbf{0},1)}} |\mathbf{y}|^2 \, M(d\mathbf{y}) + \frac{2}{R} \int_{1 < |\mathbf{y}| \leq \sqrt{R}} |\mathbf{y}| \, M(d\mathbf{y}) + M\Big(\overline{B(\mathbf{0}, \sqrt{R})}\complement\Big).$$

Hint: Write $\mathbf{Z}(t) = \mathbf{Z}_1(t) + \mathbf{Z}_2(t) + \mathbf{Z}_3(t)$, where

$$\mathbf{Z}_2(t) = \int_{1 < |\mathbf{y}| \leq \sqrt{R}} \mathbf{y} \, j(t, d\mathbf{y}, \mathbf{Z}) \quad \text{and} \quad \mathbf{Z}_3(t) = \int_{|\mathbf{y}| > \sqrt{R}} \mathbf{y} \, j(t, d\mathbf{y}, \mathbf{Z}).$$

Then,

$$\mathbb{P}\big(\|\mathbf{Z}\|_{[0,t]} \geq R\big) \leq \mathbb{P}\big(\|\mathbf{Z}_1\|_{[0,t]} \geq \tfrac{R}{2}\big) + \mathbb{P}\big(\|\mathbf{Z}_2\|_{[0,t]} \geq \tfrac{R}{2}\big) + \mathbb{P}\big(\|\mathbf{Z}_3\|_{[0,t]} \neq 0\big).$$

Apply the estimates in Lemma 4.2.10 to control the first two terms on the right, and use

$$\mathbb{P}\Big(j\big(t, \mathbb{R}^N \setminus \overline{B(\mathbf{0}, \sqrt{R})}, \mathbf{Z}\big) \neq 0\Big) = 1 - e^{-tM(\mathbb{R}^N \setminus \overline{B(\mathbf{0}, \sqrt{R})})}$$

to control the third.

Exercise 4.2.23. Let ν be a locally finite, Borel measure on $\mathbb{R}^N$. A **Poisson point process** with intensity measure ν is a random, locally finite, purely atomic measure-valued random variable $\omega \rightsquigarrow P(\,\cdot\,, \omega)$ with the properties that, for any bounded $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$, $P(\Gamma)$ is a Poisson random variable with mean value $\nu(\Gamma)$ and, for any family $\{\Gamma_1, \dots, \Gamma_n\}$ of mutually disjoint, bounded, Borel subsets of $\mathbb{R}^N$, $\{P(\Gamma_1), \dots, P(\Gamma_n)\}$ are independent. The purpose of this exercise is to show how one can construct such a Poisson point process.

(**i**) Define $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ so that $F(\mathbf{0}) = \mathbf{0}$ and $F(\mathbf{y}) = \frac{\mathbf{y}}{|\mathbf{y}|^2}$ for $\mathbf{y} \neq \mathbf{0}$. Clearly, F is 1 to 1 and onto, and both F and F^{-1} are Borel measurable. Assuming that $\nu(\{\mathbf{0}\}) = 0$, show that $M \equiv F_*\nu \in \mathfrak{M}_\infty(\mathbb{R}^N)$ and that $\nu = {F^{-1}}_* M$.

(**ii**) Continue to assume that $\nu(\{\mathbf{0}\}) = 0$, let $\{j(t, \cdot\,) : t \geq 0\}$ be a Poisson jump process associated with the M in (**i**), and set $P(\,\cdot\,, \omega) = {F^{-1}}_* j(1, \,\cdot\,, \omega)$. Show $\omega \rightsquigarrow P(\,\cdot\,, \omega)$ is a Poisson point process with intensity ν.

(**vi**) In order to handle ν's which charge $\mathbf{0}$, suppose $\nu(\{\mathbf{0}\}) > 0$. Choose a point $\mathbf{x} \in \mathbb{R}^N$ for which $\nu(\{\mathbf{x}\}) = 0$, define $\nu'(\Gamma) = \nu(\mathbf{x} + \Gamma)$, note that $\nu'(\{\mathbf{0}\}) = 0$, and construct a Poisson point process $\omega \rightsquigarrow P'(\,\cdot\,, \omega)$ with intensity measure ν'. Finally, define $P(\Gamma, \omega) = P'(\Gamma - \mathbf{x}, \omega)$, and check that $\omega \rightsquigarrow P(\,\cdot\,, \omega)$ is a Poisson point process with intensity measure ν.

Exercise 4.2.24. Let $M \in \mathfrak{M}_2(\mathbb{R}^N)$ be given, and assume that exists a decreasing sequence $\{r_n : n \geq 0\} \subseteq (0,1]$ with $r_n \searrow 0$ such that

$$\mathbf{m} = \lim_{n\to\infty} \int_{r_n < |\mathbf{y}| \leq 1} \mathbf{y}\, M(d\mathbf{y})$$

exists. Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ have Fourier transform given by (4.2.1) with this $\mathbf{m}$ and M. If $\{\mathbf{Z}(t) : t \geq 0\}$ is a Lévy process for μ, set

$$\mathbf{Z}_n(t,\omega) = \int_{|\mathbf{y}|>r_n} \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big),$$

and show that

$$\lim_{n\to\infty} \mathbb{P}\big(\|\mathbf{Z} - \mathbf{Z}_n\|_{[0,t]} \geq \epsilon\big) = 0$$

for all $t \geq 0$ and $\epsilon > 0$. Thus, after passing to a subsequence $\{n_m : m \geq 0\}$ if necessary, one sees that, $\mathbb{P}$-almost surely,

$$\mathbf{Z}(t,\omega) = \lim_{m\to\infty} \int \mathbf{y}\, j\big(t, d\mathbf{y}, \mathbf{Z}(\,\cdot\,,\omega)\big),$$

where the convergence is uniform on finite time intervals. An interesting conclusion from this is that $\mathbb{P}$-almost all the paths $t \rightsquigarrow \mathbf{Z}(t,\omega)$ are "conditionally pure jump."

§ 4.3 Coupling Lévy Processes

There is another way of thinking about the construction of the Poisson jump processes, one which is based on the transformation property described in Lemma 4.2.12. The advantage of this approach is that it provides a method of coupling Lévy processes corresponding to different Lévy measures. Indeed, it is this coupling procedure which underlies K. Itô's construction of Markov processes modeled on Lévy processes.*

§ 4.3.1. Fungibility of Measures. Let $M_0(d\mathbf{y}) = |\mathbf{y}|^{-N-1}\, d\mathbf{y}$. Our goal in this subsection is to show that every $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$ can be realized as (cf. the notation in Lemma 4.2.6) M_0^F for some Borel measurable $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfying $F(\mathbf{0}) = \mathbf{0}$.

Lemma 4.3.1. *Let E_1 and E_2 be a pair of complete, separable metric spaces and let Ω be the complete, separable metric space $E_1 \times E_2$. Given $\mu \in \mathbf{M}_1(\Omega)$, use μ_2 to denote the marginal distribution of μ on E_2: $\mu_2(\Gamma) = \mu(E_1 \times \Gamma)$ for $\Gamma \in \mathcal{B}_{E_2}$. Then there is a measurable map $x_2 \in E_2 \longmapsto \mu(x_2, \cdot\,) \in \mathbf{M}_1(E_1)$ such that $\mu(dx_1 \times dx_2) = \mu(x_2, dx_1)\, \mu_2(dx_2)$.*

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* See K. Itô's *On stochastic differential equations*, Memoirs of the A.M.S. #4 (1951) or my *Markov Processes from K. Itô's Perspective*, Annals of Math. Studies #155 (2003).

PROOF: The result here is really just a simple application of Theorem 6.1.10. Referring to that theorem, take $\mathbb{P} = \mu$, $\Sigma = \{E_1 \times \Gamma : \Gamma \in \mathcal{B}_{E_2}\}$, and let $\omega \in \Omega \longmapsto \mathbb{P}^\Sigma_\omega \in \mathbf{M}_1(\Omega)$ be the map guaranteed by the result there. Next, choose and fix a point $x_1^0 \in E_1$. Then, because $\omega \rightsquigarrow \mathbb{P}^\Sigma_\omega$ is Σ-measurable, we know that $\mathbb{P}^\Sigma_{(x_1,x_2)} = \mathbb{P}^\Sigma_{(x_1^0,x_2)}$. In addition, because Σ is countably generated, the final part of Theorem 6.1.10 guarantees that there exists a μ_2-null set $B \in \mathcal{B}_{E_2}$ such that $\mathbb{P}^\Sigma_{(x_1^0,x_2)}\big(E_1 \times \{x_2\}\big) = 1$ for all $x_2 \notin B$. Hence, if we define $x_2 \rightsquigarrow \mu(x_2, \cdot\,)$ by $\mu(x_2, \Gamma) = \mathbb{P}^\Sigma_{(x_1^0,x_2)}(\Gamma \times E_2)$, then, for any Borel measurable $\varphi : E_1 \times E_2 \longrightarrow [0,\infty)$, $\langle \varphi, \mu\rangle$ equals

$$\int \left(\int \varphi(\omega')\mathbb{P}^\Sigma(d\omega') \right) \mathbb{P}(d\omega) = \int_{E_2} \left(\int_{E_1} \varphi(x_1,x_2)\,\mu(x_2,dx_1) \right) \mu_2(dx_2). \quad \square$$

In the older literature, the result in Lemma 4.3.1 would be called a **fibering** of μ. The name derives from the idea that μ on $E_1 \times E_2$ can be decomposed into its "vertical component" μ_2 and its "restrictions" $\mu(x_2, \cdot\,)$ to "horizontal fibers" $E_1 \times \{x_2\}$.

LEMMA 4.3.2. *Let $\lambda_{[0,1)}$ be denote Lebesgue measure on $[0,1)$. For each $N \in \mathbb{Z}^+$ and $\mu \in \mathbf{M}_1(\mathbb{R}^N)$, there is a Borel measurable map $f : [0,1) \longrightarrow \mathbb{R}^N$ such that $\mu = f_*\lambda_{[0,1)}$.*

PROOF: We work by induction on $N \in \mathbb{Z}^+$. When $N = 1$, we take

$$f(u) = \inf\big\{t \in \mathbb{R} : \mu\big((-\infty,t]\big) \ge u\big\}, \quad u \in [0,1).$$

Next, assume the result is true for N, take $E_1 = \mathbb{R}$ and $E_2 = \mathbb{R}^N$ in Lemma 4.3.1, and, given $\mu \in \mathbf{M}_1(\mathbb{R}^N)$, define $\mu_2 \in \mathbf{M}_1(\mathbb{R}^N)$ and $\mathbf{y} \in \mathbb{R}^N \longmapsto \mu(\mathbf{y}, \cdot\,) \in \mathbf{M}_1(\mathbb{R})$ accordingly. By the induction hypothesis, $\mu_2 = f_2(\,\cdot\,)_*\lambda_{[0,1)}$ for some $f_2 : [0,1) \longrightarrow \mathbb{R}^N$. Thus, if $g : [0,1)^2 \longrightarrow \mathbb{R} \times \mathbb{R}^N$ is given by

$$g(u_1,u_2) = \Big(\inf\big\{t \in \mathbb{R} : \mu\big(f_2(u_2),(-\infty,t]\big) \ge u_1\big\}, f_2(u_2)\Big)$$

for $(u_1,u_2) \in [0,1)^2$, then g is Borel measurable on $[0,1)^2$ and $\mu = g_*\lambda^2_{[0,1)}$. Finally, by Lemma 1.1.6, we know that there is a Borel measurable map $u \in [0,1) \longmapsto \mathbf{U}(u) = \big(U_1(u),U_2(u)\big) \in [0,1)^2$ such that $\mathbf{U}_*\lambda_{[0,1)} = \lambda^2_{[0,1)}$, and so we can take $f(u) = g \circ \mathbf{U}$. $\square$

THEOREM 4.3.3. *For each $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$ there exists a Borel measurable map $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that $F(\mathbf{0}) = \mathbf{0}$ and*

$$M(\Gamma) = M_0^F \equiv M_0\big(F^{-1}(\Gamma \setminus \{\mathbf{0}\}\big), \quad \Gamma \in \mathcal{B}_{\mathbb{R}^N}.$$

Proof: We begin with the case when $N = 1$. Given $M \in \mathfrak{M}_\infty(\mathbb{R})$, define $\rho(r, \pm 1)$ for $r > 0$ by

$$\begin{aligned}\rho(r,1) &= \sup\bigl\{\rho \in [0,\infty) : M\bigl([\rho,\infty)\bigr) \geq r^{-1}\bigr\}\\ \rho(r,-1) &= \sup\bigl\{\rho \in (-\infty,0] : M\bigl((-\infty,-\rho)\bigr) \geq r^{-1}\bigr\},\end{aligned}$$

where we here take the supremum over the empty set to be 0. Applying Exercise 4.3.5, one sees that $M = M_0^F$ when $F(0) = 0$ and $F(y) = \rho\bigl(|y|, \frac{y}{|y|}\bigr)$ for $y \in \mathbb{R} \setminus \{0\}$.

Now assume that $N \geq 2$, and let $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$. If $M = 0$, simply take $F \equiv 0$. If $M \neq 0$, choose a non-decreasing function $h : (0,\infty) \longrightarrow (0,\infty)$ so that

$$\int h\bigl(|\mathbf{y}|\bigr)\, M(d\mathbf{y}) = 1,$$

and define $\mu \in \mathbf{M}_1\bigl((0,\infty) \times \mathbb{S}^{N-1}\bigr)$ so that

$$\langle \varphi, \mu \rangle = \int_{\mathbb{R}^N} h\bigl(|\mathbf{y}|\bigr)\varphi(\mathbf{y}) M(d\mathbf{y}).$$

Using μ_2 to denote the marginal distribution of μ on $\mathbb{S}^{N-1}$, apply Lemma 4.3.2 to find a Borel measurable $\mathbf{f} : [0,1) \longrightarrow \mathbb{R}^N$ so that $\mu_2 = \mathbf{f}_* \lambda_{[0,1)}$. Since μ_2 lives on $\mathbb{S}^{N-1}$, we may and will assume that $\mathbf{f}(u) \in \mathbb{S}^{N-1}$ for all $u \in [0,1)$. Next, use Lemma 4.3.1 to find a measurable map $\boldsymbol{\eta} \in \mathbb{S}^{N-1} \longmapsto \mu(\boldsymbol{\eta}, \cdot\,) \in \mathbf{M}_1\bigl((0,\infty)\bigr)$ so that $\mu(dr \times d\boldsymbol{\eta}) = \mu(\boldsymbol{\eta}, dr)\, \mu_2(d\boldsymbol{\eta})$, and define $\rho : (0,\infty) \times \mathbb{S}^{N-1} \longrightarrow [0,\infty)$ by

$$\rho(r,\boldsymbol{\eta}) = \sup\left\{\rho \in [0,\infty) : \int_{[\rho,\infty)} \frac{1}{h(r)}\, \mu(\boldsymbol{\eta}, dr) \geq \frac{\omega_{N-1}}{r}\right\}.$$

Then, again by Exercise 4.3.5, for any continuous $\varphi : \mathbb{R}^N \longrightarrow [0,\infty)$ which vanishes in a neighborhood of $\mathbf{0}$,

$$\int_{(0,\infty)} \frac{\varphi(r\boldsymbol{\eta})}{h(r)}\, \mu(\boldsymbol{\eta}, dr) = \omega_{N-1} \int_{(0,\infty)} \varphi\bigl(r(r,\boldsymbol{\eta})\boldsymbol{\eta}\bigr) r^{-2}\, dr, \quad \boldsymbol{\eta} \in \mathbb{S}^{N-1},$$

and so

$$\begin{aligned}\int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(dy) &= \omega_{N-1} \int_{\mathbb{S}^{N-1}} \left(\int_{(0,\infty)} \varphi\bigl(\rho(r,\boldsymbol{\eta})\boldsymbol{\eta}\bigr) r^{-2}\, dr\right) \mu_2(d\boldsymbol{\eta})\\ &= \omega_{N-1} \int_{[0,1)} \left(\int_{(0,\infty)} \varphi\bigl(\rho(r,\boldsymbol{\eta})\mathbf{f}(t)\bigr) r^{-2}\, dr\right) \lambda_{[0,1)}(dt).\end{aligned}$$

Finally, define $g : \mathbb{S}^{N-1} \longrightarrow [0, \omega_{N-1})$ by $g(\boldsymbol{\eta}) = \lambda_{\mathbb{S}^{N-1}}\bigl(\{\boldsymbol{\eta}' \in \mathbb{S}^{N-1} : \eta_1' \leq \eta_1\}\bigr)$, note that $\omega_{N-1}\lambda_{[0,1)} = g_* \lambda_{\mathbb{S}^{N-1}}$, and conclude that $M = M_0^F$ when

$$F(\mathbf{0}) = 0 \quad \text{and} \quad F(\mathbf{y}) = \rho\bigl(|\mathbf{y}|, \tfrac{\mathbf{y}}{|\mathbf{y}|}\bigr)\mathbf{f} \circ g\bigl(\tfrac{\mathbf{y}}{|\mathbf{y}|}\bigr) \text{ for } \mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \quad \square$$

§ 4.3.2. The Itô Map. We can now prove the following theorem, which is the simplest example of Itô's procedure.

THEOREM 4.3.4. *Let $\{j_0(t,\,\cdot\,) : t \geq 0\}$ be a Poisson jump process associated with M_0. Then for each $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$, there is a Borel measurable map $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with $F(\mathbf{0}) = \mathbf{0}$ and a Poisson jump process $\{j(t,\,\cdot\,) : t \geq 0\}$ associated with M such that $j(t,\,\cdot\,) = j_0^F(t,\,\cdot\,)$, $t \geq 0$, $\mathbb{P}$-almost surely.*

PROOF: Choose F as in Theorem 4.3.3 so that $M = M_0^F$. For $R > 0$, set $F_R(\mathbf{y}) = \mathbf{1}_{[R,\infty)}(\mathbf{y})F(\mathbf{y})$. By Lemma 4.2.12, we know that $\{j_0^{F_R}(t,\,\cdot\,) : t \geq 0\}$ is a Poisson jump process associated with M^{F_R}. In particular, for each $r > 0$,

$$\mathbb{E}^{\mathbb{P}}\big[j_0^F\big(t,\mathbb{R}^N \setminus B(\mathbf{0},r)\big)\big] = \lim_{R\searrow 0} \mathbb{E}^{\mathbb{P}}\big[j_0^{F_R}\big(t,\mathbb{R}^N \setminus B(\mathbf{0},r)\big)\big] = M\big(\mathbb{R}^N \setminus B(0,r)\big) < \infty.$$

Hence, there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that $t \rightsquigarrow j_0^F(t,\,\cdot\,,\omega)$ is a jump function for all $\omega \notin \mathcal{N}$. Thus, if $j(t,\,\cdot\,,\omega) = j_0^F(t,\,\cdot\,,\omega)$ if $\omega \notin \mathcal{N}$ and $j(t,\,\cdot\,,\omega) = 0$ for $\omega \in \mathcal{N}$, then $\{j(t,\,\cdot\,) : t \geq 0\}$ is a jump process associated with M and $j(t,\,\cdot\,) = j_0^F(t,\,\cdot\,)$, $t \geq 0$, for $\mathbb{P}$-almost every $\omega \in \Omega$. $\square$

Exercises for § 4.3

EXERCISE 4.3.5. Let ν be an infinite non-negative, non-atomic, Borel measure on $[0,\infty)$ with the property that $\nu\big([r_2,\infty)\big) < \nu\big([r_1,\infty)\big) < \infty$ for all $0 < r_1 < r_2 < \infty$. Given any other non-negative, Borel measure on $[0,\infty)$ with the properties that $\mu(\{0\}) = 0$ and $\mu\big([r,\infty)\big) < \infty$ for all $r > 0$, define

$$\rho(r) = \sup\big\{\rho \in (0,\infty) : \mu\big([\rho,\infty)\big) \geq \nu\big([r,\infty)\big)\big\}, \quad r \geq 0,$$

where the supremum over the empty set is taken to be 0. Show that $\mu\big([t,\infty)\big) = \nu\big(\{r : \rho(r) \geq t\}\big)$ for all $t > 0$, and therefore that $\langle \varphi, \mu \rangle = \langle \varphi \circ \rho, \nu \rangle$ for all Borel measurable $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which vanish at 0.

Hint: Deterimine $g : (0,\infty) \longrightarrow (0,\infty)$ so that $\nu\big(\big[g(r),\infty\big)\big) = r$, and check that $\{r : \rho(r) \geq t\} = \big[g\big(\mu([t,\infty))\big),\infty\big)$ for all $t > 0$.