Chapter III
Infinitely Divisible Laws

The results in this chapter are an attempt to answer the following question. Given an \( R^N \)-valued random variable \( Y \) with the property that, for each \( n \in \mathbb{Z}^+ \), \( Y = \sum_{m=1}^{n} X_m \) where \( X_1, \ldots, X_n \) are independent and identically distributed, what can one say about the distribution of \( Y \)?

Recall that the convolution \( \nu_1 * \nu_2 \) of two finite, Borel measures \( \nu_1 \) and \( \nu_2 \) on \( R^N \) is given by

\[
\nu_1 * \nu_2 (\Gamma) = \int \int_{R^N \times R^N} 1_\Gamma(x + y) \nu_1(dx) \nu_2(dy), \quad \Gamma \in B_{R^N},
\]

and that the distribution of the sum of two independent random variables is the convolution of their distributions. Thus, the analytic statement of our problem is that of describing those probability measures \( \mu \) which, for each \( n \geq 1 \), can be written as the \( n \)-fold convolution power \( \mu^* \) of some probability measure \( \mu_1 \). I will say that such a \( \mu \) is infinitely divisible and will use \( \mathcal{I}(R^N) \) to denote the class of infinitely divisible measures on \( R^N \). Since the Fourier transform takes convolution into ordinary multiplication, the Fourier formulation of this problem is that of describing \( \hat{\mu} \) when \( \mu \) is a probability measure with the property that, for each \( n \in \mathbb{Z}^+ \), there is an \( n \)th root of \( \hat{\mu} \) which is again the Fourier transform of a probability measure.

Not surprisingly, the Fourier formulation of the problem is, in many ways, the most amenable to analysis, and it is the way in which I will solve it in this chapter. On the other hand, this formulation has the disadvantage that, although it yields a quite satisfactory description of \( \hat{\mu} \), it leaves the problem of extracting information about \( \mu \) from properties of \( \hat{\mu} \). For this reason, the following chapter is devoted to developing a probabilistic understanding of the answer obtained in this one.

### 3.1 Convergence of Measures on \( R^N \)

In order to carry out our program, I will need two important facts about the convergence of probability measures on \( R^N \). The first of these is a minor modification of the classical Helly–Bray Theorem, and the second is an improvement, due to Lévy, of Lemma 2.3.3.
Say that the sequence \( \{ \mu_n : n \geq 1 \} \subseteq M_1(\mathbb{R}^N) \) converges weakly to \( \mu \in M_1(\mathbb{R}^N) \) and write \( \mu_n \rightharpoonup \mu \) when \( \langle \phi, \mu_n \rangle \to \langle \phi, \mu \rangle \) for all \( \phi \in C_0(\mathbb{R}^N; \mathbb{C}) \), and apply Lemma 2.3.3 to check that \( \mu_n \rightharpoonup \mu \) if and only if \( \hat{\mu}_n(\xi) \to \hat{\mu}(\xi) \) for every \( \xi \in \mathbb{R}^N \).

### 3.1.1. Sequential Compactness in \( M_1(\mathbb{R}^N) \)

Given a subset \( S \) of \( M_1(\mathbb{R}^N) \), I will say that \( S \) is sequentially relatively compact if for every sequence \( \{ \mu_n : n \geq 1 \} \subseteq S \) there a subsequence \( \{ \mu_{n_m} : m \geq 1 \} \) and a \( \mu \in M_1(\mathbb{R}^N) \) such that \( \mu_{n_m} \rightharpoonup \mu \).

**Theorem 3.1.1.** A subset \( S \) of \( M_1(\mathbb{R}^N) \) is sequentially relatively compact if and only if

\[
\lim_{R \to \infty} \sup_{\mu \in S} \mu(B(0, R)) = 0.
\]

**Proof:** I begin by pointing out that there is a countable set \( \{ \varphi_k : k \in \mathbb{Z}^+ \} \subseteq C_c(\mathbb{R}^N; \mathbb{R}) \) which is linear independent and whose span is dense in \( C_c(\mathbb{R}^N; \mathbb{R}) \) with respect to uniform convergence. To see this, choose \( \eta \in C_c(\mathbb{R}^N; [0, 1]) \) so that \( \eta = 1 \) on \( B(0, 1) \) and 0 off \( B(0, 2) \), and set \( \eta_R(y) = \eta(R^{-1}y) \) for \( R > 0 \). Next, for each \( \ell \in \mathbb{Z}^+ \), apply the Stone–Weierstrass Theorem to choose a countable dense subset \( \{ \psi_{j,\ell} : j \in \mathbb{Z}^+ \} \) of \( C(B(0, 2\ell); \mathbb{R}) \), and set \( \varphi_{j,\ell} = \eta_R\psi_{j,\ell} \).

Clearly \( \{ \varphi_{j,\ell} : (j, \ell) \in (\mathbb{Z}^+)^2 \} \) is dense in \( C_c(\mathbb{R}^N; \mathbb{R}) \). Finally, using lexicographic ordering of \( (\mathbb{Z}^+)^2 \), extract a linearly independent subset \( \{ \varphi_k : k \in \mathbb{Z}^+ \} \) by taking \( \varphi_k = \varphi_{j_k,t_k} \) where \( (j_1, \ell_1) = (1, 1) \) and \( (j_{k+1}, \ell_{k+1}) \) is the first \( (j, \ell) \) such that \( \varphi_{j,\ell} \) is linearly independent of \( \{ \varphi_1, \ldots, \varphi_k \} \).

Given a sequence \( \{ \mu_n : n \geq 1 \} \subseteq S \), we can use a diagonalization procedure to find a subsequence \( \{ \mu_{n_m} : m \geq 1 \} \) such that \( a_k = \lim_{m \to \infty} \langle \varphi_k, \mu_{n_m} \rangle \) exists for every \( k \in \mathbb{Z}^+ \). Next, define the linear functional \( \Lambda \) on the span of \( \{ \varphi_k : k \in \mathbb{Z}^+ \} \) so that \( \Lambda(\varphi_k) = a_k \). Notice that if \( \varphi = \sum_{k=1}^K a_k \varphi_k \), then

\[
|\Lambda(\varphi)| = \lim_{m \to \infty} \left| \sum_{k=1}^K a_k \varphi_k \mu_{n_m} \right| = \lim_{m \to \infty} \left| \langle \varphi, \mu_{n_m} \rangle \right| \leq \| \varphi \|_u
\]

and similarly \( \Lambda(\varphi) = \lim_{m \to \infty} \langle \varphi, \mu_{n_m} \rangle \geq 0 \) if \( \varphi \geq 0 \). Hence, \( \Lambda \) admits a unique extension as a non-negativity preserving linear functional on \( C_c(\mathbb{R}^N; \mathbb{R}) \) which satisfies \( |\Lambda(\varphi)| \leq \| \varphi \|_u \) for all \( \varphi \in C_c(\mathbb{R}^N; \mathbb{R}) \).

Now assume that (3.1.2) holds. For each \( \ell \in \mathbb{Z}^+ \), apply the Riesz Representation Theorem to produce a non-negative, Borel measure \( \nu_\ell \) supported on \( B(0, 2\ell) \) so that \( \langle \varphi, \nu_\ell \rangle = \Lambda(\eta_\ell \varphi) \) for \( \varphi \in C_c(\mathbb{R}^N; \mathbb{R}) \). Since \( \langle \varphi, \nu_{\ell+1} \rangle = \Lambda(\varphi) = \langle \varphi, \nu_\ell \rangle \) whenever \( \varphi \) vanishes off of \( B(0, \ell) \), it is clear that

\[
\nu_{\ell+1}(\Gamma \cap B(0, \ell + 1)) \geq \nu_{\ell+1}(\Gamma \cap B(0, \ell)) = \nu_\ell(\Gamma \cap B(0, \ell)) \quad \text{for all } \Gamma \in \mathcal{B}_{\mathbb{R}^N}.
\]
Hence, if
\[
\mu(\Gamma) \equiv \lim_{\ell \to \infty} \mu_{\ell}(\Gamma \cap B(0, \ell)) = \sum_{\ell=1}^{\infty} \mu_{\ell}(\Gamma \cap (B(0, \ell) \setminus B(0, \ell - 1))),
\]
then \(\mu\) is a non-negative, Borel measure on \(\mathbb{R}^{N}\) whose restriction to \(B(0, \ell)\) is \(\nu_{\ell}\) for each \(\ell \in \mathbb{Z}^{+}\). In particular, \(\mu(\mathbb{R}^{N}) \leq 1\) and \(\langle \varphi, \mu \rangle = \lim_{m \to \infty} \langle \varphi, \mu_{n_{m}} \rangle\) for every \(\varphi \in C_{c}(\mathbb{R}^{N}; \mathbb{R})\). Thus, by Lemma 2.1.7, all that remains is to check that \(\mu(\mathbb{R}^{N}) = 1\). But
\[
\mu(\mathbb{R}^{N}) \geq \langle \eta_{\ell}, \mu \rangle = \lim_{m \to \infty} \langle \eta_{\ell}, \mu_{n_{m}} \rangle \geq \lim_{m \to \infty} \mu_{n_{m}}(B(0, \ell))
\]
\[
= 1 - \lim_{m \to \infty} \mu_{n_{m}}(B(0, \ell) \setminus \mathbb{C}),
\]
and, by (3.1.2), the final term tends to 0 as \(\ell \to \infty\).

To prove the converse assertion, suppose that \(S\) is sequentially relatively compact. If (3.1.2) failed, then we could find an \(\theta \in (0, 1)\) and, for each \(n \in \mathbb{Z}^{+}\), a \(\mu_{n} \in S\) such that \(\mu_{n}(B(0, n)) \leq \theta\). By sequential relative compactness, this would mean that there is a subsequence \(\{\mu_{n_{m}} : m \geq 1\} \subseteq S\) and a \(\mu \in M_{1}(\mathbb{R}^{N})\) such that \(\mu_{n_{m}} \Rightarrow \mu\) and \(\mu_{n_{m}}(B(0, n_{m})) \leq \theta\). On the other hand, for any \(R > 0\),
\[
\mu(B(0, R)) \leq \langle \eta_{R}, \mu \rangle \leq \lim_{m \to \infty} \mu_{n_{m}}(B(0, n_{m})) \leq \theta,
\]
and so we would arrive at the contradiction \(1 = \lim_{R \to \infty} \mu(B(0, R)) \leq \theta\). □

3.1.2. Lévy’s Continuity Theorem. My next goal is to find a test in terms of the Fourier transform to determine when (3.1.2) holds.

Lemma 3.1.3. Define
\[
s(r) = \inf_{\theta \geq r} \left( 1 - \frac{\sin \theta}{\theta} \right) \quad \text{for } r \in (0, \infty).
\]
Then \(s\) is a strictly positive, non-decreasing, continuous function which tends to 0 as \(r \searrow 0\). Moreover, if \(\mu \in M_{1}(\mathbb{R}^{N})\), then, for all \((r, R) \in (0, \infty)^{2}\),
\[
|1 - \hat{\mu}(re)| \leq rR + 2\mu(\{y : |(e, y)_{\mathbb{R}^{N}}| \geq R\}) \quad \text{for all } e \in \mathbb{S}^{N-1},
\]
(3.1.4)
and
\[
\mu(B(0, N^{\frac{1}{2}}R) \setminus \mathbb{C}) \leq N \sup_{e \in \mathbb{S}^{N-1}} \mu(\{y : |(e, y)_{\mathbb{R}^{N}}| \geq R\})
\]
\[
\leq \frac{N}{s(rR)} \max \{ |1 - \hat{\mu}(\xi)| : |\xi| \leq r \}.
\]
(3.1.5)
In particular, for any $S \subseteq M_1(\mathbb{R}^n)$, (3.1.2) holds if and only if

$$\lim_{|\xi| \to 0} \sup_{\mu \in S} |1 - \hat{\mu}(\xi)| = 0. \quad (3.1.6)$$

**Proof:** Given (3.1.4) and (3.1.5), the final assertion is obvious. To prove (3.1.4), simply observe that

$$|1 - e^{-\frac{1}{2} r(e,y)_{\mathbb{R}^n}}| \leq 2 \wedge (r(e,y)_{\mathbb{R}^n}).$$

Turning to (3.1.5), note that

$$|1 - \hat{\mu}(\xi)| \geq \int_{\mathbb{R}^n} \left(1 - \cos(\xi, y)_{\mathbb{R}^n}\right) \mu(dy).$$

Thus, for each $e \in S^{N-1}$,

$$\frac{1}{r} \int_0^r |1 - \hat{\mu}(te)| dt \geq \int_{\mathbb{R}^n} \left(1 - \frac{\sin(r(e,y)_{\mathbb{R}^n})}{r(e,y)_{\mathbb{R}^n}}\right) \mu(dy) \geq s(rR) \mu\left(\{y : |(e,y)_{\mathbb{R}^n}| \geq R\}\right),$$

and therefore

$$\sup_{\xi \in B(0,r)} |1 - \hat{\mu}(\xi)| \geq s(rR) \mu\left(\{y : |(e,y)_{\mathbb{R}^n}| \geq R\}\right). \quad (3.1.7)$$

Since the first inequality in (3.1.5) is obvious, there is nothing more to be done.

I am now ready to prove Lévy’s crucial improvement to Lemma 2.3.3.

**Theorem 3.1.8 (Lévy’s Continuity Theorem).** Let $\{\mu_n : n \geq 1\} \subseteq M_1(\mathbb{R}^n)$, and assume that $f(\xi) = \lim_{n \to \infty} \hat{\mu}_n(\xi)$ exists for each $\xi \in \mathbb{R}^n$. Then there is a $\mu \in M_1(\mathbb{R}^n)$ such that $f = \hat{\mu}$ if and only if there is a $\delta > 0$ for which $\lim_{n \to \infty} \sup_{|\xi| \leq \delta} |\hat{\mu}_n(\xi) - f(\xi)| = 0$, in which case $\mu_n \Rightarrow \mu$. (See part (iv) of Exercise 3.1.9 for another version.)

**Proof:** The only assertion not already covered by Lemmas 2.1.7 and 2.3.3 is the “if” part of the equivalence. But, if $\mu_n \Rightarrow f$ uniformly in a neighborhood of 0, then it is easy to check that $\sup_{n \geq 1} |1 - \hat{\mu}_n(\xi)|$ must tend to zero as $|\xi| \to 0$. Hence, by the last part of Lemma 3.1.3 and Theorem 3.1.1, we know that there exists a $\mu$ and a subsequence $\{\mu_{n_m} : m \geq 1\}$ such that $\mu_{n_m} \Rightarrow \mu$. Since $\hat{\mu}$ must equal $f$, Lemma 2.1.7 says that $\mu_n \Rightarrow \mu$. □
Exercises for § 3.1

Exercise 3.1.9. One might think that to address the sort of problem posed at the beginning of this chapter, it would be helpful to know which functions \( f : \mathbb{R}^N \to \mathbb{C} \) are the Fourier transforms of a probability measure. Such a characterization is the content of Bochner’s Theorem, whose proof will be outlined in this exercise. Unfortunately, his characterization looks more useful than it is in practice. For instance, I will not use it to solve our problem, and it is difficult to see how its use would simplify matters.

In order to state Bochner’s Theorem, say that a function \( f : \mathbb{R}^N \to \mathbb{C} \) is non-negative definite if, for each \( n \geq 1 \) and \( \xi_1, \ldots, \xi_n \in \mathbb{R}^N \), the matrix \( (f(\xi_i - \xi_j))_{1 \leq i, j \leq n} \) is Hermitian and non-negative definite. Equivalently,*

\[
\sum_{i,j=1}^{n} f(\xi_i - \xi_j) \zeta_i \overline{\zeta_j} \geq 0 \quad \text{for all } \zeta_1, \ldots, \zeta_n \in \mathbb{C}.
\]

Then Bochner’s Theorem is the statement that \( f = \hat{\mu} \) for some \( \mu \in M_1(\mathbb{R}^N) \) if and only if \( f(0) = 1 \) and \( f \) is a continuous, non-negative definite function.

(i) It is ironic that the necessity assertion is the more useful even though it is nearly trivial. Indeed, if \( f = \hat{\mu} \), then it is obvious that \( f(0) = 1 \) and that \( f \) is continuous. To see that it is also non-negative definite, write

\[
\sum_{i,j=1}^{n} e^{\sqrt{-1}(\xi_i, x)_{\mathbb{R}^N}} \zeta_i \overline{\zeta_j} = \left| \sum_{i=1}^{n} e^{\sqrt{-1}(\xi_i, x)_{\mathbb{R}^N}} \zeta_i \right|^2,
\]

and integrate in \( x \) with respect to \( \mu \).

(ii) The first step in proving the sufficiency is to use the non-negative definiteness assumption to show that \( f(-x) = \overline{f(x)} \) and \( |f(x)| \leq f(0) \) for all \( x \in \mathbb{R}^N \). Obviously, this proves that \( \|f\|_u \leq 1 \). Second, using a standard Riemann approximation procedure and the continuity of \( f \), check that for any rapidly decreasing, continuous \( \psi : \mathbb{R}^N \to \mathbb{C} \),

\[
\iint_{\mathbb{R}^N \times \mathbb{R}^N} f(x - \eta) \overline{\psi(x)} \overline{\psi(\eta)} \, dx \, d\eta \geq 0.
\]

In particular, when \( f \in L^1(\mathbb{R}^N; \mathbb{C}) \), set

\[
m(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-\sqrt{-1}(x, \xi)_{\mathbb{R}^N}} f(\xi) \, d\xi,
\]

* Recall that a non-negative definite operator on a complex Hilbert space is always Hermitian.
and use Parseval’s Identity and Fubini’s Theorem, together with elementary manipulations, to arrive at

\[(2\pi)^N \int_{\mathbb{R}^N} m(x) |\omega(x)|^2 \, dx = \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(\xi - \eta) \overline{\psi(\xi)} \overline{\psi(\eta)} \, d\xi \, d\eta \geq 0\]

for all \( \psi \in L^1(\mathbb{R}^N; \mathbb{R}) \cap C_0(\mathbb{R}^N; \mathbb{R}) \) with \( \psi \in L^1(\mathbb{R}^N; \mathbb{R}) \). Conclude that \( m \) is non-negative, and use this to complete the proof in the case when \( f \in L^1(\mathbb{R}^N; \mathbb{C}) \).

(iii) It remains only to pass from the case when \( f \in L^1(\mathbb{R}^N; \mathbb{C}) \) to the general case. For each \( t \in (0, \infty) \), set \( f_t(x) = e^{-t|x|^2} f(x) \). Clearly, \( f_t(0) = 1 \) and \( f_t \in C_0(\mathbb{R}^N; \mathbb{C}) \cap L^1(\mathbb{R}^N; \mathbb{C}) \). In addition, show that

\[
\sum_{i,j=1}^n f_t(\xi_i - \xi_j) \zeta_i \zeta_j = \int_{\mathbb{R}^N} \left( \sum_{i,j=1}^n f(\xi_i - \xi_j) \zeta_i \zeta_j(x) \right) \gamma_0, \alpha(dx) \geq 0,
\]

where \( \zeta(x) \equiv \zeta e^{\sqrt{-1} \cdot \zeta \cdot x} \). Hence, \( f_t \) is also non-negative definite; and so, by part (ii), we know that \( f_t = \overline{f} \mu \) for some \( \mu \in M_1(\mathbb{R}^N) \). Finally, apply Lévy’s Continuity Theorem to see that \( f_t \longrightarrow \mu \), where \( \mu \in M_1(\mathbb{R}^N) \) satisfies \( f = \overline{f} \mu \).

(iv) Let \( \{\mu_n : n \geq 1\} \) and \( f \) be as in Theorem 3.1.8. Combining Bochner’s Theorem with Lemma 2.1.7, show that there exists a \( \mu \in M_1(\mathbb{R}^N) \) such that \( f = \overline{f} \mu \) and \( \mu_n \longrightarrow \mu \) if and only if \( f \) is continuous.

**Exercise 3.1.10.** Suppose that \( f \) is a non-negative definite function with \( f(0) = 1 \). As we have just seen, if \( f \) is continuous, then \( f = \overline{f} \mu \) for some \( \mu \in M_1(\mathbb{R}^N) \).

(i) Assuming that \( f = \overline{f} \), show that

\[ (\star) \quad \|f\|_u \leq 1 \quad \text{and} \quad |f(\eta) - f(\xi)|^2 \leq 2[1 - \Re\{f(\eta - \xi)\}], \quad \xi, \eta \in \mathbb{R}^N. \]

Next, show that \( (\star) \) follows directly from non-negative definiteness, whether or not \( f \) is continuous. Thus, a non-negative definite function is uniformly continuous everywhere if it is continuous at the origin.

**Hint:** Both parts of \( (\star) \) follow from the fact that

\[
A = \begin{pmatrix} 1 & \overline{f(\xi)} & \overline{f(\eta)} \\ f(\xi) & 1 & \overline{f(\xi - \eta)} \\ f(\eta) & \overline{f(\xi - \eta)} & 1 \end{pmatrix}
\]

is non-negative definite. To get the second part, consider the quadratic form \( \langle v, Av \rangle_{\mathbb{C}^3} \) with \( v = (v_1, 1, -1) \).

\* This choice of \( v \) was suggested to me by Linan Chen.
(ii) To understand how essential a role continuity plays in Bochner’s criterion, show that \( f = 1_{[0]} \) is non-negative definite. Even though this \( f \) cannot be the Fourier transform of any \( \mu \in M_1(\mathbb{R}^N) \), it is nonetheless the “Fourier transform” of a non-negativity preserving linear functional, one for which there is no Riesz representation. To be more precise, consider the linear functional \( \Lambda \) on the space of functions \( \varphi \in C_b(\mathbb{R}^N; \mathbb{C}) \) for which

\[
\Lambda \varphi \equiv \lim_{R \to \infty} \frac{1}{|B(0, R)|} \int_{B(0, R)} \varphi(x) \, dx \quad \text{exists},
\]

and show that \( f(\xi) = \Lambda(e_\xi) \), where \( e_\xi(x) = e^{\sqrt{-1} (\xi \cdot x)} \).

**Exercise 3.1.11.** It is important to recognize the extent to which Lévy’s Continuity Theorem and, as a byproduct, Bochner’s Theorem, are strictly finite dimensional results. For example, let \( H \) be an infinite dimensional, separable, real Hilbert space, and define \( f(h) = e^{-\frac{1}{2} \|h\|^2} \). Obviously, \( f \) is a continuous and \( f(0) = 1 \). Show that it is also non-negative definite in the sense that \( \left( (f(h_i - h_j))_{1 \leq i, j \leq n} \right) \) is a non-negative definite, Hermitian matrix for each \( n \in \mathbb{Z}^+ \) and \( h_1, \ldots, h_n \in H \). Now suppose that there were a Borel probability measure \( \mu \) on \( H \) such that

\[
\hat{\mu}(h) \equiv \int_H e^{\sqrt{-1} (h, x)} \mu(dx) = f(h), \quad h \in H.
\]

Show that for any orthonormal basis \( \{e_i : i \in \mathbb{Z}^+\} \) in \( H \), the functions \( X_i(h) = (e_i, h)_H, \quad i \in \mathbb{Z}^+ \), would be, under \( \mu \), a sequence of independent, \( N(0, 1) \)-random variables, and conclude from this that

\[
\int_H e^{-\|h\|^2} \mu(dh) = \prod_{i \in \mathbb{Z}^+} \mathbb{E}^{\mu} \left[ e^{-X_i^2} \right] = 0.
\]

Hence, no such \( \mu \) can exist. See Chapter VIII for a more thorough account of this topic.

**Hint:** The non-negative definiteness of \( f \) can be seen as a consequence of the analogous result for \( \mathbb{R}^n \).

**Exercise 3.1.12.** The **Riemann–Lebesgue Lemma** says that \( \hat{f}(\xi) \to 0 \) as \( |\xi| \to \infty \) if \( f \in L^1(\mathbb{R}^N; \mathbb{C}) \). Thus \( \hat{\mu}(\xi) \to 0 \) as \( |\xi| \to \infty \) if \( \mu \in M_1(\mathbb{R}) \) is absolutely continuous. In this exercise we will examine situations in which \( \mu \in M_1(\mathbb{R}) \) but \( \hat{\mu}(\xi) \not\to 0 \) as \( |\xi| \to \infty \).

(i) Given a symmetric \( \mu \in M_1(\mathbb{R}) \), show that \( \hat{\mu} \) is real valued, and use Bochner’s Theorem to show that \( \hat{\mu}(\xi) \) cannot tend to a strictly negative number as \( |\xi| \to \infty \).

**Hint:** Let \( \alpha > 0 \), and suppose that \( \hat{\mu}(\xi) \to -2\alpha \) as \( |\xi| \to \infty \). Choose \( R > 0 \) so that \( \hat{\mu}(\xi) \leq -\alpha \) for \( |\xi| \geq R \) and \( n \in \mathbb{Z}^+ \) so that \( (n-1)\alpha > 1 \). Set \( A = \left( (\hat{\mu}(\ell R - kR))_{1 \leq k, \ell \leq n} \right) \), and show that \( A \) cannot be non-negative definite.
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(ii) Show that $\hat{\mu}(\xi) \not\to 0$ if $\mu$ has an atom (i.e., $\mu(\{x\}) > 0$ for some $x \in \mathbb{R}$.)

**Hint:** Reduce to the case in which $\mu$ is symmetric, and therefore that $\mu = p\delta_0 + q\nu$, where $p \in (0, 1]$, $q = 1 - p$, and $\nu \in \mathcal{M}_1(\mathbb{R})$ is symmetric. If $p = 1$, $\hat{\mu}(\xi) \to 0$ for all $\xi$. If $p \in (0, 1)$, then $\hat{\mu}(\xi) \not\to 0$ as $|\xi| \to \infty$ implies $\hat{\nu}(\xi) \not\to \frac{-p}{q} < 0$.

(iii) To produce an example which is non-atomic, refer to Exercise 1.4.29, take $p \in (0, 1) \setminus \{\frac{1}{2}\}$, and let $\mu = \mu_p$, where $\mu_p$ is the measure described in that exercise. Show that $\mu$ is a non-atomic element of $\mathcal{M}_1(\mathbb{R})$ for which $\hat{\mu} \not\to 0$ as $|\xi| \to \infty$.

**Hint:** Show that $\hat{\mu}$ never vanishes and that $\hat{\mu}(2^m\pi)$ is independent of $m \in \mathbb{Z}^+$.

§ 3.2 The Lévy–Khinchine Formula

Throughout, $\mathcal{I}(\mathbb{R}^N)$ will be the set of $\mu \in \mathcal{M}_1(\mathbb{R}^N)$ which are infinitely divisible. My strategy for characterizing $\mathcal{I}(\mathbb{R}^N)$ will be to start from an easily understood subset of $\mathcal{I}(\mathbb{R}^N)$ and to get the rest by taking weak limits.

The elements of $\mathcal{I}(\mathbb{R}^N)$ which first come to mind are the Gaussian measures (cf. (2.3.6)) $\gamma_{m,C}$. Indeed, if $m \in \mathbb{R}^N$ and $C$ is a symmetric, non-negative definite transformation on $\mathbb{R}^N$, then it is clear from (2.3.7) that $\gamma_{m,C} = \gamma_{m,\mathbb{R}^N}^m \# \xi$. Unfortunately, this is not a good starting place because it is too rigid: limits of Gaussians are again Gaussian. Indeed, suppose that $\gamma_{m_n,C_n} \Rightarrow \mu$, and, using this Fourier criterion, conclude that $\mu = \gamma_{m,C}$ where $m = \lim_{n \to \infty} m_n$ and $C = \lim_{n \to \infty} C_n$. In other words, one cannot use weak convergence to escape the class of Gaussian measures.

A more fruitful choice is to start with the Poisson measures. Recall that if $\nu$ is a probability measure on $\mathbb{R}^N$ and $\alpha \in [0, \infty)$, then the Poisson measure with jump distribution $\nu$ and jumping rate $\alpha$ (see § 4.2 for an explanation of this terminology) is the measure

$$\pi_{\alpha,\nu} = e^{-\alpha \int \frac{\alpha}{n!} \nu^n}.$$

To see that $\pi_{\alpha,\nu}$ is infinitely divisible, note that

$$\pi_{\alpha,\nu}(\xi) = \exp \left( \alpha \int \left( e^{\sqrt{-1}(\xi,y)} - 1 \right) \nu(dy) \right),$$

and therefore that $\pi_{\alpha,\nu} = \pi_{\frac{\alpha}{2^n},\nu}^n$. To see why the Poisson measures provide a more hopeful choice of starting point, let $m \in \mathbb{R}^N$ and a non-negative definite, symmetric $C$ be given, and choose $\{e_1, \ldots, e_N\}$ to be an orthonormal basis of eigenvectors for $C$. Next, set $m_i = (m,e_i)_{\mathbb{R}^N}$ and $\sigma_i = \sqrt{(e_i,Ce_i)_{\mathbb{R}^N}}$, and take

$$\nu_n = \frac{1}{2N} \left( \sum_{i=1}^N \delta_{m_i\sigma_i} + \frac{1}{2} \sum_{i=1}^N \left( \delta_{m_i\sigma_i} + \delta_{-m_i\sigma_i} \right) \right).$$
§ 3.2 The Lévy–Khinchine Formula

Then the Fourier transform of $\pi_{2N,n,\nu}$ is

$$\exp\left(\sum_{i=1}^{N} n(e^{|\xi|} - 1) + \sum_{i=1}^{N} n\left(\cos\sigma_{i}(\xi,e_{i}) - 1\right)\right),$$

which tends to $\gamma_{m,C}(\xi)$ as $n \to \infty$, and so $\pi_{2N,n,\nu} \Rightarrow \gamma_{m,C}$ as $n \to \infty$. Thus, one can use weak convergence to break out to the class of Poisson measures.

As I will show below, the preceding is a special case of a result (cf. Theorem 3.2.7) which says that every infinitely divisible measure is the weak limit of Poisson measures. However, before proving that result, it will be convenient to alter our description of Poisson measures. For one thing, it should be clear that, without loss in generality, I may always assume that the jump distribution $\nu$ assigns no mass to $0$. Indeed, if $\nu(\{0\}) = 1$, then $\pi_{\alpha,\nu} = \delta_{0} = \pi_{0,\nu}$ no matter how $\alpha$ and $\nu'$ are chosen. If $\beta = \nu(\{0\}) \in (0,1)$, then $\pi_{\alpha,\nu} = \pi_{\alpha',\nu'}$ where $\alpha' = \alpha(1-\beta)$ and $\nu' = (1-\beta)^{-1}\nu$. In addition, although the segregation of the rate and jumping distribution provides probabilistic insight, there is no essential reason for doing so. Thus, nothing is lost if one replaces $\pi_{\alpha,\nu}$ by $\pi_{M}$, where $M$ is the finite measure $\alpha \mu$, in which case

$$\hat{\pi}_{M}(\xi) = \exp\left(\int (e^{|\xi|\nu} - 1)M(\text{d}y)\right).$$

With these considerations in mind, let $M_{0}(\mathbb{R}^{N})$ be the space of non-negative, finite Borel measures $M$ on $\mathbb{R}^{N}$ with $M(\{0\}) = 0$, and set $\mathcal{P}(\mathbb{R}^{N}) = \{\pi_{M} : M \in M_{0}(\mathbb{R}^{N})\}$, the space of Poisson measures on $\mathbb{R}^{N}$.

§ 3.2.1. $\mathcal{I}(\mathbb{R}^{N})$ is the Closure of $\mathcal{P}(\mathbb{R}^{N})$. Let $\overline{\mathcal{P}(\mathbb{R}^{N})}$ be the closure of $\mathcal{P}(\mathbb{R}^{N})$ under weak convergence. That is, $\mu \in \overline{\mathcal{P}(\mathbb{R}^{N})}$ if and only if there exists a sequence $\{M_{n} : n \geq 1\} \subseteq M_{0}(\mathbb{R}^{N})$ such that $\pi_{M_{n}} \Rightarrow \mu$. My goal here is to prove that

(3.2.1) $\mathcal{I}(\mathbb{R}^{N}) = \overline{\mathcal{P}(\mathbb{R}^{N})}$.

Before turning to the proof of (3.2.1), I need the following simple lemma about non-vanishing, $\mathbb{C}$-valued functions. In its statement, and elsewhere,

(3.2.2) $\log \zeta = -\sum_{m=1}^{\infty} \frac{(1 - \zeta)^{m}}{m}$ for $\zeta \in \mathbb{C}$ with $|1 - \zeta| < 1$

is the principle branch of logarithm function on the open unit disk around 1 in the complex plane.

LEMMA 3.2.3. Let $R \in (0,\infty)$ be given. If $f \in C(B(0,R)\setminus\{0\};\mathbb{C})$ with $f(0) = 1$, then there is a unique $\ell_{f} \in C(B(0,R);\mathbb{C})$ such that $\ell_{f}(0) = 0$ and
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\( f = e^{\ell t} \). Moreover, if \( \xi \in B(0,R) \), \( r \in (0,\infty) \), and \( \left| 1 - \frac{f(\eta)}{f(\xi)} \right| < 1 \) for all \( \eta \in B(\xi,r) \cap B(0,R) \), then, for each \( \eta \in B(\xi,r) \cap B(0,R) \),

\[ \ell_f(\eta) - \ell_f(\xi) = \log \frac{f(\eta)}{f(\xi)} \]

and therefore

\[ |\ell_f(\eta) - \ell_f(\xi)| \leq 2 \left| 1 - \frac{f(\eta)}{f(\xi)} \right| \quad \text{if} \quad \left| 1 - \frac{f(\eta)}{f(\xi)} \right| \leq \frac{1}{2}. \]

Finally, if \( \tilde{f} \) is a second element of \( C(B(0;R); \mathbb{C} \setminus \{0\}) \) with \( \tilde{f}(0) = 1 \) and if \( \left| 1 - \frac{\tilde{f}(\xi)}{f(\xi)} \right| \leq \frac{1}{2} \) for all \( \xi \in B(0,R) \), then

\[ |\ell_f(\xi) - \ell_{\tilde{f}}(\xi)| \leq 2 \left| 1 - \frac{\tilde{f}(\xi)}{f(\xi)} \right| \quad \text{for} \quad \xi \in B(0,R). \]

In particular, if \( \{ f_n : n \geq 1 \} \subset C(B(0;R); \mathbb{C} \setminus \{0\}) \) with \( f_n(0) = 1 \) for all \( n \geq 1 \), and if \( f_n \rightharpoonup f \in C(B(0;R); \mathbb{C} \setminus \{0\}) \) uniformly on \( B(0,R) \), then \( f(0) = 1 \) and \( \ell_{f_n} \rightharpoonup \ell_f \) uniformly on \( B(0;R) \).

PROOF: To prove the existence and uniqueness of \( \ell_f \), begin by observing that there exists an \( M \in \mathbb{Z}^+ \) and \( 0 = r_0 < r_1 < \cdots < r_M = R \) such that

\[ \left| 1 - \frac{f(\xi)}{f \left( \frac{r_{m-1} - \xi}{r_m} \right) } \right| \leq \frac{1}{2} \quad \text{for} \quad 1 \leq m \leq M \text{ and } \xi \in B(0,r_m) \setminus B(0,r_{m-1}). \]

Thus, we can define a function \( \ell_f \) on \( \overline{B(0,R)} \) so that \( \ell_f(0) = 0 \) and

\[ \ell_f(\xi) = \ell_f \left( \frac{r_{m-1} - \xi}{r_m} \right) + \log \frac{f(\xi)}{f \left( \frac{r_{m-1} - \xi}{r_m} \right) } \]

if \( 1 \leq m \leq M \) and \( \xi \in \overline{B(0,r_m)} \setminus \overline{B(0,r_{m-1})} \).

Furthermore, working by induction on \( 1 \leq m \leq M \), one sees that this \( \ell_f \) is continuous and satisfies \( f = e^{\ell t} \). Finally, for any \( \ell \in C(\overline{B(0,R)}; \mathbb{C}) \) satisfying \( \ell(0) = 0 \) and \( f = e^{\ell}, (\sqrt{-1}2\pi)^{-1}(\ell - \ell_f) \) is a continuous, \( \mathbb{Z} \)-valued function which vanishes at \( 0 \), and therefore \( \ell = \ell_f \).

Next suppose that \( \xi \in B(0,R) \) and that

\[ \left| 1 - \frac{f(\eta)}{f(\xi)} \right| < 1 \quad \text{for all} \quad \eta \in B(\xi,r) \cap B(0,R). \]
Set
\[ \ell(\eta) = \ell_f(\xi) + \log \frac{f(\eta)}{f(\xi)} \]
for \( \eta \in B(\xi, r) \cap B(0, R) \),
and check that \( \eta \sim (\sqrt{-12\pi})^{-1}(\ell(\eta) - \ell_f(\eta)) \) is a continuous, \( \mathbb{Z} \)-valued function which vanishes at \( \xi \). Hence, \( \ell = \ell_f \) on \( B(0, R) \cap B(\xi, r) \), and therefore on \( B(0, R) \cap B(\xi, r) \). Since \( |\log(1 - \zeta)| \leq 2|\zeta| \) if \( |\zeta| \leq \frac{1}{2} \), this completes the proof of the asserted properties of \( \ell_f \).

Turning to the comparison between \( \ell_f \) and \( \ell_f \) when \( |1 - \frac{f(\xi)}{f(\xi)}| \leq \frac{1}{2} \) for all \( \xi \in B(0, R) \), set \( \ell(\xi) = \ell_f(\xi) + \log \frac{f(\xi)}{f(\xi)} \), check that \( \ell(0) = 0 \) and \( f = e^\ell \), and conclude that \( \ell_f - \ell_f = \log \frac{1}{\ell} \). From this, the asserted estimate for \( |\ell_f - \ell_f| \) is immediate. \( \Box \)

**Lemma 3.2.4.** Define \( r \sim s(r) \) as in Lemma 3.1.3, and let \( \mu \in M_1(\mathbb{R}^n) \) and \( 0 < r < R \) be given. If \( |1 - \hat{\mu}(\xi)| \leq \frac{1}{2} \) for all \( \xi \in B(0, r) \) and there is an \( \nu \in M_1(\mathbb{R}^n) \) such that \( \mu = \nu^n \) for some 
\[(3.2.5) \quad n \geq \frac{16}{s\left(\frac{1}{16}\right)},\]
then \( |\hat{\mu}(\xi)| \geq 2^{-n} \) for all \( \xi \in B(0, R) \).

**Proof:** First apply Lemma 3.2.3 to see that, because \( \hat{\mu}(\xi) = \hat{\nu}(\xi)^n \), neither \( \hat{\mu} \) nor \( \hat{\nu} \) vanishes anywhere on \( B(0, r) \) and therefore that there are unique \( \hat{\ell}, \hat{\ell} \in C(B(0, r); \mathbb{C}) \) such that \( \ell(0) = 0 = \ell(0) \), \( \hat{\mu} = e^{\hat{\ell}} \), and \( \hat{\nu} = e^{\hat{\ell}} \) on \( B(0, r) \). Further, since \( \hat{\mu} = e^{\hat{\nu}} \), uniqueness requires that \( \hat{\ell} = \frac{1}{\hat{\ell}} \). Next, observe that, because \( \ell = \log \hat{\mu} \) and \( |1 - \hat{\mu}| \leq \frac{1}{2} \) on \( B(0, r) \), \( |\ell| \leq 2 \) there. Hence, because \( \Re \hat{\ell} \leq 0 \) \( |1 - \hat{\nu}| = |1 - e^{\hat{\ell}}| \leq \frac{1}{2} \) on \( B(0, r) \). Using this in (3.1.7), we have, for any \( \rho > 0 \) and \( e \in \mathbb{S}^{n-1} \), that
\[(3.2.6) \quad |\nu(\mathcal{B})(\zeta, \zeta)^n| \leq \frac{1}{s(\rho)} \max_{\zeta \in B(0, r)} |1 - \hat{\nu}(\xi)| \leq \frac{2}{ns(\rho)} ,\]
which, by (3.1.4), leads to \( |1 - \hat{\nu}(\xi)| \leq \rho R + \frac{4}{ns(\rho)} \) for \( \xi \in B(0, R) \). Finally take \( \rho = \frac{1}{16} \) and use (3.2.5) and \( \hat{\mu}(\xi) = \hat{\nu}(\xi)^n \) to check that this gives the desired conclusion. \( \Box \)

I now have everything that I need to prove the equality (3.2.1).

**Theorem 3.2.7.** For each \( \mu \in I(\mathbb{R}^n) \) there is a unique \( \ell_\mu \in C(\mathbb{R}^n; \mathbb{C}) \) satisfying \( \ell_\mu(0) = 0 \) and \( \ell_\mu = e^{\ell_\mu} \). Moreover, for each \( n \in \mathbb{Z}^+ \), \( e^{\ell_\mu} \) is the Fourier transform of the unique \( \mu_n \in M_1(\mathbb{R}^n) \) such that \( \mu = \mu_n^n \). In addition, if \( M_n \in M_1(\mathbb{R}^n) \) is defined by
\[(3.2.8) \quad M_n(\Gamma) \equiv n \mu_n \left( \Gamma \cap (\mathbb{R}^n \setminus \{0\}) \right) \quad \text{for} \quad \Gamma \in \mathcal{B}(\mathbb{R}^n) ,\]
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then \( \pi_{M, \pi} \rightarrow \mu \). Finally, \( I(\mathbb{R}^N) \) is closed in the sense that \( \mu \in I(\mathbb{R}^N) \) if there exists a sequence \( \{ \mu_k : k \geq 1 \} \subseteq I(\mathbb{R}^N) \) such that \( \mu_k \rightarrow \mu \). In particular, \( \mu_{\pi} \) is uniquely determined and (3.2.1) holds.

PROOF: Let \( \mu \in I(\mathbb{R}^N) \) be given. Since there is an \( r > 0 \) such that \( |1 - \mu(\xi)| \leq \frac{1}{2} \) for all \( \xi \in B(0, r) \) and, for all \( n \in \mathbb{Z}^+ \), \( \mu = \mu^n_{\pi} \) for some \( \mu_{\pi} \in M_1(\mathbb{R}^N) \), Lemma 3.2.4 guarantees that \( \hat{\mu} \) never vanishes. Hence, by Lemma 3.2.3, both the existence and uniqueness of \( \ell_\mu \) follow. Moreover, if \( \mu = \mu^n_{\pi} \), then, from \( \hat{\mu}(\xi) = \hat{\mu^n_{\pi}}(\xi)^n \), we know first that \( \hat{\mu^n_{\pi}} \) never vanishes and then that \( \ell_\mu = n\ell \), where \( \ell \) is the unique element of \( C(\mathbb{R}^N; \mathbb{C}) \) satisfying \( (0) = 0 \) and \( \hat{\mu^n_{\pi}} = e^\ell \). Now define \( M_n \) as in the statement, and observe that

\[
\hat{\pi_{M_n}}(\xi) = \exp\left(n(\hat{\mu^n_{\pi}}(\xi) - 1)\right) = \exp\left(n(e^{\ell_\mu}(\xi) - 1)\right) \longrightarrow e^{\ell_\mu}(\xi) = \hat{\mu}(\xi)
\]
as \( n \rightarrow \infty \). Hence, \( \pi_{M_n} \rightarrow \mu \). In particular, this proves that \( I(\mathbb{R}^N) \subseteq \overline{P}(\mathbb{R}^N) \), and therefore, since we already know that \( P(\mathbb{R}^N) \subseteq I(\mathbb{R}^N) \), the final statement will follow once we check that \( I(\mathbb{R}^N) \) is closed.

To prove that \( I(\mathbb{R}^N) \) is closed, suppose that \( \{ \mu_k : k \geq 1 \} \subseteq I(\mathbb{R}^N) \) and that \( \mu_k \rightarrow \mu \). The first step in checking that \( \mu \in I(\mathbb{R}^N) \) is to show that \( \hat{\mu} \) never vanishes. To this end, use the fact that \( \hat{\mu_k} \rightarrow \hat{\mu} \) uniformly on compacts to see that there must exist an \( r > 0 \) such that \( |1 - \mu_k(\xi)| \leq \frac{1}{2} \) for all \( k \in \mathbb{Z}^+ \) and \( \xi \in B(0, r) \). Hence, because each of the \( \mu_k \)'s is infinitely divisible, one can use Lemma 3.2.4 to see that, for each \( R \in (0, \infty) \),

\[
\inf\{|\mu_k(\xi)| : k \in \mathbb{Z}^+ \text{ and } \xi \in B(0, R)\} > 0,
\]

and clearly this is more than enough to show that \( \hat{\mu} \) never vanishes. Thus we can choose a unique \( \ell \in C(\mathbb{R}^N; \mathbb{C}) \) so that \( (0) = 0 \) and \( \hat{\mu} = e^\ell \). Moreover, if \( \ell_k = \ell_{\mu_k} \), then, by Lemma 3.2.3, \( \ell_k \rightarrow \ell \) uniformly on compacts. Now let \( n \in \mathbb{Z}^+ \) be given, and choose \( \{ \mu_{k, \pi} : k \geq 1 \} \subseteq M_1(\mathbb{R}^N) \) so that \( \mu_k = \mu^n_{\pi, \pi} \). Then we know that \( \hat{\mu_{k, \pi}} = e^{\frac{1}{n}\ell_k} \), and so, as \( k \rightarrow \infty \), \( \hat{\mu_{k, \pi}} \rightarrow e^{\frac{1}{n}\ell} \) uniformly on compacts. Hence, by Lévy’s Continuity Theorem, \( e^{\frac{1}{n}\ell} = \hat{\mu^n_{\pi}} \) for some \( \mu^n_{\pi} \in M_1(\mathbb{R}^N) \). Since this means that \( \mu = \mu^n_{\pi} \), we have shown that \( \mu \in I(\mathbb{R}^N) \). \( \square \)

§ 3.2.2. The Formula. Theorem 3.2.7 provides interesting information, but it fails to provide a concrete characterization of the infinitely divisible laws. In this subsection I will give an explicit formula for \( \hat{\mu} \) when \( \mu \in I(\mathbb{R}^N) \), which, in view of the first part of Theorem 3.2.7, is equivalent to characterizing the functions in \( \{ \ell_\mu : \mu \in I(\mathbb{R}^N) \} \).

In order to understand what follows, it may be helpful to first guess what the characterization might be. We already know two families of measures which are contained in \( I(\mathbb{R}^N) \): the Gaussian measures \( \gamma_{m, C} \) for \( m \in \mathbb{R}^N \) and symmetric, non-negative definite \( C \in \text{Hom}(\mathbb{R}^N; \mathbb{R}^N) \), and the Poisson measures \( \pi_M \) for
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M ∈ \mathcal{M}_0(\mathbb{R}^N)$. Further, it is obvious that $\mu, \nu ∈ \mathcal{I}(\mathbb{R}^N) \implies \mu \ast \nu ∈ \mathcal{I}(\mathbb{R}^N)$, and we know that $\mu ∈ \mathcal{I}(\mathbb{R}^N)$ if $\mu_n \to \nu$ for some $\{\mu_n : n \geq 1\} ⊆ \mathcal{I}(\mathbb{R}^N)$. Finally, Theorem 3.2.7 tells us that every element of $\mathcal{I}(\mathbb{R}^N)$ is the limit of Poisson measures. Thus, by Lévy’s Continuity Theorem, what we should be asking is what sort of functions can arise as the locally uniform limit of functions of the form

(*) \quad \xi \mapsto \ell = \sqrt{-1}(\xi, m)_{\mathbb{R}^N} - \frac{1}{2}(\xi, C\xi)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 \right] M(dy),

and, as I already noted, only the Poisson component $M$ offers much flexibility.

With this in mind, I introduce for each $\alpha ∈ [0, \infty)$ the class $\mathcal{M}_\alpha(\mathbb{R}^N)$ of Borel measures $M$ on $\mathbb{R}^N$ such that

$$M(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|y|^{\alpha}}{1 + |y|^\alpha} M(dy) < \infty.$$ 

When $M ∈ \mathcal{M}_0(\mathbb{R}^N)$, the function $\ell$ in (*) equals $\ell_\mu$ for $\mu = \gamma_{m,c} \ast \pi_M$. More generally, even if $M ∈ \mathcal{M}_\alpha(\mathbb{R}^N) \setminus \mathcal{M}_0(\mathbb{R}^N)$, for each $r > 0$, $M_r$ given by $M(dy) = 1_{[r, \infty)}(|y|)M(dy)$ is an element of $\mathcal{M}_0(\mathbb{R}^N)$. Furthermore, if $M ∈ \mathcal{M}_1(\mathbb{R}^N)$, then it is clear that, as $r \downarrow 0$,

$$\int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 \right] M_r(dy) \to \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 \right] M(dy)$$

uniformly on compacts. Thus, by Lévy’s Continuity Theorem, when $M ∈ \mathcal{M}_1(\mathbb{R}^N)$, the function $\ell$ in (*) is $\ell_\mu$ for a $\mu ∈ \mathcal{I}(\mathbb{R}^N)$. In order to handle $M ∈ \mathcal{M}_\alpha(\mathbb{R}^N)$ for $\alpha > 1$, we must make the integrand $M$-integrable at 0 by substracting off the next term in the Taylor expansion of $e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}}$. Thus, choose a Borel measurable function $\eta : \mathbb{R}^N \to [0, 1]$ which equals 1 in a neighborhood of 0, and set $\ell_r(\xi)$ equal to

$$\sqrt{-1}(\xi, m)_{\mathbb{R}^N} - \frac{1}{2}(\xi, C\xi)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\xi,y)_{\mathbb{R}^N} \right] M_r(dy).$$

Because

$$\ell_r(\xi) = \sqrt{-1}(\xi, m_r)_{\mathbb{R}^N} - \frac{1}{2}(\xi, C\xi)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 \right] M_r(dy)$$

where $m_r = m - \int_{\mathbb{R}^N} \eta(y)y M_r(dy),$

we know that $\ell_r = \ell_\mu_r$ for $\mu_r = \gamma_{m_r,c} \ast \pi_{M_r}$. In addition, if $M ∈ \mathcal{M}_2(\mathbb{R}^N)$,

$$\ell_r(\xi) \to \ell(\xi) \equiv \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\xi,y)_{\mathbb{R}^N} \right] M(dy)$$
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uniformly on compacts. Hence, again by Lévy’s Continuity Theorem, we know that, for each $M \in \mathcal{M}_2(\mathbb{R}^N)$, the function

$$
\xi \mapsto \ell(\xi) \equiv \sqrt{-1}(\xi, \mathbf{m})_{\mathbb{R}^N} - \frac{1}{2}(\xi, C\xi)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi, y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right] M(dy)
$$

equals $\ell_{\mu}$ for some $\mu \in \mathcal{I}(\mathbb{R}^N)$.

Having successfully extended to $M \in \mathcal{M}_2(\mathbb{R}^N)$, one might hope that one can extend to $M \in \mathcal{M}_3(\mathbb{R}^N)$ by subtracting off the next term in the Taylor expansion. That is, one would replace

$$
\int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi, y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right] M_r(dy)
$$

by

$$
\int_{\mathbb{R}^N} \left[ e^{\sqrt{-1}(\xi, y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} + \frac{1}{2}\eta(y)(\xi, y)^2_{\mathbb{R}^N} \right] M_r(dy)
$$

in the expression for $\ell_r$. However, to re-write this $\ell_r$ in the form given in (**), one would have to replace $C$ by

$$
C - \int_{\mathbb{R}^N} \eta(y)y \otimes y M_r(dy),
$$

which would destroy non-negative definiteness as $r \searrow 0$.

The preceding discussion is evidence for the conjecture that the functions $\ell$ of the form in (***) coincide with $\{\ell_{\mu} : \mu \in \mathcal{I}(\mathbb{R}^N)\}$, and the rest of this subsection is devoted to the verification of this conjecture. Because of their role here, elements of $\mathcal{M}_2(\mathbb{R}^N)$ are called Lévy measures.

The strategy which I will adopt derives from the observation that $\ell_{\mu}(\xi) = \lim_{n \to \infty} n(\mu_{\frac{1}{n}}(\xi) - 1)$. Thus, if we can understand the operation

$$
A_{\mu}\varphi = \lim_{n \to \infty} n \left( \langle \varphi, \mu_{\frac{1}{n}} \rangle - \varphi(0) \right)
$$

for a sufficiently rich class of functions $\varphi$, then we can understand $\ell_{\mu}(\xi)$ by applying $A_{\mu}$ to $x \mapsto e^{\sqrt{-1}(\xi, x)_{\mathbb{R}^N}}$. It turns out that, for technical reasons, the class of $\varphi$’s on which it is easiest to understand $A_{\mu}$ is the Schwartz test function space $\mathcal{S}(\mathbb{R}^N; \mathbb{C})$ (the space of smooth $\mathbb{C}$-valued functions which, together with all of their derivatives, are rapidly decreasing) even though $x \mapsto e^{\sqrt{-1}(\xi, x)_{\mathbb{R}^N}}$ is not an element of $\mathcal{S}(\mathbb{R}^N; \mathbb{C})$. The basic reason why $\mathcal{S}(\mathbb{R}^N; \mathbb{C})$ is well suited to our analysis is that the Fourier transform maps $\mathcal{S}(\mathbb{R}^N; \mathbb{C})$ onto itself. Further, once we understand how $A_{\mu}$ acts on $\mathcal{S}(\mathbb{R}^N; \mathbb{C})$, it is a relatively simple matter to use that understanding to compute $\ell_{\mu}(\xi)$. 
Lemma 3.2.9. Let \( \mu \in \mathcal{I}(\mathbb{R}^N) \) be given. For each \( r \in (0, \infty) \) there exists a \( C(r) < \infty \) such that \( |\ell_\mu(\xi)| \leq C(r)(1 + |\xi|^2) \) for all \( \xi \in \mathbb{R}^N \) whenever \( \mu \in \mathcal{I}(\mathbb{R}^N) \) satisfies \(|1 - \hat{\mu}(\xi)| \leq \frac{1}{2} \) for \( \xi \in \overline{B(0, r)} \). Moreover,

\[
A_\mu(c1 + \varphi) \equiv \lim_{n \to \infty} n \left( \langle c1 + \varphi, \mu_\frac{1}{n} \rangle - (c + \varphi(0)) \right)
\]

(3.2.10)

\[
= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \ell_\mu(\xi) \hat{\varphi}(\xi) \, d\xi
\]

for each \( c \in \mathbb{C} \) and \( \varphi \in \mathcal{S}(\mathbb{R}^N; \mathbb{C}) \).

Proof: Suppose that \( \mu \in \mathcal{I}(\mathbb{R}^N) \) satisfies \(|1 - \hat{\mu}(\xi)| \leq \frac{1}{2} \) for \( \xi \in \overline{B(0, r)} \). Applying (3.1.4) and the second inequality in (3.2.6) with \( \nu = \mu_\frac{1}{n} \), we know that, for any \((\rho, R) \in (0, \infty)^2\),

\[
\sup_{|\xi| \leq R} |1 - \hat{\mu}_\frac{1}{n}(\xi)| \leq \rho R + \frac{4}{ns(r \rho)}.
\]

Hence, if \( R \geq r \), then, by taking \( \rho = \frac{1}{16} \), we obtain \( \sup_{|\xi| \leq R} |1 - \hat{\mu}_\frac{1}{n}(\xi)| \leq \frac{1}{2} \) and therefore \( \sup_{|\xi| \leq R} |\frac{1}{t} \ell_\mu(\xi)| \leq 2 \) if \( n \) satisfies (3.2.5). Finally, observe that there is a \( c > 0 \) such that \( s(t) \geq ct^2 \) for \( t \in (0, 1] \), and therefore that \( |\ell_\mu(\xi)| \leq 2(1 + \frac{64\pi^2}{c^2}) \) for \(|\xi| \leq R \), which completes the proof of the first assertion.

Clearly it suffices to prove (3.2.10) when \( c = 0 \). Thus, let \( \varphi \in \mathcal{S}(\mathbb{R}^N; \mathbb{C}) \) be given. Then, by (2.3.4),

\[
(2\pi)^N n \left( \langle \varphi, \mu_\frac{1}{n} \rangle - \varphi(0) \right) = \int_{\mathbb{R}^N} n \left( e^{\frac{4\pi^2}{c^2}} - 1 \right) \hat{\varphi}(\xi) \, d\xi
\]

\[
= \int_0^1 \left( \int_{\mathbb{R}^N} e^{\frac{4\pi^2}{c^2}} \ell_\mu(\xi) \hat{\varphi}(\xi) \, d\xi \right) \, dt \longrightarrow \int_{\mathbb{R}^N} \ell_\mu(\xi) \hat{\varphi}(\xi) \, d\xi,
\]

where (keeping in mind that \( |e^{\frac{4\pi^2}{c^2}}| = |\hat{\mu}_\frac{1}{n}(\xi)| \leq 1 \), \( \ell_\mu(\xi) \) has a most quadratic growth, and \( \hat{\varphi}(\xi) \) is rapidly decreasing) the passage to the second line is justified by Fubini’s Theorem and the limit is an application of Lebesgue’s Dominated Convergence Theorem. \( \square \)

Lemma 3.2.9, especially (3.2.10), provides us with two critical pieces of information about \( A_\mu \). Namely, it tells us that \( A_\mu \) satisfies the minimum principle and that it is quasi-local. To be precise, set \( \mathbf{D} = \mathbb{R} \oplus \mathcal{S}(\mathbb{R}^N; \mathbb{R}) \). That is \( \varphi \in \mathbf{D} \) if and only if there is a \( \varphi(\infty) \in \mathbb{R} \) such that \( \varphi - \varphi(\infty)1 \in \mathcal{S}(\mathbb{R}^N; \mathbb{R}) \). I will say that a real-valued linear functional \( A \) on \( \mathbf{D} \) satisfies the minimum principle if

\[
A \varphi \geq 0 \text{ if } \varphi \in \mathbf{D} \text{ and } \varphi(0) = \min_{x \in \mathbb{R}^N} \varphi(x)
\]

(3.2.11)
and that $A$ is quasi-local if

\[(3.2.12) \lim_{R \to \infty} A\varphi_R = 0 \quad \text{for all } \varphi \in \mathcal{D},\]

where $\varphi_R(x) = \varphi \left( \frac{x}{R} \right)$ for $R > 0$. Notice that by applying the minimum principle to both $1$ and $-1$, one knows that $A1 = 0$. To see that $A_\mu$ satisfies both these conditions, first observe that if $\varphi(0) = \min_{x \in \mathbb{R}^n} \varphi(x)$, then $\langle \varphi, \mu_R \rangle - \varphi(0) \geq 0$ for all $n \in \mathbb{Z}^+$ and therefore that $A_\mu \varphi \geq 0$. Secondly, to check that $A_\mu$ is quasi-local, note that it suffices to treat $\varphi \in \mathcal{S}(\mathbb{R}^N; \mathbb{R})$ and that for such a $\varphi$, $\hat{\varphi}_R(\xi) = R^N \hat{\varphi}(R\xi)$. Thus,

\[
(2\pi)^N A_\mu \varphi_R = \int_{\mathbb{R}^N} \ell_\mu((R^{-1}\xi) \hat{\varphi}(\xi)) \, d\xi \to 0,
\]

since $\ell_\mu(0) = 0$ and $\xi \mapsto \sup_{R \geq 1} |\ell_\mu((R^{-1}\xi) \hat{\varphi}(\xi))|$ is rapidly decreasing.

As I am about to show, these two properties allow us to say a great deal about $A_\mu$. Before explaining this, first observe that if $M \in \mathcal{M}_0(\mathbb{R}^N)$, then for every Borel measurable $\varphi : \mathbb{R}^N \to \mathbb{C}$

\[
(3.2.13) \quad \sup_{y \in \mathbb{R}^N \setminus \{ 0 \}} \frac{|\varphi(y)|}{1 + |y|^n} < \infty \implies \varphi \in L^1(M; \mathbb{C}).
\]

Using (3.2.13), one can easily check that if $\varphi \in C^2_b(\mathbb{R}^N; \mathbb{C})$ and $\eta \in \mathcal{S}(\mathbb{R}^N; \mathbb{R})$ equals $1$ in a neighborhood of $0$, then

\[
y \mapsto \varphi(y) - \varphi(0) - \eta(y) \langle \nabla \varphi(0), y \rangle_{\mathbb{R}^N}
\]

is $M$-integrable for every $M \in \mathcal{M}_2(\mathbb{R}^N)$.

Second, in preparation for the proof of the next lemma, I have to introduce the following partition of unity for $\mathbb{R}^N \setminus \{ 0 \}$. Choose $\psi \in C^\infty(\mathbb{R}^N; [0, 1])$ so that $\psi$ has compact support in $B(0, 2) \setminus B(0, \frac{1}{4})$ and $\psi(y) = 1$ when $\frac{1}{2} \leq |y| \leq 1$, and set $\psi_m(y) = \psi(2^m y)$ for $m \in \mathbb{Z}$. Then, if $y \in \mathbb{R}^N$ and $2^{-m-1} \leq |y| \leq 2^{-m}$, $\psi_m(y) = 1$ unless $-m - 2 \leq n \leq -m + 1$. Hence, if $\Psi(y) = \sum_{m \in \mathbb{Z}} \psi_m(y)$ for $y \in \mathbb{R}^N \setminus \{ 0 \}$, then $\Psi$ is a smooth function with values in $[1, 4]$; and therefore, for each $m \in \mathbb{Z}$, the function $\chi_m$ given by $\chi_m(0) = 0$ and $\chi_m(y) = \frac{\psi_m(y)}{\Psi(y)}$ for $y \in \mathbb{R}^N \setminus \{ 0 \}$ is a smooth, $[0, 1]$-valued function which vanishes off of $B(0, 2^{-m-1}) \setminus B(0, 2^{-m-2})$. In addition, for each $y \in \mathbb{R}^N \setminus \{ 0 \}$, $\sum_{m \in \mathbb{Z}} \chi_m(y) = 1$ and $\chi_m(y) = 0$ unless $2^{-m-2} \leq |y| \leq 2^{-m+1}$.

Finally, given $n \in \mathbb{Z}^+$ and $\varphi \in C^n(\mathbb{R}^N; \mathbb{C})$, define $\nabla^n \varphi(x)$ to be the multilinear map on $(\mathbb{R}^N)^n$ into $\mathbb{C}$ by

\[
\left[ \nabla^n \varphi(x) \right](\xi_1, \ldots, \xi_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \varphi \left( x + \sum_{m=1}^n t_m \xi_m \right) \bigg|_{t_1 = \cdots = t_n = 0}.
\]

Obviously, $\nabla \varphi$ and $\nabla^2 \varphi$ can be identified as the gradient of $\varphi$ and Hessian of $\varphi$. 
LEMMA 3.2.14. Let $D$ be the space of functions described above. If $A : D \rightarrow \mathbb{R}$ is a linear functional on $D$ which satisfies (3.2.11) and (3.2.12), then there is a unique $M \in \mathcal{M}_2(\mathbb{R}^N)$ such that $A \varphi = \int_{\mathbb{R}^N} \varphi(y) M(dy)$ for $\varphi \in \mathcal{F}(\mathbb{R}^N; \mathbb{R})$ which satisfy $\varphi(0) = 0$, $\nabla \varphi(0) = 0$, and $\nabla^2 \varphi(0) = 0$. Next, given $\eta \in C_c^\infty(\mathbb{R}^N; [0, 1])$ satisfying $\eta = 1$ in a neighborhood of 0, set $\eta_\varphi(y) = \eta(y)(\xi, y)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$, and define $m^\eta \in \mathbb{R}^N$ and $C \in \text{Hom}(\mathbb{R}^N; \mathbb{R}^N)$ by

$$\eta \phi(y) = \int_{\mathbb{R}^N} (\varphi(y) - \varphi(0) - \eta(y)(\xi, y)_{\mathbb{R}^N}) M(dy).$$

Then $C$ is symmetric, non-negative definite, and independent of the choice of $\eta$. Finally, for any $\varphi \in D$,

$$A \varphi = \frac{1}{2} \text{Trace}(C \nabla^2 \varphi(0)) + (m^\eta, \nabla \varphi(0))_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\varphi(y) - \varphi(0) - \eta(y)(\xi, y)_{\mathbb{R}^N}) M(dy).$$

PROOF: Referring to the partition of unity described above, define $\Lambda_m \varphi = A(\chi_m \varphi)$ for $\varphi \in C^\infty(\overline{B(0, 2^{m+1}) \setminus B(0, 2^{-m-2})}; \mathbb{R})$, where

$$\chi_m(y) = \begin{cases} \chi_m(y) \varphi(y) & \text{if } 2^{m-2} \leq |y| \leq 2^{-m+1} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\Lambda_m$ is linear. In addition, if $\varphi \geq 0$, then $\chi_m \varphi \geq 0 = \chi_m \varphi(0)$, and so, by (3.2.11), $\Lambda_m \varphi \geq 0$. Similarly, for any $\varphi \in C^\infty(\overline{B(0, 2^{m+1}) \setminus B(0, 2^{-m-2})}; \mathbb{R})$,

$$\|\varphi\|_a \chi_m \pm \chi_m \varphi \geq 0 = (\|\varphi\|_a \chi_m \pm \chi_m \varphi)(0),$$

and therefore $|\Lambda_m \varphi| \leq K_m \|\varphi\|_a$, where $K_m = A \chi_m$. Hence, $\Lambda_m$ admits a unique extension as a continuous linear functional on $C(\overline{B(0, 2^{m+1}) \setminus B(0, 2^{-m-2})}; \mathbb{R})$ which is non-negativity preserving and has norm $K_m$; and so, by the Riesz Representation Theorem, we now know that there is a unique non-negative Borel measure $M_m$ on $\mathbb{R}^N$ such that $M_m$ is supported on $\overline{B(0, 2^{m+1}) \setminus B(0, 2^{-m-2})}$, $K_m = M_m(\mathbb{R}^N)$, and

$$A(\chi_m \varphi) = \int_{\mathbb{R}^N} \varphi(y) M_m(dy) \text{ for all } \varphi \in \mathcal{F}(\mathbb{R}^N; \mathbb{R}).$$

Now define the non-negative, Borel measure $M$ on $\mathbb{R}^N$ by $M = \sum_{m \in \mathbb{Z}} M_m$. Clearly, $M(\{0\}) = 0$. In addition, if $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$, then there is an $n \in \mathbb{Z}^+$ such that $\chi_m \varphi \equiv 0$ unless $|m| \leq n$. Thus,

$$A \varphi = \sum_{m=-n}^n A(\chi_m \varphi) = \sum_{m=-n}^n \int_{\mathbb{R}^N} \varphi(y) M_m(dy)$$

$$= \int_{\mathbb{R}^N} \left( \sum_{m=-n}^n \chi_m(y) \varphi(y) \right) M(dy) = \int_{\mathbb{R}^N} \varphi(y) M(dy)$$

and therefore

$$A \varphi = \int_{\mathbb{R}^N} \varphi(y) M(dy)$$
for \( \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \).

Before taking the next step, observe that, as an application of (3.2.11), if \( \varphi_1, \varphi_2 \in \mathcal{D} \), then

\[
\varphi_1 \leq \varphi_2 \text{ and } \varphi_1(0) = \varphi_2(0) \implies A\varphi_1 \leq A\varphi_2.
\]

Indeed, by linearity, this reduces to the observation that, by (3.2.11), if \( \varphi \in \mathcal{D} \) is non-negative and \( \varphi(0) = 0 \), then \( A\varphi \geq 0 \).

With these preparations, I can show that, for any \( \varphi \in \mathcal{D} \),

\[
\varphi \geq 0 = \varphi(0) \implies \int_{\mathbb{R}^N} \varphi(y) M(dy) \leq A\varphi.
\]

To check this, apply (*) to \( \varphi_n = \sum_{m=n}^{\infty} \chi_m \varphi \) and \( \varphi \), and use (3.2.17) together with the Monotone Convergence Theorem to conclude that

\[
\int_{\mathbb{R}^N} \varphi(y) M(dy) = \lim_{n \to \infty} \int_{\mathbb{R}^N} \varphi_n(y) M(dy) = \lim_{n \to \infty} A\varphi_n \leq A\varphi.
\]

Now let \( \eta \) be as in the statement of the lemma, and set \( \eta_R(y) = \eta(R^{-1}y) \) for \( R > 0 \). By (**) with \( \varphi(y) = |y|^2 \eta(y) \) we know that

\[
\int_{\mathbb{R}^N} |y|^2 \eta(y) M(dy) \leq A\varphi < \infty.
\]

At the same time, by (3.2.17) and (*),

\[
\int_{\mathbb{R}^N} (1 - \eta(y)) \eta_R(y) M(dy) \leq A(1 - \eta)
\]

for all \( R > 0 \), and therefore, by Fatou’s Lemma,

\[
\int_{\mathbb{R}^N} (1 - \eta(y)) M(dy) \leq A(1 - \eta) < \infty.
\]

Hence, I have proved that \( M \in \mathfrak{M}_1(\mathbb{R}^N) \).

I am now in a position to show that (3.2.17) continues to hold for any \( \varphi \in \mathcal{S}(\mathbb{R}^N; \mathbb{R}) \) which vanishes along with its first and second order derivatives at \( 0 \).

To this end, first suppose that \( \varphi \) vanishes in a neighborhood of \( 0 \). Then, for each \( R > 0 \), (3.2.17) applies to \( \eta_R \varphi \), and so

\[
\int_{\mathbb{R}^N} \eta_R(y) \varphi(y) M(dy) = A(\eta_R \varphi) = A\varphi + A((1 - \eta_R) \varphi).
\]

By (*) applied to \( \pm (1 - \eta_R) \varphi \) and \( (1 - \eta_R) \| \varphi \|_u \),

\[
|A((1 - \eta_R) \varphi)| \leq \| \varphi \|_u A(1 - \eta_R) = -\| \varphi \|_u A\eta_R \implies 0 \quad \text{as } R \to \infty,
\]
§ 3.2 The Lévy–Khinchine Formula

where I used (3.2.12) to get the limit assertion. Thus,

\[ A \varphi = \lim_{R \to \infty} \int_{\mathbb{R}^N} \eta_R(y) \varphi(y) M(dy) = \int_{\mathbb{R}^N} \varphi(y) M(dy), \]

because, since \( M \) is finite on the support of \( \varphi \) and therefore \( \varphi \) is \( M \)-integrable, Lebesgue’s Dominated Convergence Theorem applies. I still have to replace the assumption that \( \varphi \) vanishes in a neighborhood of \( \mathbf{0} \) by the assumption that it vanishes to second order there. For this purpose, first note that, by (3.2.13), \( \varphi \) is certainly \( M \)-integrable, and therefore

\[ \int_{\mathbb{R}^N} \varphi(y) M(dy) = \lim_{R \searrow 0} A((1 - \eta_R) \varphi) = A \varphi - \lim_{R \searrow 0} A(\eta_R \varphi). \]

By our assumptions about \( \varphi \) at \( \mathbf{0} \), we can find a \( C < \infty \) such that

\[ |\eta_R \varphi(y)| \leq CR|y|^2 \eta(y) \]

for all \( R \in (0, 1] \). Hence, by (*), it is clearly non-negative definite. Finally, to see that it is independent of the \( \eta \) chosen, let \( \eta' \) be a second choice, note that \( \eta' \xi = \eta \xi \) in a neighborhood of \( \mathbf{0} \), and apply (3.2.17).

To complete the proof from here, let \( \varphi \in \mathcal{S}(\mathbb{R}^N; \mathbb{R}) \) be given, and set

\[ \tilde{\varphi}(x) = \varphi(x) - \varphi(0) - \eta(x)(y, \nabla \varphi(0))_{\mathbb{R}^N} - \frac{1}{2}\eta(x)^2(x, \nabla^2 \varphi(0)x)_{\mathbb{R}^N}. \]

Then, by the preceding, (3.2.17) holds for \( \tilde{\varphi} \) and, after one re-arranges terms, says that (3.2.16) holds. Thus, all the remains is to prove the properties of \( C \).

Remark 3.2.18. A careful examination at the proof of Lemma 3.2.14 reveals a lot. Specifically, it shows why the operation performed by the linear functional \( A \) cannot be of order greater than 2. The point is, that, because of the minimum principle, \( A \) acts as a bounded, non-negative linear functional on the difference between \( \varphi \) and its second order Taylor polynomial, and, because of quasi-locality, this action can be represented by integration against a non-negative measure. The reason why the second order Taylor polynomial suffices is that second order polynomials are, apart from constants, the lowest order polynomial which can have a definite sign.

In order to complete the program, I need to introduce the notion of a Lévy system, which is a triple \((m, C, M)\) consisting of an \( m \in \mathbb{R}^N \), a symmetric, non-negative definite transformation \( C \) on \( \mathbb{R}^N \), and a Lévy measure \( M \in \mathcal{M}_2(\mathbb{R}^N) \). Given a Lévy system \((m, C, M)\) and a bounded, Borel measurable \( \eta : \mathbb{R}^N \to [0, 1] \) satisfying

\[ (3.2.19) \quad \left( \sup_{y \in B(0,1) \setminus \{0\}} |y|^{-1}(1 - \eta(y)) \right) \vee \left( \sup_{y \notin B(0,1)} \eta(y)|y| \right) < \infty, \]
we will need to know that

\[
\frac{1}{1 + |\xi|^2} \int_{\mathbb{R}^N} \left| e^{\sqrt{-1}(\xi \cdot y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right| M(dy)
\]

is bounded and tends to 0 as $|\xi| \to \infty$.

To see this, note that, for each $r \in (0, 1]$,

\[
\int_{\mathbb{R}^N} \left| e^{\sqrt{-1}(\xi \cdot y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right| M(dy) 
\leq \int_{B(0, r)} \left| e^{\sqrt{-1}(\xi \cdot y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right| M(dy) 
+ |\xi| \int_{B(0, r)} (1 - \eta(y)) |y| M(dy) + \int_{B(0, r)^c} (2 + |\xi|\eta(y)|y|) M(y) 
\leq \frac{|\xi|^2}{2} \int_{B(0, r)} |y|^2 M(dy) + |\xi| \int_{B(0, r)} (1 - \eta(y)) M(dy) 
+ |\xi| \int_{B(0, r)^c} \eta(y)|y| M(dy) + 2M(B(0, r)\mathbb{C}).
\]

Obviously, this proves the boundedness in (3.2.20). In addition, it shows that, for each $r \in (0, 1]$, the limit there as $|\xi| \to \infty$ is dominated by $\frac{1}{2} \int_{B(0, r)} |y|^2 M(dy)$, which tends to 0 as $r \downarrow 0$.

Knowing (3.2.20), we can define

\[
\ell_{(m, c, M)}^\eta(\xi) = \sqrt{-1}(m, \xi)_{\mathbb{R}^N} - \frac{1}{2}(\xi, c^T \xi)_{\mathbb{R}^N} 
+ \int_{\mathbb{R}^N} \left( e^{\sqrt{-1}(\xi \cdot y)_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(y)(\xi, y)_{\mathbb{R}^N} \right) M(dy)
\]

for any Lévy system $(m, c, M)$ and Borel measurable $\eta : \mathbb{R}^N \to [0, 1]$ which satisfies (3.2.19). Furthermore, because $\ell_{(m, c, M)}^\eta \to \ell_{(m, c, M)}^\eta$ uniformly on compacts when $M_r(dy) = 1_{[r, \infty)}(|y|) M(dy)$, it is clear that $\ell_{(m, c, M)}^\eta$ is continuous.

**Theorem 3.2.22 (Lévy–Khinchine).** For each $\mu \in \mathcal{I}(\mathbb{R}^N)$, there is a unique $\ell_{\mu} \in C(\mathbb{R}^N; \mathbb{C})$ such that $\ell_{\mu}(0) = 0$ and $\tilde{\mu} = e^{\ell_{\mu}}$, and, for each $n \in \mathbb{Z}^+$, $e^{\ell_{\mu}}$ is the Fourier transform of the unique $\mu_{1/2}^\eta \in \mathcal{M}_1(\mathbb{R}^N)$ satisfying $\mu = \mu_{1/2}^\eta$. Next, let be a $\eta : \mathbb{R}^N \to [0, 1]$ be a Borel measurable function which satisfies (3.2.19). Then, for each $\mu \in \mathcal{I}(\mathbb{R}^N)$, there is a unique Lévy system $(m_{\eta}^\mu, c_{\mu}, M_{\mu})$ such that $\ell_{\mu} = \ell_{(m_{\eta}^\mu, c_{\mu}, M_{\mu})}^\eta$, and, for each Lévy system $(m, c, M)$, there is a unique $\mu \in \mathcal{I}(\mathbb{R}^N)$ such that $\ell_{\mu} = \ell_{(m, c, M)}^\eta$. In fact, if $\eta_0 \in C^\infty_c(\mathbb{R}^N; [0, 1])$ satisfies
Proof: The initial assertion is covered by Theorem 3.2.7.

To prove the second assertion, let \( \eta \in C_c^\infty(\mathbb{R}^N; [0,1]) \) with \( \eta = 1 \) in \( B(0,1) \) be given. For \( \mu \in \mathcal{I}(\mathbb{R}^N) \), I will show that \( \ell_\mu = \ell_{(\mathfrak{m}^\eta,\mathcal{C},M)} \), where \( \mathfrak{m}^\eta \), \( \mathcal{C} \), and \( M \) are determined from (cf. (3.2.10)) \( A_\mu \) as in Lemma 3.2.14. To this end, define \( \epsilon_\xi \) for \( \xi \in \mathbb{R}^N \) by \( \epsilon_\xi(x) = e^{\sqrt{-1} \xi \langle x \rangle} \), and set \( \eta_R(x) = \eta(R^{-1}x) \) for \( R > 0 \). The idea is to show that, as \( R \to \infty \), \( A_\mu(\eta_R \epsilon_\xi) \) tends to both \( \ell_\mu(\xi) \) and to \( \ell_{(\mathfrak{m}^\eta,\mathcal{C},M)}(\xi) \).

To check the first of these, use (3.2.10) to see that

\[
(2\pi)^N A_\mu(\eta_R \epsilon_\xi) = \int_{\mathbb{R}^N} \ell_\mu(\xi') \eta_R(\xi' + \xi) \, d\xi' = \int_{\mathbb{R}^N} \ell_\mu(R^{-1}\xi' - x) \eta(\xi') \, d\xi'.
\]

Hence, since \( \ell_\mu \) is continuous and, by Lemma 3.2.9, \( \sup_{R \geq 1} |\ell_\mu(R^{-1} \xi) \eta(\xi)| \) is rapidly decreasing, Lebesgue's Dominated Convergence Theorem says that

\[
\lim_{R \to \infty} A_\mu(\eta_R \epsilon_\xi) = \ell_\mu(-\xi)(2\pi)^{-N} \int_{\mathbb{R}^N} \eta_R(\xi') \, d\xi' = \ell_\mu(\xi).
\]

To prove that \( A_\mu(\eta_R \epsilon_\xi) \) also tends to \( \ell_{(\mathfrak{m}^\eta,\mathcal{C},M)}(\xi) \), use Lemma (3.2.16) to write

\[
A_\mu(\eta_R \epsilon_\xi) = \ell_{(\mathfrak{m}^\eta,\mathcal{C},M)}(\xi) - \int_{\mathbb{R}^N} (1 - \eta_R(y)) \epsilon_\xi(y) M(dy),
\]

and observe that the last term is dominated by \( M(B(0,R)\xi) \to 0 \).

So far we know that for each \( \mu \in \mathcal{I}(\mathbb{R}^N) \) there is a Lévy system \((\mathfrak{m}^\eta,\mathcal{C},M)\) such that \( \ell_\mu(\xi) = \ell_{(\mathfrak{m}^\eta,\mathcal{C},M)} \). Moreover, in the preliminary discussion at the beginning of this subsection, it was shown that for each Lévy system \((\mathfrak{m},\mathcal{C},M)\) there exists a \( \mu \in \mathcal{I}(\mathbb{R}^N) \) for which \( \ell_{(\mathfrak{m},\mathcal{C},M)} = \ell_\mu \).
Finally, let $\eta_0$ be as in the statement of this theorem. Given $\mu \in \mathcal{T}(\mathbb{R}^N)$, let $\mathbf{m}_\mu^{\eta_0} \in \mathbb{R}^N$, $\mathbf{C}_\mu \in \text{Hom}(\mathbb{R}^N; \mathbb{R}^N)$, and $M_\mu \in \mathfrak{M}_2(\mathbb{R}^N)$ be associated with $A_\mu$ as in Lemma (3.2.16) when $\eta = \eta_0$. As we have just seen, $\ell_\mu = \ell^{\eta_0}_{(\mathbf{m}_\mu^{\eta_0}, \mathbf{C}_\mu, M_\mu)}$. Further, by that lemma and (3.2.10), we know that

$$
\int_{\mathbb{R}^N} \varphi(y) M_\mu(dy) = A_\mu \varphi = \lim_{n \to \infty} n \langle \varphi, \mu_\pi \rangle
$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$ which vanishes to second order at 0. In addition, by that same lemma and (3.2.10), we know that

$$
\mathbf{C}_\mu = \lim_{n \to \infty} n \int_{\mathbb{R}^N} \eta_0(y)^2 \otimes y \mu_\pi - \int_{\mathbb{R}^N} \eta_0(y)^2 y \otimes y M_\mu(dy)
$$

and that

$$
\mathbf{m}_\mu^{\eta_0} = \lim_{n \to \infty} n \int_{\mathbb{R}^N} \eta_0(y) \mu_\pi(dy).
$$

In particular, $\mathbf{m}_\mu^{\eta_0}$, $\mathbf{C}_\mu$, and $M_\mu$ are all uniquely determined by $\mu$ and $\eta_0$. In addition, if $\eta: \mathbb{R}^N \to [0, 1]$ is any other Borel measurable function satisfying (3.2.19), then the preceding combined with

$$
\ell^n_{(\mathbf{m}(\mathbf{m}, \mathbf{C}, M))}(\mathbf{C}) = \ell_{(\mathbf{m}, \mathbf{C}, M)}(\mathbf{C}) + \sqrt{-1} \int_{\mathbb{R}^N} (\eta_0(y) - \eta(y)) (\xi, y) \otimes y M_\mu(dy).
$$

Hence, $\ell_{(\mathbf{m}, \mathbf{C}, M)} = \ell_\mu$ if and only if $\mathbf{m} = \mathbf{m}_\mu^{\eta_0} + \int_{\mathbb{R}^N} (\eta(y) - \eta_0(y)) M(dy)$, $\mathbf{C} = \mathbf{C}_\mu$, and $M = M_\mu$. \qed

The expression in (3.2.21) for $\ell_\mu$ in terms of a Lévy system is known as the Lévy–Khinchine formula.

**Exercises for § 3.2**

**Exercise 3.2.23.** Referring to (3.2.21), suppose that $\mu \in \mathcal{T}(\mathbb{R}^N)$ with $\ell_\mu = \ell^n_{(\mathbf{m}, \mathbf{C}, M)}$ for some Lévy system $(\mathbf{m}, \mathbf{C}, M)$ whose Lévy measure $M$ satisfies $\int_{|y| \geq 1} e^{\lambda|y|} M(dy) < \infty$ for all $\lambda \in (0, \infty)$. Show that $\ell_\mu$ admits a unique extension as an analytic function on $\mathbb{C}^N$ and that $\ell_\mu(\xi)$ continues to be given by (3.2.21) when the $\mathbb{R}^N$-inner product of $(\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$ with $(\xi'_1, \ldots, \xi'_N) \in \mathbb{C}^N$ is $\sum_{i=1}^N \xi_i \xi'_i$. Further, show that

$$
\int_{\mathbb{R}^N} e^{\langle \xi, y \rangle} \mu(dy) = e^{\ell_\mu(-\sqrt{-1} \xi)} \quad \text{for all } \xi \in \mathbb{C}^N.
$$

**Hint:** The first part is completely elementary complex analysis. To handle the second part, begin by arguing that it is enough to treat the cases when either $M = 0$ or $\mathbf{C} = 0$. The case $M = 0$ is trivial, and the case when $\mathbf{C} = 0$ can be further reduced to the one in which $\mu = \pi_M$ for an $M \in \mathfrak{M}_0(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N \setminus \{0\}$. Finally, use the representation $\pi_M = e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \nu^m$ to complete the computation in this case.
Exercise 3.2.24. Given $\mu \in I(\mathbb{R}^N)$ and knowing (3.2.20), show that

$$\langle \xi, C_\mu \xi \rangle_{\mathbb{R}^N} \equiv -2 \lim_{t \to \infty} t^{-2} \ell_\mu(t \xi) \quad \text{for all } \mu \in I(\mathbb{R}^N) \text{ and } \xi \in \mathbb{R}^N.$$ 

Similarly, when $M_\mu \in \mathfrak{M}_1(\mathbb{R}^N)$, show that

$$m_\mu \equiv m^g_\mu - \int_{\mathbb{R}^N} \eta(y) y M_\mu(dy)$$

is independent of the choice of $\eta$ satisfying (3.2.19) and, for each $\xi \in \mathbb{R}^N$,

$$\langle \xi, m_\mu \rangle = -\frac{1}{2} \langle \xi, C_\mu \xi \rangle_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left( e^{\sqrt{-1} \langle \xi, y \rangle_{\mathbb{R}^N}} - 1 \right) M_\mu(dy).$$

Finally, if $\mu \in I(\mathbb{R}^N)$ is symmetric, show that $M_\mu$ is also symmetric and that

$$\ell_\mu(\xi) = -\frac{1}{2} \langle \xi, C_\mu \xi \rangle + \int_{\mathbb{R}^N} \left( \cos(\langle \xi, y \rangle_{\mathbb{R}^N} - 1 \right) M_\mu(dy).$$

Exercise 3.2.25. Given $\mu \in I(\mathbb{R})$, show that $\mu((\infty, 0)) = 0$ if and only if $C_\mu = 0$, $M_\mu \in \mathfrak{M}_1(\mathbb{R})$, $M_\mu((\infty, 0)) = 0$, and (cf. the preceding exercise) $m_\mu \geq 0$. The following are steps which you might follow.

(i) To prove the “if” assertion, set $M_r(dy) = 1_{[r, \infty)}(y) M_\mu(dy)$ for $r > 0$, and show that $\delta_{m_\mu} \ast \pi_M, ((\infty, 0)) = 0$ for all $r > 0$ and $\delta_{m_\mu} \ast \pi_M, \Rightarrow \mu$ as $r \searrow 0$. Conclude from these that $\mu((\infty, 0)) = 0$.

(ii) Now assume that $\mu((\infty, 0)) = 0$. To see that $C_\mu = 0$, show that if $\sigma > 0$, then $\gamma_{0, \sigma} \ast \nu, ((\infty, 0)) > 0$ for any $\nu \in \mathfrak{M}_1(\mathbb{R})$.

(iii) Continuing (ii), show that $\mu((\infty, 0)) \geq \mu_{\frac{1}{2}} (\infty, 0), \ast \nu, ((\infty, 0))^{n}$, and conclude first that $\mu_{\frac{1}{2}} ((\infty, 0)) = 0$ for all $n \in \mathbb{Z}^+$ and then that

$$M_\mu((\infty, 0)) = 0 \text{ and } m_\mu \geq \int_{\mathbb{R}^N} \eta(y) y M_\mu(dy).$$

Finally, deduce from these that $M_\mu \in \mathfrak{M}_1(\mathbb{R})$ and that $m_\mu \geq 0$.

(iv) Suppose that $X \in N(0, 1)$, and show that the distribution of $|X|$ cannot be infinitely divisible.
III Infinitely Divisible Laws

Exercise 3.2.26. The **Gamma distributions** is an interesting source of infinitely divisible laws. Namely, consider the family \( \{ \mu_t : t \in (0, \infty) \} \subseteq M_1(\mathbb{R}) \) given by

\[
\mu_t(dx) = 1_{(0, \infty)}(x) \frac{x^{t-1}e^{-x}}{\Gamma(t)} \, dx.
\]

(i) Show by direct computation that

\[
\mu_s * \mu_t(dx) = \frac{B(s,t)}{\Gamma(s)\Gamma(t)} 1_{(0, \infty)}(x)x^{s+t-1}e^{-x} \, dx,
\]

where

\[
B(s,t) = \int_{(0,1)} \xi^{s-1}(1-\xi)^{t-1} \, d\xi
\]

is Euler’s **Beta function**, and conclude that \( \mu_{s+t} = \mu_s * \mu_t \). In particular, one gets, as a dividend, the famous identity

\[
B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.
\]

(ii) As a consequence of (i), we know that the \( \mu_t \)'s are infinitely divisible. Show that their Lévy–Khinchine representation is

\[
\hat{\mu}_t(\xi) = \exp \left[ t \int_{(0,\infty)} \left( e^{\sqrt{-1}\xi y} - 1 \right) e^{-y} \frac{dy}{y} \right].
\]

Exercise 3.2.27. Suppose that \( \mu \in M_1(\mathbb{R}^N) \) for which there exists a strictly increasing sequence \( \{ n_m : m \geq 1 \} \subseteq \mathbb{Z}^+ \) and a sequence \( \{ \mu_{\frac{n}{m}} : m \geq 1 \} \) such that \( \mu = \mu_{\frac{n}{m}} \) for all \( m \geq 1 \). Show that \( \mu \in I(\mathbb{R}^N) \).

**Hint:** First use Lemma 3.2.4 to show that \( \hat{\mu} \) never vanishes and therefore that there is a unique \( \ell_\mu \in C(\mathbb{R}^N; \mathbb{C}) \) such that \( \ell_\mu(0) = 0 \) and \( \hat{\mu} = e^{\ell_\mu} \). Next, proceed as in the proof of Theorem 3.2.7 to show that \( \mu \in I(\mathbb{R}^N) \), and apply that theorem to conclude that \( \mu \in I(\mathbb{R}^N) \).

§ 3.3 Stable Laws

Recall from Exercise 2.3.23 the maps \( T_\alpha : M_1(\mathbb{R}^N) \to M_1(\mathbb{R}^N) \) given by the prescription

\[
T_\alpha \mu(\Gamma) = \int_{\mathbb{R}^N \times \mathbb{R}^N} 1_{\Gamma} \left( \frac{x+y}{2^{\frac{N}{2}}} \right) \mu(dx)\mu(dy),
\]

and let \( F_\alpha(\mathbb{R}^N) \) denote be the set of non-trivial fixed points of \( T_\alpha \). That is, \( F_\alpha(\mathbb{R}^N) = \{ \mu \in M_1(\mathbb{R}^N) \setminus \{ \delta_0 \} : \mu = T_\alpha \mu \} \). If \( \mu \in F_\alpha(\mathbb{R}^N) \) and \( \mu_{2^{-n}} \) denotes the distribution of \( x \sim \mathcal{N} \) under \( \mu \), then \( \mu = \mu_{2^{-n}} \) for all \( n \). Hence, by the result in Exercise 3.2.27, \( \mu \in I(\mathbb{R}^N) \), and so \( F_\alpha(\mathbb{R}^N) \subseteq I(\mathbb{R}^N) \) for all \( \alpha \in (0, \infty) \).

In this section, I will study the Lévy systems which associated with elements of \( F_\alpha(\mathbb{R}^N) \).
§ 3.3.1. General Results. Knowing that \( F_\alpha(\mathbb{R}^N) \subseteq \mathcal{I}(\mathbb{R}^N) \), we can phrase the condition \( \mu = T_\alpha \mu \) in terms of the associated Lévy systems. Namely, \( \mu \in F_\alpha(\mathbb{R}^N) \) if and only if \( \mu \in \mathcal{I}(\mathbb{R}^N) \setminus \{ \delta_0 \} \) and \( \ell_\mu(\xi) = 2\ell_\mu\left(2^{-\frac{\alpha}{2}}\xi\right) \) for all \( \xi \in \mathbb{R}^N \). Next, using this and Exercise 3.2.24, we see that for \( \mu \in F_\alpha(\mathbb{R}^N) \)

\[
\ell_\mu(\xi) = 2^{-\eta}\ell_\mu\left(2^{2\eta}\xi\right) = 2^n(\frac{2^n}{2} - 1)2^{-\frac{2n}{2}}\ell_\mu\left(2^{-\frac{\alpha}{2}}\xi\right) \rightarrow \begin{cases} 0 & \text{if } \alpha > 2 \\ \frac{1}{2}\xi, C_\mu \xi & \text{if } \alpha = 2 \end{cases}
\]

as \( n \to \infty \). Thus, we have already recovered the results in Exercises 2.3.21 and 2.3.23.

I next will examine \( F_\alpha(\mathbb{R}^N) \) for \( \alpha \in (0, 2) \) in greater detail. For this purpose, define \( \tilde{T}_\alpha M \) for \( M \in \mathcal{M}_2(\mathbb{R}^N) \) to be the Borel measure determined by

\[
(3.3.1) \quad \int_{\mathbb{R}^N} \varphi(y) \tilde{T}_\alpha M(dy) = 2 \int_{\mathbb{R}^N} \varphi(2^{-\frac{\alpha}{2}}y) M(dy)
\]

for Borel measurable \( \varphi : \mathbb{R}^N \to [0, \infty) \). It is easy to check that \( \tilde{T}_\alpha \) maps \( \mathcal{M}_2(\mathbb{R}^N) \) into itself.

**Lemma 3.3.2.** For any \( \alpha \in (0, 2) \),

\[
\mu \in F_\alpha(\mathbb{R}^N) \cup \{ \delta_0 \} \iff \begin{cases} C_\mu = 0, & M_\mu = \tilde{T}_\alpha M_\mu, \text{ and} \\ (1 - 2^{1 - \frac{\alpha}{2}}) C_\mu = \int_{\mathbb{R}^N} (\eta(y) - \eta(\frac{2}{\alpha}y)) y \mu(dy). \end{cases}
\]

In addition, if \( M \in \mathcal{M}_2(\mathbb{R}^N) \setminus \{0\} \) satisfies \( M = \tilde{T}_\alpha M \) for some \( \alpha \in (0, 2) \), then \( M \in \mathcal{M}_\beta(\mathbb{R}^N) \) for all \( \beta > \alpha \) but \( M \notin \mathcal{M}_\alpha(\mathbb{R}^N) \).

**Proof:** From the uniqueness of the Lévy system associated with an element of \( \mathcal{I}(\mathbb{R}^N) \), it is clear that, for any \( \mu \in \mathcal{I}(\mathbb{R}^N) \), \( M_{T_\alpha \mu} = \tilde{T}_\alpha M_\mu \), \( C_{T_\alpha \mu} = 2^{1 - \frac{\alpha}{2}} C_\mu \), and

\[
m_{T_\alpha \mu} = 2^{1 - \frac{\alpha}{2}} m_\mu + \int_{\mathbb{R}^N} (\eta(y) - \eta(\frac{2}{\alpha}y)) y \tilde{T}_\alpha M_\mu(dy).
\]

Hence, \( \mu \in F_\alpha(\mathbb{R}^N) \cup \{ \delta_0 \} \) if and only if \( M_\mu = \tilde{T}_\alpha M_\mu \), \( C_\mu = 2^{1 - \frac{\alpha}{2}} C_\mu \), and, for any \( \eta \) satisfying (3.2.19),

\[
(1 - 2^{1 - \frac{\alpha}{2}}) m_\mu = \int_{\mathbb{R}^N} (\eta(y) - \eta(\frac{2}{\alpha}y)) y \mu(dy).
\]

In particular, when \( \alpha \in (0, 2) \), \( C_\mu = 0 \), and so the first assertion follows.

The second assertion turns on the fact that, for all \( n \in \mathbb{Z}^+ \),

\[
M = \tilde{T}_\alpha M \implies M(\overline{B(\mathbf{0}, 2^{-\frac{\alpha}{2}})}) = M(\overline{B(\mathbf{0}, 2^{-\frac{\alpha}{2}})}).
\]

From this we see that \( \kappa \equiv M(\overline{B(\mathbf{0}, 1)} \setminus B(\mathbf{0}, 2^{-\frac{\alpha}{2}})) > 0 \) unless \( M = 0 \) and that the \( M \)-integral of \(|y|^\beta\) over \( B(\mathbf{0}, 1) \) is bounded below by \( 2^{-1} \kappa 2^{n(1 - \frac{\alpha}{2})} \) and above by \( \kappa \sum_{n=0}^{\infty} 2^{n(1 - \frac{\alpha}{2})} \). □
Theorem 3.3.3. \( \mu \in F_2(\mathbb{R}^N) \) if and only if \( \mu = \gamma_0 \cdot C \) for some non-negative definite, symmetric \( C \in \text{Hom}(\mathbb{R}^N; \mathbb{R}^N) \setminus \{0\} \). If \( \alpha \in (1, 2) \), then \( \mu \in F_\alpha(\mathbb{R}^N) \) if and only if \( \mu \in \mathcal{I}(\mathbb{R}^N) \) and \( \ell_\mu(\xi) \) equals

\[
\frac{\sqrt{-1}}{1 - 2^{1 - \frac{\alpha}{2}}} \int_{\frac{2^{-\frac{\alpha}{2}}}{2} < |y| \leq 1} (\xi, y)_{\mathbb{R}^N} M(dy)
+ \int_{\mathbb{R}^N} \left( e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 - \sqrt{-1} t_{1[0,1]}(|y|) (\xi, y)_{\mathbb{R}^N} \right) M(dy)
\]

for some \( M \in \left( \bigcap_{\beta > \alpha} \mathcal{M}_\beta(\mathbb{R}^N) \right) \setminus \mathcal{M}_\alpha(\mathbb{R}^N) \) satisfying \( M = \hat{T}_\alpha M \). If \( \alpha \in (0, 1) \), then \( \mu \in F_\alpha(\mathbb{R}^N) \) if and only if \( \mu \in \mathcal{I}(\mathbb{R}^N) \) and \( \ell_\mu(\xi) \) equals

\[
\int_{\mathbb{R}^N} \left( e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 \right) M(dy)
\]

for some \( M \in \left( \bigcap_{\beta > \alpha} \mathcal{M}_\beta(\mathbb{R}^N) \right) \setminus \mathcal{M}_\alpha(\mathbb{R}^N) \) satisfying \( M = \hat{T}_\alpha M \). Finally, \( \mu \in F_1(\mathbb{R}^N) \) if and only if \( \mu \in \mathcal{I}(\mathbb{R}^N) \) and either \( \mu = \delta_m \) for some \( m \in \mathbb{R}^N \setminus \{0\} \) or \( \ell_\mu(\xi) = \sqrt{-1}(m, \xi)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left( e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^N}} - 1 - \sqrt{-1} t_{1[0,1]}(|y|) (\xi, y)_{\mathbb{R}^N} \right) M(dy) \)

for some \( m \in \mathbb{R}^N \) and \( M \in \left( \bigcap_{\beta \in (1, 2]} \mathcal{M}_\beta(\mathbb{R}^N) \right) \setminus \mathcal{M}_1(\mathbb{R}^N) \) satisfying \( M = \hat{T}_1 M \) and

\[
\int_{\frac{2^{-\frac{\alpha}{2}}}{2} < |y| \leq 1} y M(dy) = 0.
\]

Proof: The first assertion requires no comment. When \( \alpha \in (0, 2) \), the “if” assertions can be proved by checking that, in each case, \( \ell_\mu(\xi) = 2 \ell_\mu(2^{-\frac{\alpha}{2}} \xi) \). When \( \alpha \in [1, 2) \), the “only if” assertion follows immediately from Lemma 3.3.2 with \( \eta = \mathbf{1}_{B(0,1)} \), and when \( \alpha \in (0, 1) \), it follows from that lemma combined with the observation that \( M = \hat{T}_\alpha M \) implies that

\[
(1 - 2^{1 - \frac{\alpha}{2}}) \int_{B(0,1)} y M(dy) = \int_{(2^{-\frac{\alpha}{2}})^2 < |y| \leq 1} y M(dy).
\]

§ 3.3.2. \( \alpha \)-Stable Laws. The most studied elements of \( F_\alpha(\mathbb{R}^N) \) are the \( \alpha \)-stable laws. That is, those \( \mu \in \mathcal{I}(\mathbb{R}^N) \setminus \{0\} \) such that \( \ell_\mu(t\xi) = t^\alpha \ell_\mu(\xi) \) for all \( t \in (0, \infty) \), not just for \( t = 2 \). Equivalently, if \( \mu \in \mathcal{M}_1(\mathbb{R}^N) \) is \( \alpha \)-stable if and only if \( \mu \in \mathcal{I}(\mathbb{R}^N) \) \( \{0\} \) and, for all non-negative, Borel measurable functions \( \varphi \),

\[
\int_{\mathbb{R}^N} \varphi(y) \mu_\gamma(dy) = \int_{\mathbb{R}^N} \varphi(t^\frac{\alpha}{2} y) \mu(dy), \quad t \in (0, \infty),
\]

where \( \mu(\xi) = e^{\alpha t}. \) Thus, there are no \( \alpha \)-stable laws if \( \alpha > 2 \), and \( \mu \) is \( 2 \)-stable if and only if \( \mu = \gamma_0 \cdot C \) for some \( C \neq 0 \). To examine the \( \alpha \)-stable laws when \( \alpha \in (0, 2) \), I will need the computations contained in the following lemmas.
Lemma 3.3.4. Assume that \( M \in \mathfrak{M}_2(\mathbb{R}^N) \) and that \( \alpha \in (0, 2) \), and define the finite, Borel measure \( \nu \) on \( S^{N-1} \) by

\[
\langle \varphi, \nu \rangle = \frac{1}{\Gamma(2 - \alpha)} \int_{\mathbb{R}^N \setminus \{0\}} \varphi(\frac{y}{|y|}) |y|^2 e^{-|y|} M(dy)
\]

for bounded, Borel measurable \( \varphi : S^{N-1} \to \mathbb{C} \). Then, \( M \) satisfies

\[
(3.3.5) \quad \int_{\mathbb{R}^N} \varphi(ty) M(dy) = t^\alpha \int_{\mathbb{R}^N} \varphi(y) M(dy), \quad t \in (0, \infty)
\]

for all \( \varphi \in C_c(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \) if and only if

\[
\int_{\mathbb{R}^N} \varphi(y) M(dy) = \int_{S^{N-1}} \left( \int_{(0, \infty)} \varphi(r\omega) \frac{dr}{r^{1+\alpha}} \right) \nu(d\omega)
\]

for all \( \varphi \in C_c(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \).

Proof: The “if” assertion is obvious. In addition, the “only if” assertion will follow once I prove it for \( \varphi \)'s such that \( \varphi(y) = \varphi_1(\frac{y}{|y|}) \varphi_2(|y|) \), where \( \varphi_1 \in C(S^{N-1}; [0, \infty)) \) and \( \varphi_2 \in C_c((0, \infty); \mathbb{R}) \). Given \( \varphi_1 \in C(S^{N-1}; [0, \infty)) \), determine the Borel measure \( \rho \) on \( (0, \infty) \) by

\[
\langle \varphi_2, \rho \rangle = \int_{\mathbb{R}^N \setminus \{0\}} \varphi_1(\frac{y}{|y|}) \varphi_2(|y|) |y|^2 M(dy)
\]

for \( \varphi_2 \in C_c((0, \infty); \mathbb{R}) \). Then (3.3.5) implies that

\[
\int_{(0, \infty)} e^{-tr} \rho(dr) = t^{\alpha-2} \int_{(0, \infty)} e^{-r} \rho(dr) = t^{\alpha-2} \Gamma(2 - \alpha) \langle \varphi, \nu \rangle
\]

for \( t \in (0, \infty) \). Hence, since

\[
\int_{(0, \infty)} r^{1-\alpha} e^{-tr} dr = \Gamma(2 - \alpha) t^{\alpha-2}, \quad t \in (0, \infty),
\]

uniqueness of the Laplace transform (cf. Exercise 1.2.12) implies that \( \rho(dr) = \langle \varphi_1, \nu \rangle r^{1-\alpha} dr \), and therefore that

\[
\int_{\mathbb{R}^N \setminus \{0\}} \varphi_1(\frac{y}{|y|}) M(dy) = \int_{(0, \infty)} \frac{\varphi_2(r)}{r^2} \rho(dr) = \langle \varphi_1, \nu \rangle \int_{(0, \infty)} \varphi_1(r) \frac{dr}{r^{1+\alpha}}. \quad \square
\]
Lemma 3.3.6. Let $\mu \in \mathcal{I}(\mathbb{R}^N)$. Then $\mu$ is 2-stable if and only if $\mu = \gamma_0 \mathbf{C}$ for some symmetric, non-negative definite $\mathbf{C} \neq 0$; $\mu$ is $\alpha$-stable for some $\alpha \in (0,1)$ if and only if there is a finite, non-negative Borel measure $\nu \neq 0$ on $S^{N-1}$ such that

$$
\ell_\mu(\xi) = \int_{S^{N-1}} \left( \int_{(0,\infty)} \left[ e^{\sqrt{-1}(\xi,r\omega)} - 1 \right] \frac{dr}{r^{1+\alpha}} \right) \nu(d\omega);
$$

$\mu$ is 1-stable if and only if there exists a finite, non-negative, Borel measure $\nu$ on $S^{N-1}$ and an $\mathbf{m} \in \mathbb{R}^N$ satisfying

$$
|\mathbf{m}| + \nu(S^{N-1}) > 0 \text{ and } \int_{S^{N-1}} \omega \nu(d\omega) = 0
$$
such that $\ell_\mu(\xi)$ equals

$$
\sqrt{-1}(\xi,\mathbf{m})_{\mathbb{R}^N} + \int_{S^{N-1}} \left( \int_{(0,\infty)} \left[ e^{\sqrt{-1}(\xi,r\omega)} - 1 - \sqrt{-1}1_{[0,1]}(r)(\xi,r\omega)_{\mathbb{R}^N} \right] \frac{dr}{r^2} \right) \nu(d\omega);
$$

and $\mu$ is $\alpha$-stable for some $\alpha \in (1,2)$ if and only if there is a finite, non-negative, Borel measure $\nu \neq 0$ on $S^{N-1}$ such that $\ell_\mu(\xi)$ equals

$$
\frac{\sqrt{-1}}{1-\alpha} \int_{S^{N-1}} (\xi,\omega)_{\mathbb{R}^N} \nu(d\omega) + \int_{S^{N-1}} \left( \int_{(0,\infty)} \left[ e^{\sqrt{-1}(\xi,r\omega)} - 1 - \sqrt{-1}1_{[0,1]}(r)(\xi,r\omega)_{\mathbb{R}^N} \right] \frac{dr}{r^{1+\alpha}} \right) \nu(d\omega).
$$

Proof: The sufficiency part of each case is easy to check directly or as a consequence of Theorem 3.3.3. To prove the necessity, first check that if $\mu$ is $\alpha$-stable and therefore $\ell_\mu(\xi) = t^\alpha \ell_\mu(\xi)$, then $M$ must have the scaling property in (3.3.5) and therefore have the form described in Lemma 3.3.4. Second, when $M$ has this form, check that in each case the result in Theorem 3.3.3 translates into the result here. \(\square\)

In the following, $\mathbb{C}_+$ denotes the open upper halfspace $\{ \zeta \in \mathbb{C} : \Im(\zeta) > 0 \}$ in $\mathbb{C}$, and $\mathbb{C}_+$ denotes its closure $\{ \zeta \in \mathbb{C} : \Im(\zeta) \geq 0 \}$. In addition, given $\zeta \in \mathbb{C}$ and $\alpha \in (0,2)$, we take $\zeta^\alpha = |\zeta|^\alpha e^{\sqrt{-1}\arg\zeta}$, where $\arg\zeta$ is 0 if $\zeta = 0$ and is the unique $\theta \in (-\pi,\pi]$ such that $\zeta = |\zeta|e^{\sqrt{-1}\theta}$ if $\zeta \neq 0$.

Lemma 3.3.7. If $\alpha \in (0,1)$, then

$$
\int_{(0,\infty)} \frac{e^{\sqrt{-1}\zeta r} - 1}{r^{1+\alpha}} dr = \frac{\Gamma(1-\alpha)}{\alpha^\alpha} \left( \frac{\zeta}{\sqrt{-1}} \right)^\alpha \text{ for } \zeta \in \mathbb{C}_+.
$$
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In particular,

\[
a_\alpha \equiv \int_{(0, \infty)} \frac{\cos r - 1}{r^{1+\alpha}} \, dr = \begin{cases} \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} \cos \frac{\alpha\pi}{2} & \text{if } \alpha \in (1, 2) \\ -\frac{\Gamma(1-\alpha)}{\alpha} \cos \frac{\alpha\pi}{2} & \text{if } \alpha \in (0, 1) \\ -\frac{\pi}{2} & \text{if } \alpha = 1 \end{cases}
\]

and

\[
b_\alpha \equiv \int_{(0, \infty)} \frac{\sin r}{r^{1+\alpha}} \, dr = \frac{\Gamma(1-\alpha)}{\alpha} \sin \frac{\alpha\pi}{2} \quad \text{if } \alpha \in (0, 1).
\]

**Proof:** Let \( f_\alpha(\zeta) \) denote the integral on the left hand side of the first equation. Clearly \( f_\alpha \) is continuous on \( \mathbb{C}_+ \) and analytic on \( \mathbb{C}_+ \). In addition, \( f_\alpha(\xi) = \xi^\alpha f_\alpha(1) \) for \( \xi \in (0, \infty) \), and \( \Re(f_\alpha(1)) < 0 \). Hence, there exist \( c > 0 \) and \( \theta \in (0, \frac{\pi}{2}) \) such that \( f_\alpha(\xi) = -ce^{-\theta\xi}\xi^\alpha \) for \( \xi \in (0, \infty) \). Since \( \zeta \in \mathbb{C}_+ \mapsto \zeta^\alpha \in \mathbb{C} \) is the unique continuous extension of \( \xi \in (0, \infty) \mapsto \xi^\alpha \in (0, \infty) \) to \( \mathbb{C}_+ \), which is analytic on \( \mathbb{C}_+ \), we know that \( f_\alpha(\zeta) = -ce^{-\theta\zeta}\zeta^\alpha \) for \( \zeta \in \mathbb{C}_+ \). In addition

\[
f_\alpha(\sqrt{-1}) = \int_{(0, \infty)} \frac{e^{-r} - 1}{r^{1+\alpha}} \, dr = -\frac{1}{\alpha} \int_{(0, \infty)} r^{-\alpha} e^{-r} \, dr = -\frac{\Gamma(1-\alpha)}{\alpha}.
\]

Hence, \( c = \frac{\Gamma(1-\alpha)}{\alpha} \) and \( \theta = -\frac{\alpha\pi}{2} \).

When \( \alpha \in (0, 1) \), the values of \( a_\alpha \) and \( b_\alpha \) follow immediately from the evaluation of \( f_\alpha(1) \). When \( \alpha \in (1, 2) \), one can find the value of \( a_\alpha \) by first observing that

\[
\int_{(0, \infty)} \frac{\cos(\xi r) - 1}{r^{1+\alpha}} \, dr = \xi^\alpha \int_{(0, \infty)} \frac{\cos r - 1}{r^{1+\alpha}} \, dr \quad \text{for } \xi \in (0, \infty),
\]

and then differentiating this with respect to \( \xi \) to get

\[
\alpha \int_{(0, \infty)} \frac{\cos r - 1}{r^{1+\alpha}} \, dr = -\int_{(0, \infty)} \frac{\sin r}{r^{\alpha}} \, dr = -b_{\alpha-1}.
\]

To evaluate \( a_1 \), simply note that

\[
a_1 = \lim_{\alpha \searrow 1} a_\alpha = -\lim_{\alpha \searrow 1} \frac{\Gamma(2-\alpha)}{\alpha} \cos \frac{\alpha\pi}{2} \cdot \frac{1}{1-\alpha} = -\frac{\pi}{2}.
\]

**Theorem 3.3.8.** Let \( \mu \in \mathcal{I}(\mathbb{R}^N) \). If \( \alpha \in (0, 2) \setminus \{1\} \), then \( \mu \) is \( \alpha \)-stable if and only if there exists a finite, non-negative, Borel measure \( \nu \neq 0 \) on \( \mathbb{S}^{N-1} \) such that

\[
\ell_\mu(\xi) = (-1)^{1(0,1)}(\alpha) \int_{\mathbb{S}^{N-1}} \left( \frac{(\xi, \omega)_{\mathbb{R}^N}}{\sqrt{-1}} \right)^\alpha \nu(\omega).
\]
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On the other hand, $\mu$ is 1-stable if and only if there exist an $m \in \mathbb{R}^n$ and a finite, non-negative, Borel measure $\nu$ on $\mathbb{S}^{n-1}$ such that $|m| + \nu(\mathbb{S}^{n-1}) > 0$,

$$\int_{\mathbb{S}^{n-1}} \omega \nu(d\omega) = 0,$$

and

$$\ell_{\mu}(\xi) = \sqrt{-1}(\xi, m)_{\mathbb{R}^n} - \sqrt{-1} \int_{\mathbb{S}^{n-1}} (\xi, \omega)_{\mathbb{R}^n} \log(\xi, \omega)_{\mathbb{R}^n} \nu(d\omega),$$

where $\zeta \log \zeta = \zeta \log |\zeta| + \sqrt{-1} \arg \zeta$ for $\zeta \in \mathbb{C}$.

**Proof:** When $\alpha \in (0, 1)$, the conclusion is a simple application of the corresponding results in Lemmas 3.3.6 and 3.3.7. When $\alpha \in (1, 2)$, one has to massage the corresponding expression in Lemma 3.3.6. Specifically, begin with the observation that

$$\frac{\sqrt{-1} \xi}{1 - \alpha} + \int_{(0, \infty)} \left[ e^{\sqrt{-1} \xi r} - 1 - \sqrt{-1} \xi [r] \right] \frac{dr}{r^{1 + \alpha}}$$

$$= |\xi|^\alpha \left( \int_{(0, \infty)} \left[ e^{\sqrt{-1} \text{sgn}(\xi) r} - 1 - \sqrt{-1} \text{sgn}(\xi) [r] \right] \frac{dr}{r^{1 + \alpha}} + \frac{\sqrt{-1} \text{sgn}(\xi)}{1 - \alpha} \right)$$

for $\xi \in \mathbb{R}$. Thus, we can write the expression for $\ell_{\mu}(\xi)$ as

$$\int_{\mathbb{S}^{n-1}} |(\xi, \omega)_{\mathbb{R}^n}|^\alpha g_{\alpha}(\text{sgn}(\xi, \omega)_{\mathbb{R}^n}) \nu(d\omega),$$

where (cf. Lemma 3.3.7)

$$g_{\alpha}(\pm 1) = \int_{(0, \infty)} \left[ e^{\pm \sqrt{-1} \xi r} - 1 - \sqrt{-1} [r] \right] \frac{dr}{r^{1 + \alpha}} \pm \frac{\sqrt{-1}}{1 - \alpha}$$

$$= a_{\alpha} \pm \sqrt{-1} \int_{(0, \infty)} (\sin r - 1 [r]) \frac{dr}{r^{1 + \alpha}} \pm \frac{\sqrt{-1}}{1 - \alpha}. $$

Next use integration by parts over the intervals $(0, 1]$ and $[1, \infty)$ to check that

$$\int_{(0, \infty)} (\sin r - 1 [r]) \frac{dr}{r^{1 + \alpha}} = \frac{1}{\alpha - 1} + \frac{1}{\alpha} \int_{(0, \infty)} \frac{\cos r - 1}{r^\alpha} dr = \frac{1}{\alpha - 1} + a_{\alpha - 1}. $$

Hence, since $a_{\alpha - 1} = -\frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} \sin \frac{\alpha \pi}{2},$

$$g_{\alpha}(\pm 1) = \frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} e^{\mp \frac{\alpha \pi}{2}}.$$

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and therefore

\[ g_\alpha \left( \text{sgn}(x, \omega)_{\mathbb{R}^N} \right) \left| (\xi, \omega)_{\mathbb{R}^N} \right|^\alpha = \frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} \left( \frac{(\xi, \omega)_{\mathbb{R}^N}}{\sqrt{-1}} \right)^\alpha. \]

Thus, all that we need to do is replace the \( \nu \) in Theorem 3.3.8 by \( \frac{\alpha}{1 - \alpha} \nu \).

Turning to the case \( \alpha = 1 \), note that, because of the mean zero condition on \( \nu \),

\[
\int_{S^{N-1}} \left( \int_{(0, \infty)} \left[ e^{\sqrt{-1}(\xi, \omega)_{\mathbb{R}^N} r} - 1 - \sqrt{-1} \right] \frac{dr}{r^{1+\alpha}} \right) \nu(d\omega)
\]

\[
= \lim_{\alpha \to 1} \int_{S^{N-1}} \left( \int_{(0, \infty)} \left[ e^{\sqrt{-1}(\xi, \omega)_{\mathbb{R}^N} r} - 1 \right] \frac{dr}{r^{1+\alpha}} \right) \nu(d\omega)
\]

\[
= \frac{1}{\sqrt{-1}} \lim_{\alpha \to 1} \frac{1}{1 - \alpha} \int_{S^{N-1}} \left[ (\xi, \omega)_{\mathbb{R}^N}^\alpha - (\xi, \omega)_{\mathbb{R}^N} \right] \nu(d\omega)
\]

\[
= -\sqrt{-1} \int_{S^{N-1}} (\xi, \omega)_{\mathbb{R}^N} \log(\xi, \omega)_{\mathbb{R}^N} \nu(d\omega),
\]

where I have used \( (1 - \alpha)\Gamma(1 - \alpha) = \Gamma(2 - \alpha) \to 1. \)

I close this section with a couple of examples of particularly important stable laws.

**Corollary 3.3.9.** For any \( \alpha \in (0, 2] \), \( \mu \) is a symmetric, \( \alpha \)-stable law if and only if there is a finite, non-negative a symmetric, Borel measure \( \nu \neq 0 \) on \( S^{N-1} \) such that

\[ \ell_\mu(\xi) = -\int_{S^{N-1}} \left| (\xi, \omega)_{\mathbb{R}^N} \right|^\alpha \nu(d\omega). \]

Moreover, \( \mu \) is a rotationally invariant, \( \alpha \)-stable law if and only if \( \ell_\mu(\xi) = -t|\xi|^\alpha \) for some \( t \in (0, \infty) \).

**Proof:** If \( \mu \) is 2-stable, then \( \mu = \gamma_{0, C} \) for some \( C \neq 0 \) and is therefore symmetric. In addition, by defining \( \nu \) on \( S^{N-1} \) so that

\[ \langle \varphi, \nu \rangle = \frac{1}{2} \int_{\mathbb{R}^N} |y|^2 \varphi \left( \frac{y}{|y|} \right) \gamma_{0, C}(d\omega), \]

we see that

\[ \ell_\mu(\xi) = -\frac{1}{2} (\xi, C\xi)_{\mathbb{R}^N} = \int_{S^{N-1}} \left| (\xi, \omega)_{\mathbb{R}^N} \right|^2 \nu(d\omega). \]
If $\alpha \in (0, 2) \setminus \{1\}$, then for every non-zero, symmetric $\nu$ on $\mathbb{S}^{N-1}$,

$$-\int_{\mathbb{S}^{N-1}} |(\xi, \omega)_\mathbb{R}^N|^{\alpha} \nu(d\omega) = (-1)^{1(0,1)(\alpha)} \left|\csc\frac{\alpha \pi}{2}\right| \int_{\mathbb{S}^{N-1}} \left(\frac{(\xi, \omega)_\mathbb{R}^N}{\sqrt{1 - \tau}}\right)^{\alpha} \nu(d\omega)$$

is $\ell_\mu(\xi)$ for a symmetric, $\alpha$-stable $\mu$. Conversely, if $\mu$ is symmetric and $\alpha$-stable for some $\alpha \in (0, 1) \setminus \{1\}$, then, because $\ell_\mu(\xi) = \ell_\mu(-\xi)$, the associated $\nu$ in Theorem 3.3.8 can be chosen to be symmetric, in which case $\ell_\mu(\xi)$ equals

$$(-1)^{1(0,1)(\alpha)} \int_{\mathbb{S}^{N-1}} \left(\frac{(\xi, \omega)_\mathbb{R}^N}{\sqrt{1 - \tau}}\right)^{\alpha} \nu(d\omega) = -\left|\cos\frac{\alpha \pi}{2}\right| \int_{\mathbb{S}^{N-1}} |(\xi, \omega)_\mathbb{R}^N|^{\alpha} \nu(d\omega).$$

To handle the case when $\alpha = 1$, first suppose that $\nu \neq 0$ on $\mathbb{S}^{N-1}$ is symmetric. Then

$$-\int_{\mathbb{S}^{N-1}} |(\xi, \omega)_\mathbb{R}^N| \nu(d\omega) = 2 \int_{\{\omega : (\xi, \omega)_\mathbb{R}^N < 0\}} (\xi, \omega)_\mathbb{R}^N \nu(d\omega)$$

$$= \frac{2}{\pi \sqrt{-1}} \int_{\{\omega : (\xi, \omega)_\mathbb{R}^N < 0\}} \left[(\xi, \omega)_\mathbb{R}^N \log((\xi, \omega)_\mathbb{R}^N) + (\xi, -\omega)_\mathbb{R}^N \log((\xi, -\omega)_\mathbb{R}^N)\right] \nu(d\omega)$$

$$= -\sqrt{-1} \int_{\mathbb{S}^{N-1}} (\xi, \omega)_\mathbb{R}^N \log((\xi, \omega)_\mathbb{R}^N) \nu(d\omega),$$

which is $\ell_\mu(\xi)$ for a symmetric, 1-stable $\mu$. Conversely, if $\mu$ is symmetric and 1-stable, one can use $\ell_\mu(\xi) = \ell_\mu(-\xi)$ to see that $m = 0$ and $\nu$ is symmetric in the expression for $\ell_\mu(\xi)$ in Theorem 3.3.8. Hence, by the preceding calculation, $\ell_\mu(\xi)$ has the desired form.

Finally, if $\mu$ a rotationally invariant, $\alpha$-stable law, then $\ell_\mu(\xi)$ is a rotationally invariant function of $\xi$ and therefore the preceding leads to

$$\ell_\mu(\xi) = -\int_{\mathbb{S}^{N-1}} \left(\int_{\mathbb{S}^{N-1}} |\xi'|^\alpha |(\omega, \omega')|^{\alpha} \nu(d\omega')\right) \overline{\lambda_{\mathbb{S}^{N-1}}}(d\omega) = -t|\xi|^\alpha,$$

where $\overline{\lambda_{\mathbb{S}^{N-1}}}$ is normalized surface measure on $\mathbb{S}^{N-1}$ and

$$t = \nu(\mathbb{S}^{N-1}) \int_{\mathbb{S}^{N-1}} \int_{\mathbb{S}^{N-1}} |(\xi, \omega)_\mathbb{R}^N|^{\alpha} \overline{\lambda_{\mathbb{S}^{N-1}}}(d\omega)$$

for any $e \in \mathbb{S}^{N-1}$. Conversely, by taking $\nu$ to be an appropriate multiple of $\overline{\lambda_{\mathbb{S}^{N-1}}}$, one sees that, for any $t \in (0, \infty)$, $-t|\xi|^\alpha$ is $\ell_\mu(\xi)$ for a symmetric, $\alpha$-stable $\mu$. □
Exercises for § 3.3

**Exercise 3.3.10.** Given $\alpha \in (0, 2)$, define $S_\alpha \nu$ for finite, non-negative, Borel measures $\nu$ on $\mathcal{M}(\mathbb{R}^N) \setminus \mathcal{M}(\mathbb{R}^N)$ by

$$S_\alpha \nu(\Gamma) = \sum_{m \in \mathbb{Z}} 2^{-m} \int_{\mathbb{R}^N} 1_{\Gamma}(2^m y) \nu(dy),$$

and show that this map is one-to-one and onto the set of $M \in \mathcal{M}(\mathbb{R}^N)$ satisfying (cf. (3.3.1)) $M = \hat{T}_\alpha M$. Conclude that, for each $\alpha \in (0, 2)$, $F_\alpha(\mathbb{R}^N)$ contains lots of elements!

**Exercise 3.3.11.** Here are a few further properties of elements of $F_\alpha(\mathbb{R}^N)$.

(i) Show that there is $\mu \in F_1(\mathbb{R}^N)$ such that $\mu(\{ y : (e, y)_{\mathbb{R}^N} < 0 \}) = 0$ for some $e \in S^{N-1}$ if and only if $\alpha \in (0, 1)$.

**Hint:** Reduce to the case when $N = 1$, and look at Exercise 3.2.24.

(ii) If $\mu \in F_1(\mathbb{R}^N)$, show that $\mu(\{ y : (e, y)_{\mathbb{R}^N} < 0 \}) > 0$ for every $e \in S^{N-1}$.

(iii) If $\alpha \in (1, 2)$, show that for each $\epsilon > 0$ there is a $\mu \in F_\alpha(\mathbb{R})$ such that $\mu((-\infty, -\epsilon]) = 0$.

**Exercise 3.3.12.** Take $N = 1$. This exercise is about an important class of stable laws known as one-sided stable laws. That is, stable laws which are supported on $[0, \infty)$.

(i) Show that there exists a one-sided $\alpha$-stable law only if $\alpha \in (0, 1)$.

(ii) If $\alpha \in (0, 1)$, show that $\mu$ is a one-sided $\alpha$-stable law if and only if $\ell_\mu(\xi) = -t \left( \frac{\xi}{\sqrt{t-1}} \right)^\alpha$ for some $t \in (0, \infty)$.

(iii) Let $\alpha \in (0, 1)$, and use $\nu^\alpha_\mu$ to denote the one-sided, $\alpha$-stable law with $\ell_{\nu^\alpha_\mu}(\xi) = -t \left( \frac{\xi}{\sqrt{t-1}} \right)^\alpha$. Show that

$$\int_{[0, \infty)} e^{\sqrt{-1} \xi y} \nu^\alpha_\mu(dy) = \exp \left[ -t \left( \frac{\zeta}{\sqrt{t-1}} \right)^\alpha \right] \text{ for } \zeta \in \mathbb{C} \text{ with } \Re(\zeta) \geq 0.$$

In particular, use Exercise 1.2.12 to conclude that $\nu^\alpha_\mu$ is characterized by the facts that it is supported on $[0, \infty)$ and its Laplace transform is given by

$$\int_{[0, \infty)} e^{-\lambda y} \nu^\alpha_\mu(dy) = e^{-t \lambda^\alpha}, \quad \lambda \geq 0.$$
EXERCISE 3.3.13. Given $\alpha \in (0, 2]$, let $\mu^\alpha_t$ denote the symmetric $\alpha$-stable law, described in Corollary 3.3.9, with $f_{\mu^\alpha_t}(\xi) = -t|\xi|^\alpha$. Clearly $\mu^2_t = \gamma_{0,2tI}$. When $\alpha \in (0, 2)$, show that

$$\mu^\alpha_t = \int_{[0,\infty)} \gamma_{0,2tI} \nu^\alpha_t (d\tau),$$

where $\nu^\alpha_t$ is the one-sided $\frac{\alpha}{2}$-stable law in part (iii) of the preceding exercise. This representation is an example of subordination, and, as we will see below, can be used to good effect.

EXERCISE 3.3.14. Because their Fourier transforms are rapidly decreasing, we know that each of the measures $\nu^\alpha_t$ in part (iii) of Exercise 3.3.11 admits a smooth density with respect to Lebesgue measure $\lambda_\mathbb{R}$ on $\mathbb{R}$. In this exercise, we examine these densities.

(i) For $\alpha \in (0, 1)$, set

$$h^\alpha_t = \frac{d\nu^\alpha_t}{d\lambda_\mathbb{R}}$$

for $t \in (0, \infty)$, and show that

$$\int_0^\infty e^{-\lambda \tau} h^\alpha_t (\tau) d\tau = e^{-t \lambda^\alpha}, \quad \lambda \in [0, \infty),$$

and that $h^\alpha_t (\tau) \equiv t^{-\frac{1}{2}} h^\alpha_1 (t^{-\frac{1}{2}} \tau)$.

(ii) Only when $\alpha = \frac{1}{2}$ is an explicit expression for $h^\alpha_t$ readily available. To find this expression, first note that, by the uniqueness of the Laplace transform (cf. Exercise 1.2.12) and (i), $h^\frac{1}{2}_1$ is uniquely determined by

$$\int_0^\infty e^{-\lambda^{\frac{1}{2}} \tau} h^\frac{1}{2}_1 (\tau) d\tau = e^{-\lambda}, \quad \lambda \in [0, \infty).$$

Next, show that

$$\int_0^\infty \tau^{-\frac{1}{4}} e^{-(\frac{a^2}{\pi \tau} + b^2 \tau)} d\tau = \frac{\pi \frac{1}{2} e^{-2ab}}{b} \quad \text{and} \quad \int_0^\infty \tau^{-\frac{3}{4}} e^{-(\frac{a^2}{\pi \tau} + b^2 \tau)} d\tau = \frac{\pi \frac{1}{2} e^{-2ab}}{a}$$

for all $(a, b) \in (0, \infty)^2$, and conclude from the second of these that

$$h^\frac{1}{2}_1 (\tau) = \frac{1_{(0,\infty)}(\tau) e^{-\frac{1}{\tau}}}{\sqrt{4\pi \tau^2}}.$$  

**Hint:** To prove the first identity, try the change of variables $x = a\tau^{-\frac{1}{2}} - b\tau^{\frac{1}{2}}$, and get the second by differentiating the first with respect to $a$.  

III Infinitely Divisible Laws
In this exercise we will discuss the densities of the symmetric stable laws \( \mu_\alpha^t \) for \( \alpha \in (0, 2) \) (cf. Exercise 3.3.13). Once again, we know that each \( \mu_\alpha^t \) admits a smooth density with respect to Lebesgue measure \( \lambda_\mathbb{R}^N \) on \( \mathbb{R}^N \). Further, it is clear that this density is symmetric and that

\[
\frac{d\mu_\alpha^t}{d\lambda_\mathbb{R}^N}(x) = t^{-\frac{\alpha}{2}} \frac{d\mu_\alpha}{d\lambda_\mathbb{R}^N}(t^{-\frac{\alpha}{2}}x) \quad \text{for } t \in (0, \infty).
\]

(i) Referring to 3.3.14 and using Exercises 3.3.12, show that

\[
(3.3.18) \quad \frac{d\mu_\alpha}{d\lambda_\mathbb{R}^N}(x) = \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^\infty \tau^{-\frac{N}{2}} e^{\frac{-|x|^2}{4\tau}} h_\tau^2(\tau) \, d\tau.
\]

(ii) Because we have an explicit expression for \( h_1^2 \), we can use (3.3.18) to get an explicit expression for \( \frac{d\mu_1}{d\lambda_\mathbb{R}^N} \). In fact, show that

\[
(3.3.19) \quad \frac{d\mu_1}{d\lambda_\mathbb{R}^N}(x) = \pi_1^\mathbb{R}^N(x) \equiv \frac{2t^N}{\omega_N(t^2 + |x|^2)^{\frac{N+1}{2}}}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,
\]

where \( \omega_N = 2\pi^{\frac{N+1}{2}} \Gamma\left( \frac{N+1}{2} \right)^{-1} \) is the surface area of \( S^N \) in \( \mathbb{R}^{N+1} \). The function \( \pi_1^\mathbb{R}^N \) is the density for what probabilists call the Cauchy distribution. For general \( N \)'s, \( (t, x) \in (0, \infty) \times \mathbb{R}^N \mapsto \pi_1^\mathbb{R}^N(x) \) is what analysts call the Poisson kernel for the right half space in \( \mathbb{R}^{N+1} \). That is (cf. Exercise 10.2.22), if \( f \in C_b(\mathbb{R}^N; \mathbb{R}) \), then

\[
(t, x) \mapsto u_f(t, x) = \int_{\mathbb{R}^N} f(x - y) \pi_1^\mathbb{R}^N(y) \, dy
\]

is the unique, bounded harmonic extension of \( f \) to the right half space.

(iii) Given \( \alpha \in (0, 2) \), show that

\[
\|f\|_\alpha^2 = \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{-|y - x|^{2\alpha}} f(x) \overline{f(y)} \, dx \, dy = \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \mu_\alpha^0(\xi) \, d\xi
\]

for \( f \in L^1(\mathbb{R}^N; \mathbb{C}) \). This can be used to prove that \( \| \cdot \|_\alpha \) determines a Hilbert norm on \( C_c(\mathbb{R}^N; \mathbb{C}) \).