Chapter III

Infinite Divisible Laws

The results in this chapter are an attempt to answer the following question. Given an $\mathbb{R}^N$-valued random variable $\mathbf{Y}$ with the property that, for each $n \in \mathbb{Z}^+$, $\mathbf{Y} = \sum_{m=1}^n \mathbf{X}_m$ where $\mathbf{X}_1, \dots, \mathbf{X}_n$ are independent and identically distributed, what can one say about the distribution of $\mathbf{Y}$?

After recalling the relationship between addition of independent random variables and convolution of their distributions, one can phrase the same question more analytically as that of describing those probability measures μ which, for each $n \geq 1$, can be written as the n-fold convolution power $\mu_{\frac{1}{n}}^{\star n}$ of some probability measure $\mu_{\frac{1}{n}}$. We will say that such a μ is **infinitely divisible** and will use $\mathcal{I}(\mathbb{R}^N)$ to denote the class of infinitely divisible measures on $\mathbb{R}^N$.

Since the Fourier transform takes convolution into ordinary multiplication, the Fourier formulation of this problem is that of describing $\hat{\mu}$ when μ is a probability measure with the property that, for each $n \in \mathbb{Z}^+$ there is an nth root of $\hat{\mu}$ which is again the Fourier transform of a probability measure.

Not surprisingly, this latter formulation is, in many ways, the most amenable to analysis, and it is the way in which we will solve it in this chapter. On the other hand, this formulation has the disadvantage that, although it yields a quite satisfactory description of $\hat{\mu}$, it leaves the problem of extracting information about μ from properties of $\hat{\mu}$. For this reason, the following chapter is devoted to developing a probabilistic understanding of the answer obtained in this one.

Throughout, $\mathbf{M}_1(\mathbb{R}^N)$ will denote the space of Borel probability measures on $\mathbb{R}^N$ and $\mathcal{I}(\mathbb{R}^N)$ will be the space of $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ which are infinitely divisible.

In view of the comments made above, it is reasonable to begin by writing down a family of probability measures which are obviously infinitely divisible, and then start taking limits of these. The measures which first come to mind are the Gaussian measures (cf. (2.3.6)) $\gamma_{\mathbf{m},\mathbf{C}}$. Indeed, if $\mathbf{m} \in \mathbb{R}^N$ and $\mathbf{C}$ is a symmetric, non-negative definite transformation on $\mathbb{R}^N$, then it is clear from (2.3.7) that $\gamma_{\mathbf{m},\mathbf{C}} = \gamma_{\frac{\mathbf{m}}{n},\frac{\mathbf{C}}{n}}^{\star n}$. Unfortunately, this is not a good choice because it is too rigid: limits of Gaussians are again Gaussian. To be more precise, say that the sequence $\{\mu_n : n \geq 1\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$ **converges weakly** to $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ and write $\mu_n \Longrightarrow \mu$ when $\langle \varphi, \mu_n \rangle \longrightarrow \langle \varphi, \mu \rangle$ for all $\varphi \in C_{\mathrm{b}}(\mathbb{R}^N; \mathbb{C})$, and apply Lemma 2.3.3 to check that $\mu_n \Longrightarrow \mu$ if and only if $\widehat{\mu_n}(\boldsymbol{\xi}) \longrightarrow \hat{\mu}(\boldsymbol{\xi})$ for every $\boldsymbol{\xi} \in \mathbb{R}^N$. Now suppose that $\gamma_{\mathbf{m}_n,\mathbf{C}_n} \Longrightarrow \mu$, and, using this Fourier criterion, conclude that $\mu = \gamma_{\mathbf{m},\mathbf{C}}$ where $\mathbf{m} = \lim_{n\to\infty} \mathbf{m}_n$ and $\mathbf{C} = \lim_{n\to\infty} \mathbf{C}_n$. In other words, one cannot use weak convergence to escape the class of Gaussian measures.

A more fruitful choice is to start with the Poisson measures. Recall that if ν is a probability measure on $\mathbb{R}^N$ and $\alpha \in [0,\infty)$, then the **Poisson measure** with jump distribution ν and jumping rate α (see § 3.2 for an explanation of this terminology) is the measure

$$\pi_{\alpha,\nu} = e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^n}{n!} \nu^{\star n}.$$

To see that $\pi_{\alpha,\nu}$ is infinitely divisible, note that

$$\widehat{\pi_{\alpha,\nu}}(\boldsymbol{\xi}) = \exp\left(\alpha \int \left(e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\right) \nu(d\mathbf{y})\right),$$

and therefore that $\pi_{\alpha,\nu} = \pi^{\star n}_{\frac{\alpha}{n},\nu}$. To see why the Poisson measures provide a more hopeful choice of starting point, let $\mathbf{m} \in \mathbb{R}^N$ and a non-negative definite, symmetric $\mathbf{C}$ be given, and choose $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ to be an orthonormal basis of eigenvectors for $\mathbf{C}$. Next, set $m_i = (\mathbf{m}, \mathbf{e}_i)_{\mathbb{R}^N}$ and $\sigma_i = \sqrt{(\mathbf{e}_i, \mathbf{C}\mathbf{e}_i)_{\mathbb{R}^N}}$, and take

$$\nu_n = \frac{1}{2N}\left(\sum_{i=1}^N \delta_{\frac{m_i \mathbf{e}_i}{n}} + \frac{1}{2}\sum_{i=1}^N \left(\delta_{\frac{\sigma_i \mathbf{e}_i}{\sqrt{n}}} + \delta_{-\frac{\sigma_i \mathbf{e}_i}{\sqrt{n}}}\right)\right).$$

Then the Fourier transform of π_{2Nn,ν_n} is

$$\exp\left(\sum_{i=1}^N n\left(e^{\sqrt{-1} m_i (\boldsymbol{\xi},\mathbf{e}_i)_{\mathbb{R}^N}} - 1\right) + \sum_{i=1}^N n\left(\cos\frac{\sigma_i(\boldsymbol{\xi},\mathbf{e}_i)_{\mathbb{R}^N}}{n^{\frac{1}{2}}} - 1\right)\right),$$

which tends to $\widehat{\gamma_{\mathbf{m},\mathbf{C}}}(\boldsymbol{\xi})$ as $n \to \infty$, and so $\pi_{2Nn,\nu_n} \Longrightarrow \gamma_{\mathbf{m},\mathbf{C}}$ as $n \to \infty$. Thus, one can use weak convergence to break out to the class of Poisson measures.

As we will show below, the preceding is a special case of a result (cf. Theorem 3.2.7) which says that every infinitely divisible measure is the limit of Poisson measures. However, before proving that result, it will be convenient to alter our description of Poisson measures. For one thing, it should be clear that, without loss in generality, we may always assume that the jump distribution ν assigns no mass to $\mathbf{0}$. If $\nu(\{0\}) = 1$, then $\pi_{\alpha,\nu} = \delta_{\mathbf{0}} = \pi_{0,\nu'}$ no matter how α and ν' are chosen. If $\beta = \nu(\{0\}) \in (0,1)$, then $\pi_{\alpha,\nu} = \pi_{\alpha',\nu'}$ where $\alpha' = \alpha(1-\beta)$ and $\nu' = (1-\beta)^{-1}\nu$. In addition, although the segregation of the rate and jumping distribution provides probabilistic insight, there is no essential reason for doing so. Thus, nothing is lost if one replaces $\pi_{\alpha,\nu}$ by π_M, where M is the finite measure $\alpha\mu$, in which case

$$\widehat{\pi_M}(\boldsymbol{\xi}) = \exp\left(\int \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\right) M(d\mathbf{y})\right).$$

With these considerations in mind, let $\mathfrak{M}_0(\mathbb{R}^N)$ be the space of non-negative, finite Borel measures M on $\mathbb{R}^N$ with $M(\{\mathbf{0}\}) = 0$, and set $\mathcal{P}(\mathbb{R}^N) = \{\pi_M : M \in \mathfrak{M}_0(\mathbb{R}^N)\}$, the space of Poisson measures on $\mathbb{R}^N$.

3.1 Convergence of Measures on $\mathbb{R}^N$

In order to carry out our program, we will need two important facts about the convergence of probability measures on $\mathbb{R}^N$. The first of these is a minor modification of the classical Helly–Bray Theorem, and the second is an improvement, due to Lévy, of Lemma 2.3.3.

3.1.1. Lévy's Continuity Theorem. Given a subset of S is $\mathbf{M}_1(\mathbb{R}^N)$, we will say the S is **sequentially relatively compact** if for every sequence $\{\mu_n : n \geq 1\} \subseteq S$ there a subsequence $\{\mu_{n_m} : m \geq 1\}$ and a $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ such that $\mu_{n_m} \Longrightarrow \mu$.

Theorem 3.1.1. *A subset S of $\mathbf{M}_1(\mathbb{R}^N)$ is sequentially relatively compact if and only if*

$$\lim_{R\to\infty} \sup_{\mu\in S} \mu\big(B(0,R)\complement\big) = 0. \tag{3.1.2}$$

Proof: We begin by pointing out that there is a countable set $\{\varphi_k : k \in \mathbb{Z}^+\} \subseteq C_c(\mathbb{R}^N;\mathbb{R})$ which is linear independent and whose span is dense in $C_c(\mathbb{R}^N;\mathbb{R})$ with respect to uniform convergence. To see this, choose $\eta \in C_c\big(\mathbb{R}^N;[0,1]\big)$ so that $\eta = 1$ on $\overline{B(\mathbf{0},1)}$ and 0 off $B(0,2)$, and set $\eta_R(\mathbf{y}) = \eta(R^{-1}\mathbf{y})$ for $R > 0$. Next, for each $\ell \in \mathbb{Z}^+$, apply the Stone–Weierstrass Theorem to choose a countable dense subset $\{\psi_{j,\ell} : j \in \mathbb{Z}^+\}$ of $C\big(\overline{B(\mathbf{0},2\ell)};\mathbb{R}\big)$, and set $\varphi_{j,\ell} = \eta_\ell\psi_{j,\ell}$. Clearly $\{\varphi_{j,\ell} : (j,\ell) \in (\mathbb{Z}^+)^2\}$ is dense in $C_c(\mathbb{R}^N;\mathbb{R})$. Finally, using lexicographic ordering of $(\mathbb{Z}^+)^2$, extract a linearly independent subset $\{\varphi_k : k \in \mathbb{Z}^+\}$ by taking $\varphi_k = \varphi_{j_k,\ell_k}$ where $(j_1,\ell_1) = (1,1)$ and (j_{k+1},ℓ_{k+1}) is the first (j,ℓ) such that $\varphi_{j,\ell}$ is linearly independent of $\{\varphi_1,\dots,\varphi_k\}$.

Now suppose that (3.1.2) holds for S. Given a sequence $\{\mu_n : n \geq 1\} \subseteq S$, we can use a diagonalization procedure to find a subsequence $\{\mu_{n_m} : m \geq 1\}$ such that $a_k = \lim_{m\to\infty}\langle\varphi_k,\mu_{n_m}\rangle$ exists for each $k \in \mathbb{Z}^+$. Next, define the linear functional Λ on the span of $\{\varphi_k : k \in \mathbb{Z}^+\}$ so that $\Lambda(\varphi_k) = a_k$. Notice that if $\varphi = \sum_{k=1}^K \alpha_k\varphi_k$, then

$$\big|\Lambda(\varphi)\big| = \lim_{m\to\infty}\left|\sum_{k=1}^K \alpha_k\langle\varphi_k,\mu_{n_m}\rangle\right| = \lim_{m\to\infty}\big|\langle\varphi,\mu_{n_m}\rangle\big| \leq \|\varphi\|_{\mathrm{u}}$$

and similarly that $\Lambda(\varphi) = \lim_{m\to\infty}\langle\varphi,\mu_{n_m}\rangle \geq 0$ if $\varphi \geq 0$. Hence, Λ admits a unique extension as a non-negativity preserving linear functional on $C_c(\mathbb{R}^N;\mathbb{R})$ which satisfies $|\Lambda(\varphi)| \leq \|\varphi\|_{\mathrm{u}}$ for all $\varphi \in C_c(\mathbb{R}^N;\mathbb{R})$.

For each $\ell \in \mathbb{Z}^+$, apply the Riesz Representation Theorem to produce a non-negative, Borel measure μ_ℓ supported on $\overline{B(\mathbf{0},\ell)}$ so that $\langle\varphi,\mu_\ell\rangle = \Lambda(\varphi)$ for

$\varphi \in C_c(\mathbb{R}^N;\mathbb{R})$ which vanish off of $B(\mathbf{0},\ell)$. Since $\langle\varphi,\mu_{\ell+1}\rangle = \Lambda(\varphi) = \langle\varphi,\mu_\ell\rangle$ whenever φ vanishes off of $B(\mathbf{0},\ell)$, it is clear that

$$\mu_{\ell+1}(\Gamma) \geq \mu_{\ell+1}\big(\Gamma\cap B(\mathbf{0},\ell)\big) = \mu_\ell(\Gamma) \quad \text{for all } \Gamma\in\mathcal{B}_{\mathbb{R}^N}.$$

Hence, if

$$\mu(\Gamma) \equiv \lim_{\ell\to\infty}\mu_\ell(\Gamma) = \mu\big(\Gamma\cap\{\mathbf{0}\}\big) + \sum_{\ell=1}^{\infty}\mu_\ell\Big(\Gamma\cap\big(B(\mathbf{0},\ell)\setminus\overline{B(\mathbf{0},\ell-1)}\big)\Big),$$

then μ is a non-negative, Borel measure on $\mathbb{R}^N$ whose restriction to $B(\mathbf{0},\ell)$ is μ_ℓ for each $\ell\in\mathbb{Z}^+$. In particular, $\mu(\mathbb{R}^N)\leq 1$ and $\langle\varphi,\mu\rangle = \lim_{m\to\infty}\langle\varphi,\mu_{n_m}\rangle$ for every $\varphi\in C_c(\mathbb{R}^N;\mathbb{R})$. Thus, by Lemma 2.1.7, all that remains is to check that $\mu(\mathbb{R}^N) = 1$. But

$$\begin{aligned}\mu(\mathbb{R}^N) \geq \langle\eta_\ell,\mu\rangle = \lim_{m\to\infty}\langle\eta_\ell,\mu_{n_m}\rangle &\geq \varlimsup_{m\to\infty}\mu_{n_m}\big(\overline{B(\mathbf{0},\ell)}\big)\\ &= 1 - \varliminf_{m\to\infty}\mu_{m_m}\big(B(\mathbf{0},\ell)\complement\big),\end{aligned}$$

and, by (3.1.2), the final term tends to 0 as $\ell\to\infty$.

To prove the converse assertion, suppose that S is sequentially relatively compact. If (3.1.2) failed, then we could find an $\theta\in(0,1)$ and, for each $n\in\mathbb{Z}^+$, a $\mu_n\in S$ such that $\mu_n\big(B(\mathbf{0},n)\big)\leq\theta$. By sequential relative compactness, this would mean that there is a subsequence $\{\mu_{n_m}: m\geq 1\}\subseteq S$ and a $\mu\in\mathbf{M}_1(\mathbb{R}^N)$ such that $\mu_{n_m}\implies\mu$ and $\mu_{n_m}\big(B(\mathbf{0},n_m)\big)\leq\theta$. On the other hand, for any $R>0$,

$$\mu\big(B(\mathbf{0},R)\big) \leq \langle\eta_R,\mu\rangle \leq \varlimsup_{m\to\infty}\mu_{n_m}\big(B(\mathbf{0},n_m)\big)\leq\theta,$$

and so we would arrive at the contradiction $1 = \lim_{R\to\infty}\mu\big(B(\mathbf{0},R)\big)\leq\theta$. □

Our next goal is to find out how to find a test in terms of the Fourier transform to determine when (3.1.2) holds.

Lemma 3.1.3. *Define*

$$s(r) = \inf_{\theta\geq r}\left(1-\frac{\sin\theta}{\theta}\right) \quad \text{for } r\in(0,\infty).$$

Then s is a strictly positive, non-decreasing, continuous function which tends to 0 as $r\searrow 0$. Moreover, if $\mu\in\mathbf{M}_1(\mathbb{R}^N)$, then, for all $(r,R)\in(0,\infty)^2$,

$$\big|1-\hat{\mu}(r\mathbf{e})\big| \leq rR + \mu\big(\{\mathbf{y}: |(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\geq R\}\big) \quad \text{for all } \mathbf{e}\in\mathbb{S}^{N-1}, \tag{3.1.4}$$

and

$$\begin{aligned}\mu\big(B(\mathbf{0},N^{\frac{1}{2}}R)\complement\big) &\leq N\sup_{\mathbf{e}\in\mathbb{S}^{N-1}}\mu\big(\{\mathbf{y}: |(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\geq R\}\big)\\ &\leq \frac{N}{s(rR)}\max\big\{\big|1-\hat{\mu}(\boldsymbol{\xi})\big| : |\xi|\leq r\big\}.\end{aligned} \tag{3.1.5}$$

In particular, for any $S \subseteq \mathbf{M}_1(\mathbb{R}^N)$, (3.1.2) holds if and only if

$$\lim_{|\boldsymbol{\xi}|\searrow 0} \sup_{\mu\in S} \big|1-\hat{\mu}(\boldsymbol{\xi})\big| = 0. \tag{3.1.6}$$

Proof: Given (3.1.4) and (3.1.5), the final assertion is obvious. To prove (3.1.4), simply observe that $\big|1-e^{\sqrt{-1}(r\mathbf{e},\mathbf{y})_{\mathbb{R}^N}}\big| \le 1 \wedge \big(r|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\big)$.

Turning to (3.1.5), note that

$$\big|1-\hat{\mu}(\boldsymbol{\xi})\big| \ge \int_{\mathbb{R}^N} \Big(1-\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\Big)\,\mu(d\mathbf{y}).$$

Thus, for each $\mathbf{e}\in\mathbb{S}^{N-1}$,

$$\begin{aligned}\frac{1}{r}\int_0^r \big|1-\hat{\mu}(t\mathbf{e})\big|\,dt &\ge \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} \left(1-\frac{\sin\big(r(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}\big)}{r(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}}\right)\mu(d\mathbf{y})\\ &\ge s(rR)\mu\big(\{\mathbf{y}:\,|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\ge R\}\big),\end{aligned}$$

and therefore

$$\sup_{\boldsymbol{\xi}\in\overline{B(\mathbf{0},r)}} \big|1-\hat{\mu}(\boldsymbol{\xi})\big| \ge s(rR)\mu\big(\{\mathbf{y}:\,|(\mathbf{e},\mathbf{y})_{\mathbb{R}^N}|\ge R\}\big). \tag{3.1.7}$$

Since the first inequality in (3.1.5) is obvious, there is nothing more to be done. □

We are now ready to prove Lévy's crucial improvement to Lemma 2.3.3.

Theorem 3.1.8 (**Lévy's Continuity Theorem**). *Let $\{\mu_n : n\ge 1\}\subseteq \mathbf{M}_1(\mathbb{R}^N)$, and assume that $f(\boldsymbol{\xi}) = \lim_{n\to\infty}\hat{\mu}_n(\boldsymbol{\xi})$ exists for each $\boldsymbol{\xi}\in\mathbb{R}^N$. Then there is a $\mu\in\mathbf{M}_1(\mathbb{R}^N)$ such that $f=\hat{\mu}$ if and only if there is a $\delta>0$ for which $\lim_{n\to\infty}\sup_{|\boldsymbol{\xi}|\le\delta}\big|\hat{\mu}_n(\boldsymbol{\xi})-f(\boldsymbol{\xi})\big| = 0$, in which case $\mu_n \Longrightarrow \mu$.*

Proof: The only assertion not already covered by Lemmas 2.1.7 and 2.3.3 is the "if" part of the equivalence. But, if $\hat{\mu}_n \longrightarrow f$ uniformly in a neighborhood of $\mathbf{0}$, then it is easy to check that $\sup_{n\ge 1}|1-\hat{\mu}_n(\boldsymbol{\xi})|$ must tend to zero as $|\boldsymbol{\xi}|\to 0$. Hence, by the last part of Lemma 3.1.1, we know that there exists a μ and a subsequence $\{\mu_{n_m} : m\ge 1\}$ such that $\mu_{n_m}\Longrightarrow\mu$. Since $\hat{\mu}$ must equal f, we are done. □

Exercises for § 3.1

Exercise 3.1.9. One might think that to address the sort of problem posed at the beginning of this chapter, it would be helpful to know which functions $f:\mathbb{R}^N\longrightarrow\mathbb{C}$ are the Fourier transforms of a probability measure. Such a characterization is the content of **Bochner's Theorem**, whose proof we will outline in this exercise. Unfortunately, his characterization looks more useful than it is in practice. For instance, we will not use it to solve our problem, and it is difficult to see how its use would simplify matters.

In order to state Bochner's Theorem, say that a function $f : \mathbb{R}^N \longrightarrow \mathbb{C}$ is **non-negative definite** if, for each $n \geq 1$ and $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n \in \mathbb{R}^N$, the matrix $\big(\big(f(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)\big)\big)_{1\leq i,j\leq n}$ is Hermitian and non-negative definite. Equivalently,

$$\sum_{i,j=1}^{n} f(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)\zeta_i\bar{\zeta}_j \geq 0 \quad \text{for all } \zeta_1, \dots, \zeta_n \in \mathbb{C}.$$

Then Bochner's Theorem is the statement that $f = \hat{\mu}$ *for some* $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ *if and only if* $f(\mathbf{0}) = 1$ *and* f *is a continuous, non-negative definite function.*

(**i**) It is ironic that the necessity assertion is the more useful even though it is nearly trivial. Indeed, if $f = \hat{\mu}$, then it is obvious that $f(\mathbf{0}) = 1$ and that f is continuous. To see that it is also non-negative definite, write

$$\sum_{i,j=1}^{n} e^{\sqrt{-1}(\boldsymbol{\xi}_j - \boldsymbol{\xi}_i, \mathbf{x})_{\mathbb{R}^N}} \zeta_i\bar{\zeta}_i = \left|\sum_{i=1}^{n} e^{\sqrt{-1}(\boldsymbol{\xi}_i, \mathbf{x})_{\mathbb{R}^N}} \zeta_i\right|^2,$$

and integrate in $\mathbf{x}$ with respect to μ.

(**ii**) The first step in proving the sufficiency is to use the non-negative definiteness assumption to show that $f(-\mathbf{x}) = \overline{f(\mathbf{x})}$ and $\big|f(\mathbf{x})\big| \leq f(\mathbf{0})$ for all $\mathbf{x} \in \mathbb{R}^N$. In particular, this proves that $\|f\|_{\mathrm{u}} \leq 1$. Second, using an obvious Riemann approximation procedure and the continuity of f, check that for any rapidly decreasing, continuous $\hat{\psi} : \mathbb{R}^N \longrightarrow \mathbb{C}$,

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} f(\mathbf{x} - \boldsymbol{\eta})\hat{\psi}(\mathbf{x})\overline{\hat{\psi}(\boldsymbol{\eta})}\, d\mathbf{x}\, d\boldsymbol{\eta} \geq 0.$$

In particular, when $f \in L^1\big(\mathbb{R}^N; \mathbb{C}\big)$, set

$$m(\mathbf{x}) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-\sqrt{-1}\,(\mathbf{x}, \boldsymbol{\xi})_{\mathbb{R}^N}} f(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

and use Parseval's Identity and Fubini's Theorem, together with elementary manipulations, to arrive at

$$(2\pi)^N \int_{\mathbb{R}^N} m(\mathbf{x})\, \psi(\mathbf{x})^2\, d\mathbf{x} = \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} f(\boldsymbol{\xi} - \boldsymbol{\eta})\hat{\psi}(\boldsymbol{\xi})\overline{\hat{\psi}(\boldsymbol{\eta})}\, d\boldsymbol{\xi}\, d\boldsymbol{\eta} \geq 0$$

for all $\psi \in \mathscr{S}(\mathbb{R}^N; \mathbb{R})$. Conclude that m is nonnegative, and use this to complete the proof in the case when $f \in L^1\big(\mathbb{R}^N; \mathbb{C}\big)$.

(**iii**) It remains only to pass from the case when $f \in L^1\big(\mathbb{R}^N;\mathbb{C}\big)$ to the general case. For each $t \in (0,\infty)$, set $f_t(\mathbf{x}) = e^{-t\frac{|\mathbf{x}|^2}{2}} f(\mathbf{x})$ for $t > 0$. Clearly, $f_t(\mathbf{0}) = 1$ and $f_t \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C}) \cap L^1(\mathbb{R}^N;\mathbb{C})$. In addition, show that

$$\sum_{i,j=1}^{n} f_t\big(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\big)\zeta_i\bar{\zeta}_j = \int_{\mathbb{R}^N} \left(\sum_{i,j=1}^{n} f\big(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\big)\zeta_i(\mathbf{x})\bar{\zeta}_j(\mathbf{x})\right) \gamma_{\mathbf{0},t\mathbf{I}}(d\mathbf{x}) \geq 0,$$

where $\zeta_i(\mathbf{x}) \equiv \zeta_i e^{\sqrt{-1}\,(\boldsymbol{\xi}_i,\mathbf{x})_{\mathbb{R}^N}}$. Hence, f_t is also nonnegative definite; and so, by part (**ii**), we know that$f_t = \widehat{\mu_t}$ for some $\mu_t \in \mathbf{M}_1(\mathbb{R}^N)$. Finally, apply Lévy's Continuity Theorem to see that $\mu_t \Longrightarrow \mu$, where $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ satisfies $f = \hat{\mu}$.

(**iv**) Suppose that f is a non-negative definite function with $f(\mathbf{0}) = 1$. As we have just seen, if f is continuous, then $f = \hat{\mu}$ for some $\mu \in \mathbf{M}_1(\mathbb{R}^N)$. Using this representation, show that

$$(*) \qquad \|f\|_{\mathrm{u}} \leq 1 \quad \text{and} \quad |f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})|^2 \leq 2\big[1 - \mathfrak{Re}\big(f(\boldsymbol{\eta} - \boldsymbol{\xi})\big)\big], \quad \boldsymbol{\xi},\, bh \in \mathbb{R}^N.$$

Next, show that (*) follows directly from non-negative definiteness, whether or not f is continuous. Thus, a non-negative definite function is uniformly continuous everwhere if it is continuous at the origin.

Hint: Both parts of (*) follow from the fact that

$$A = \begin{pmatrix} 1 & \overline{f(\boldsymbol{\xi})} & \overline{f(\boldsymbol{\eta})} \\ f(\boldsymbol{\xi}) & 1 & f(\boldsymbol{\xi} - \boldsymbol{\eta}) \\ f(\boldsymbol{\eta}) & \overline{f(\boldsymbol{\xi} - \boldsymbol{\eta})} & 1 \end{pmatrix}$$

is non-negative definite. To get the second part, consider the quadratic form $\big(\mathbf{v}, A\mathbf{v}\big)_{\mathbb{C}^3}$ with $\mathbf{v} = (v_1, 1, -1)$.[*]

(**v**) To understand the how essential role of continuity in Bochner's criterion, show that $f = \mathbf{1}_{\{\mathbf{0}\}}$ is non-negative definite. Even though this f cannot be the Fourier transform of any $\mu \in \mathbf{M}_1(\mathbb{R}^N)$, it is nonetheless the "Fourier transform" of a non-negativity preserving linear functional, one for which there is no Riesz representation. To be more precise, consider the linear functional Λ on the space of functions $\varphi \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ for which

$$\Lambda\varphi \equiv \lim_{R\to\infty} \frac{1}{|B(\mathbf{0},R)|} \int_{B(\mathbf{0},R)} \varphi(\mathbf{x})\, d\mathbf{x} \quad \text{exists,}$$

and show that $f(\boldsymbol{\xi}) = \Lambda\big(e_{\boldsymbol{\xi}}\big)\big)$, where $e_{\boldsymbol{\xi}}(\mathbf{x}) = e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}$.

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[*] This was made to me by Linan Chen.

Exercise 3.1.10. It is important to recognize that the extent to which L'evy's Continuity Theorem and, as a by product, Bochner's Theorem, are strictly finite dimensional results. For example, let H be an infinite dimensional, separable, real Hilbert space, and define $f(h)=e^{\frac{1}{2}\|h\|_H^2}$. Obviously, f is a continuous and $f(0)=1$. Show that it is also non-negative in the sense that $\big(\big(f(h_i-f_j)\big)\big)_{1\le i,h\le n}$ is a non-negative definite, Hermitian matrix for each $n\in\mathbb{Z}^+$ and $h_1,\dots,h_n\in H$. Now suppose that there were a Borel probability measure μ on H such that

$$\hat\mu(h)\equiv\int_H e^{\sqrt{-1}(h,x)_H}\,\mu(dx)=f(h),\quad h\in H.$$

Show that for any orthonormal basis $\{e_i : i\in\mathbb{Z}^+\}$ in H, the functions $X_i(h)=(e_i,h)_H$, $i\in\mathbb{Z}^+$ would be, under μ, a sequence of independent, $\mathcal{N}(0,1)$-random variables, and conclude from this that

$$\int_H e^{-\|h\|_H^2}\,\mu(dh)=\prod_{i\in\mathbb{Z}^+}\mathbb{E}^\mu\big[e^{-X_i^2}\big]=0.$$

Hence, no such μ can exist.

Hint: The non-negative definiteness of f can be seen as a consequence of the analogous result for $\mathbb{R}^n$.

§ 3.2 The Lévy–Khinchine Formula

In this section we will solve the problem posed in the introduction to this chapter.

§ 3.2.1. $\mathcal{I}(\mathbb{R}^N)$ is the Closure of $\mathcal{P}(\mathbb{R}^N)$. Let $\overline{\mathcal{P}(\mathbb{R}^N)}$ be the closure of $\mathcal{P}(\mathbb{R}^N)$ under weak convergence. That is, $\mu\in\overline{\mathcal{P}(\mathbb{R}^N)}$ if and only if there exists a sequence $\{M_n : n\ge 1\}\subseteq\mathfrak{M}_0(\mathbb{R}^N)$ such that $\pi_{M_n}\Longrightarrow\mu$. Our goal here is to prove that

$$\mathcal{I}(\mathbb{R}^N)=\overline{\mathcal{P}(\mathbb{R}^N)}. \tag{3.2.1}$$

Before turning to the proof of (3.2.1), we need the following simple lemma about non-vanishing, $\mathbb{C}$-valued functions. In its statement, and elsewhere,

$$\log\zeta=-\sum_{m=1}^\infty\frac{(1-\zeta)^m}{m}\quad\text{for }\zeta\in\mathbb{C}\text{ with }|1-\zeta|<1 \tag{3.2.2}$$

is the principle branch of logarithm function on the open unit disk around 1 in the complex plane.

Lemma 3.2.3. *Let $R\in(0,\infty)$ be given. If $f\in C\big(\overline{B(\mathbf{0},R)};\mathbb{C}\setminus\{\mathbf{0}\}\big)$ with $f(\mathbf{0})=1$, then there is a unique $\ell=\ell_f\in C\big(\overline{B(\mathbf{0};R)};\mathbb{C}\big)$ such that $\ell(\mathbf{0})=0$*

and $f = e^{\ell}$. *Moreover, if* $\boldsymbol{\xi} \in \overline{B(\mathbf{0};R)}$, $r \in (0,\infty)$, *and* $\left|1 - \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}\right| < 1$ *for all* $\boldsymbol{\eta} \in \overline{B(\boldsymbol{\xi},r) \cap B(\mathbf{0},R)}$, *then, for each* $\boldsymbol{\eta} \in \overline{B(\boldsymbol{\xi},r) \cap B(\mathbf{0},R)}$,

$$\ell_f(\boldsymbol{\eta}) - \ell_f(\boldsymbol{\xi}) = \log \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}$$

and therefore

$$|\ell_f(\boldsymbol{\eta}) - \ell_f(\boldsymbol{\xi})| \le 2\left|1 - \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}\right| \quad \textit{if} \ \left|1 - \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}\right| \le \frac{1}{2}.$$

Finally, if $\tilde{f}$ *is a second element of* $C\big(\overline{B(\mathbf{0};R)};\mathbb{C}\setminus\{0\}\big)$ *with* $\tilde{f}(\mathbf{0}) = 1$ *and if* $\left|1 - \frac{\tilde{f}(\boldsymbol{\xi})}{f(\boldsymbol{\xi})}\right| \le \frac{1}{2}$ *for all* $\boldsymbol{\xi} \in \overline{B(\mathbf{0},R)}$, *then*

$$\big|\ell_{\tilde{f}}(\boldsymbol{\xi}) - \ell_f(\boldsymbol{\xi})\big| \le 2\left|1 - \frac{\tilde{f}(\boldsymbol{\xi})}{f(\boldsymbol{\xi})}\right| \quad \textit{for } \boldsymbol{\xi} \in \overline{B(\mathbf{0},R)}.$$

In particular, if $\{f_n : n \ge 1\} \subseteq C\big(\overline{B(\mathbf{0},R)};\mathbb{C}\setminus\{0\}\big)$ *with* $f_n(\mathbf{0}) = 1$ *for all* $n \ge 1$, *and if* $f_n \longrightarrow f \in C\big(\overline{B(\mathbf{0};R)};\mathbb{C}\setminus\{0\}\big)$ *uniformly on* $\overline{B(\mathbf{0},R)}$, *then* $f(0) = 1$ *and* $\ell_{f_n} \longrightarrow \ell_f$ *uniformly on* $\overline{B(\mathbf{0};R)}$.

PROOF: To prove the existence and uniqueness of ℓ_f, begin by observing that there exists an $M \in \mathbb{Z}^+$ and $0 = r_0 < r_1 < \cdots < r_M = R$ such that

$$\left|1 - \frac{f(\boldsymbol{\xi})}{f\left(\frac{r_{m-1}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}\right| \le \frac{1}{2} \quad \text{for } 1 \le m \le M \text{ and } \boldsymbol{\xi} \in \overline{B(\mathbf{0},r_m)} \setminus \overline{B(\mathbf{0},r_{m-1})}.$$

Thus, we can define a function ℓ_f on $\overline{B(\mathbf{0},R)}$ so that $\ell_f(\mathbf{0}) = 0$ and

$$\ell_f(\boldsymbol{\xi}) = \ell_f\left(\frac{r_{m-1}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) + \log \frac{f(\boldsymbol{\xi})}{f\left(\frac{r_{m-1}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}$$
$$\text{if } 1 \le m \le M \text{ and } \boldsymbol{\xi} \in \overline{B(\mathbf{0},r_m)} \setminus \overline{B(\mathbf{0},r_{m-1})}.$$

Furthermore, working by induction of $1 \le m \le M$, one sees that this ℓ_f is continuous and satisfies $f = e^{\ell_f}$. Finally, for any $\ell \in C\big(\overline{B(\mathbf{0},R)};\mathbb{C}\big)$ satisfying $\ell(\mathbf{0}) = 0$ and $f = e^{\ell}$, $(\sqrt{-1}2\pi)^{-1}(\ell - \ell_f)$ is a continuous, $\mathbb{Z}$-valued function which vanishes at $\mathbf{0}$, and therefore $\ell = \ell_f$.

Next suppose that $\boldsymbol{\xi} \in B(\mathbf{0},R)$ and that

$$\left|1 - \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}\right| < 1 \quad \text{for all } \boldsymbol{\eta} \in \overline{B(\boldsymbol{\xi},r) \cap B(\mathbf{0},R)}.$$

Set

$$\ell(\boldsymbol{\eta}) = \ell_f(\boldsymbol{\xi}) + \log \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})} \quad \text{for } \boldsymbol{\eta} \in B(\boldsymbol{\xi}, r) \cap B(\mathbf{0}, R),$$

and check that $\boldsymbol{\eta} \rightsquigarrow (\sqrt{-1}2\pi)^{-1}\big(\ell(\boldsymbol{\eta}) - \ell_f(\boldsymbol{\eta})\big)$ is a continuous, $\mathbb{Z}$-valued function which vanishes at $\boldsymbol{\xi}$. Hence, $\ell_f(\boldsymbol{\eta}) - \ell(\boldsymbol{\xi}) = \log \frac{f(\boldsymbol{\eta})}{f(\boldsymbol{\xi})}$ for $\boldsymbol{\eta} \in B(\mathbf{0}, R) \cap B(\boldsymbol{\xi}, r)$, and therefore on $\overline{B(\mathbf{0}, R) \cap B(\mathbf{0}, r)}$. Since $|\log \zeta_2 - \log \zeta_1| \le 2|\zeta_2 - \zeta_1|$ if $|1 - \zeta_1| \vee |1 - \zeta_1| \le \frac{1}{2}$, this completes the proof of the asserted properties of ℓ_f.

Turning to the comparison of ℓ_f and $\ell_{\tilde{f}}$ when $\left|1 - \frac{\tilde{f}(\boldsymbol{\xi})}{f(\boldsymbol{\xi})}\right| \le \frac{1}{2}$ for all $\boldsymbol{\xi} \in \overline{B(\mathbf{0}, R)}$, set $\ell(\boldsymbol{\xi}) = \ell_f(\boldsymbol{\xi}) + \log \frac{\tilde{f}(\boldsymbol{\xi})}{f(\boldsymbol{\xi})}$, check that $\ell(\mathbf{0}) = 0$ and $\tilde{f} = e^{\ell}$, and conclude that $\ell_{\tilde{f}} - \ell_f = \log \frac{\tilde{f}}{f}$. From this, the asserted estimate for $\|\ell_{\tilde{f}} - \ell_f\|_{\mathrm{u}}$ is immediate. □

Lemma 3.2.4. *Define $r \rightsquigarrow s(r)$ as in Lemma 3.1.3, and let $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ and $0 < r < R$ be given. If $|1 - \hat{\mu}(\boldsymbol{\xi})| \le \frac{1}{2}$ for all $\boldsymbol{\xi} \in \overline{B(\mathbf{0}, r)}$ and there is an $\nu \in \mathbf{M}_1(\mathbb{R}^N)$ such that $\mu = \nu^{\star n}$ for some*

$$n \ge \frac{4e^2}{s\left(\frac{r}{4R}\right)}, \tag{3.2.5}$$

then $|\hat{\mu}(\xi)| \ge 2^{-n}$ for all $\xi \in \overline{B(\mathbf{0}, R)}$.

Proof: First apply Lemma 3.2.3 to see that, because $\hat{\mu}(\boldsymbol{\xi}) = \hat{\nu}(\boldsymbol{\xi})^n$, neither $\hat{\mu}$ nor $\hat{\nu}$ vanishes on $\overline{B(\mathbf{0}, r)}$ and therefore that there are unique $\ell, \tilde{\ell} \in C\big(\overline{B(\mathbf{0}, r)}; \mathbb{C}\big)$ such that $\ell(\mathbf{0}) = 0 = \tilde{\ell}(\mathbf{0})$, $\hat{\mu} = e^{\ell}$, and $\hat{\nu} = e^{\tilde{\ell}}$ on $\overline{B(\mathbf{0}, r)}$. Further, since $\hat{\mu} = e^{n\tilde{\ell}}$, uniqueness requires that $\tilde{\ell} = \frac{1}{n}\ell$. Next, observe that $\ell = \log \hat{\mu}$ and $|1 - \hat{\mu}| \le \frac{1}{2}$ on $\overline{B(\mathbf{0}, r)}$, and therefore that $|\ell| \le 2$ there. In particular, this means that $|1 - \hat{\nu}| \le \frac{e^2}{n}$ on $\overline{B(\mathbf{0}, r)}$. Using this in (3.1.7), we have, for any $\rho > 0$ and $\mathbf{e} \in \mathbb{S}^{N-1}$, that

$$\nu\big(\{\mathbf{y} : |(\mathbf{e}, \mathbf{y})_{\mathbb{R}^N}| \ge \rho\}\big) \le \frac{1}{s(r\rho)} \max_{\boldsymbol{\xi} \in \overline{B(\mathbf{0}, r)}} \big|1 - \hat{\nu}(\boldsymbol{\xi})\big| \le \frac{e^2}{ns(r\rho)}, \tag{3.2.6}$$

which, by (3.1.4), leads to

$$\big|1 - \hat{\nu}(\boldsymbol{\xi})\big| \le \rho R + \frac{e^2}{ns(r\rho)} \quad \text{for } \boldsymbol{\xi} \in \overline{B(\mathbf{0}, R)}.$$

Finally take $\rho = \frac{1}{4R}$, and use (3.2.5) and $\hat{\mu}(\boldsymbol{\xi}) = \hat{\nu}(\boldsymbol{\xi})^n$ to check that this gives the desired conclusion. □

We now have everything that we need to prove the equality (3.2.1).

THEOREM 3.2.7. *For each* $\mu \in \mathcal{I}(\mathbb{R}^N)$ *there is a unique* $\ell_\mu \in C(\mathbb{R}^N;\mathbb{C})$ *satisfying* $\ell_\mu(\mathbf{0}) = 0$ *and* $\hat{\mu} = e^{\ell_\mu}$. *Moreover, for each* $n \in \mathbb{Z}^+$, $e^{\frac{1}{n}\ell_\mu}$ *is the Fourier transform of the unique* $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$ *such that* $\mu = \mu_{\frac{1}{n}}^{\star n}$. *In addition, if* $M_n \in \mathfrak{M}_0(\mathbb{R}^N)$ *is defined by*

$$M_n(\Gamma) \equiv n\mu_{\frac{1}{n}}\big(\Gamma \cap (\mathbb{R}^N \setminus \{\mathbf{0}\})\big) \quad \textit{for } \Gamma \in \mathcal{B}_{\mathbb{R}^N}, \tag{3.2.8}$$

then $\pi_{M_n} \Longrightarrow \mu$. *Finally,* $\mathcal{I}(\mathbb{R}^N)$ *is closed in the sense that* $\mu \in \mathcal{I}(\mathbb{R}^N)$ *if there exists a sequence* $\{\mu_k : k \ge 1\} \subseteq \mathcal{I}(\mathbb{R}^N)$ *such that* $\mu_k \Longrightarrow \mu$. *In particular,* $\mu_{\frac{1}{n}}$ *is uniquely determined and (3.2.1) holds.*

PROOF: Let $\mu \in \mathcal{I}(\mathbb{R}^N)$ be given. Since there is an $r > 0$ such that $|1-\hat{\mu}(\boldsymbol{\xi})| \le \frac{1}{2}$ for all $\boldsymbol{\xi} \in \overline{B(\mathbf{0},r)}$ and, for all $n \in \mathbb{Z}^+$, $\mu = \mu_{\frac{1}{n}}^{\star n}$ for some $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$, Lemma 3.2.4 guarantees that $\hat{\mu}$ never vanishes. Hence, by Lemma 3.2.3, both the existence and uniqueness of ℓ_μ follow. Moreover, if $\mu = \mu_{\frac{1}{n}}^{\star n}$, then, from $\hat{\mu}(\boldsymbol{\xi}) = \widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})^n$, we know first that $\widehat{\mu_{\frac{1}{n}}}$ never vanishes and that $\ell_\mu = n\ell$, where ℓ is the unique element of $C(\mathbb{R}^N;\mathbb{C})$ satisfying $\ell(\mathbf{0}) = 0$ and $\widehat{\mu_{\frac{1}{n}}} = e^{\ell}$. Now define M_n as in the statement, and observe that

$$\begin{aligned}\widehat{\pi_{M_n}}(\boldsymbol{\xi}) &= \exp\Big(n\big(\hat{\mu}_{\frac{1}{n}}(\boldsymbol{\xi}) - 1\big)\Big)\\ &= \exp\Big(n\big(e^{\frac{1}{n}\ell_\mu(\boldsymbol{\xi})} - 1\big)\Big) \longrightarrow e^{\ell_\mu(\boldsymbol{\xi})} = \hat{\mu}(\boldsymbol{\xi})\end{aligned}$$

as $n \to \infty$. Hence, $\pi_{M_n} \Longrightarrow \mu$. In particular, this proves that $\mathcal{I}(\mathbb{R}^N) \subseteq \overline{\mathcal{P}(\mathbb{R}^N)}$, and therefore, since we already know that $\mathcal{P}(\mathbb{R}^N) \subseteq \mathcal{I}(\mathbb{R}^N)$, the final statement will follow once we check that $\mathcal{I}(\mathbb{R}^N)$ is closed.

To prove that $\mathcal{I}(\mathbb{R}^N)$ is closed, suppose that $\{\mu_k : k \ge 1\} \subseteq \mathcal{I}(\mathbb{R}^N)$ and that $\mu_k \Longrightarrow \mu$. The first step in checking that $\mu \in \mathcal{I}(\mathbb{R}^N)$ is to show that $\hat{\mu}$ never vanishes. To this end, use the fact that $\hat{\mu}_k \longrightarrow \hat{\mu}$ uniformly on compacts to see that there must exist an $r > 0$ such that $|1 - \hat{\mu}_k(\boldsymbol{\xi})| \le \frac{1}{2}$ for all $k \in \mathbb{Z}^+$ and $\boldsymbol{\xi} \in \overline{B(\mathbf{0},r)}$. Hence, because each of the μ_k's is infinitely divisible, one can use Lemma 3.2.4 to see that, for each $R \in (0,\infty)$, $\inf\{|\hat{\mu}_k(\boldsymbol{\xi})| : k \in \mathbb{Z}^+ \text{ and } \boldsymbol{\xi} \in \overline{B(\mathbf{0},R)}\} > 0$, and clearly this is more than enough to show that $\hat{\mu}$ never vanishes. Thus we can choose a unique $\ell \in C(\mathbb{R}^N;\mathbb{C})$ so that $\ell(\mathbf{0}) = 0$ and $\hat{\mu} = e^{\ell}$. Moreover, if $\ell_k = \ell_{\mu_k}$, then, by Lemma 3.2.3, $\ell_k \longrightarrow \ell$ uniformly on compacts. Now let $n \in \mathbb{Z}^+$ be given, and choose $\{\mu_{k,n} : k \ge 1\} \subseteq \mathbf{M}_1(\mathbb{R}^N)$ so that $\mu_k = \mu_{k,n}^{\star n}$. Then, we know that $\widehat{\mu_{k,n}} = e^{\frac{1}{n}\ell_k}$, and so, as $k \to \infty$, $\hat{\mu}_{k,n} \longrightarrow e^{\frac{1}{n}\ell}$ uniformly on compacts. Hence, by Lévy's Continuity Theorem, $e^{\frac{1}{n}\ell} = \hat{\mu}_{\frac{1}{n}}$ for some $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$. Since this means that $\mu = \mu_{\frac{1}{n}}^{\star n}$, we have shown that $\mu \in \mathcal{I}(\mathbb{R}^N)$. $\square$

§ 3.2.2. The Formula. Theorem 3.2.7 provides interesting information, but it fails to provide a concrete characterization of the infinitely divisible laws. In this subsection we will give a complete characterization of $\hat{\mu}$ for $\mu \in \mathcal{I}(\mathbb{R}^N)$, which, in view of the first part of Theorem 3.2.7, is equivalent to characterizing of the functions in $\{\ell_\mu : \mu \in \mathcal{I}(\mathbb{R}^N)\}$.

In the following lemma, and elsewhere, $\mathscr{S}(\mathbb{R}^{\mathscr{N}};\mathbb{C})$ and $\mathscr{S}(\mathbb{R}^{\mathscr{N}};\mathbb{R})$ denote the Schwartz test function spaces of smooth, respectively, $\mathbb{C}$-valued and $\mathbb{R}$-valued functions which, together with all of their derivatives, are rapidly decreasing. Recall that the Fourier transform maps $\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ onto itself.

Lemma 3.2.9. *For each $r \in (0,\infty)$ there exists a $C(r) < \infty$ such that $|\ell_\mu(\boldsymbol{\xi})| \le C(r)(1+|\boldsymbol{\xi}|^2)$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ whenever $\mu \in \mathcal{I}(\mathbb{R}^N)$ satisfies $|1-\hat{\mu}(\boldsymbol{\xi})| \le \frac{1}{2}$ for $\boldsymbol{\xi} \in \overline{B(\mathbf{0},r)}$. Moreover, if $\mu \in \mathcal{I}(\mathbb{R}^N)$, then*

$$A_\mu\varphi \equiv \lim_{n\to\infty} n\big(\langle\varphi,\mu_{\frac{1}{n}}\rangle - \varphi(\mathbf{0})\big) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \overline{\ell_\mu(\boldsymbol{\xi})}\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi} \tag{3.2.10}$$

for each $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$.

Proof: Suppose that $\mu \in \mathcal{I}(\mathbb{R}^N)$ satisfies $|1-\hat{\mu}(\boldsymbol{\xi})| \le \frac{1}{2}$ for $\boldsymbol{\xi} \in \overline{B(\mathbf{0},r)}$. Applying the second inequality in (3.2.6) and (3.1.4) with $\nu = \mu_{\frac{1}{n}}$, we know that, for any $(\rho,R) \in (0,\infty)^2$,

$$\sup_{|\boldsymbol{\xi}|\le R} |1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})| \le \rho R + \frac{e^2}{ns(r\rho)}.$$

Hence, if $R \ge r$, then, by taking $\rho = \frac{1}{4R}$, we obtain $\sup_{|\boldsymbol{\xi}|\le R} |1-\widehat{\mu_{\frac{1}{n}}}(\boldsymbol{\xi})| \le \frac{1}{2}$ and therefore $\sup_{|\boldsymbol{\xi}|\le R} |\frac{1}{n}\ell_\mu(\boldsymbol{\xi})| \le 2$ if n satisfies (3.2.5). Finally, observe that that there is an $\epsilon > 0$ such that $s(t) \ge \epsilon t^2$ for $t \in (0,1]$, and therefore that $\ell_\mu(\boldsymbol{\xi}) \le 2\left(1+\frac{16e^2R^2}{\epsilon r^2}\right)$ for $|\boldsymbol{\xi}| \le R$, which completes the proof of the first assertion.

Now suppose that $\mu \in \mathcal{I}(\mathbb{R}^N)$ and that $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. Then, by (2.3.4),

$$\begin{aligned}(2\pi)^N n\big(\langle\varphi,\mu_{\frac{1}{n}}\rangle - \varphi(\mathbf{0})\big) &= \int_{\mathbb{R}^N} n\big(e^{\frac{1}{n}\overline{\ell_\mu(\boldsymbol{\xi})}}-1\big)\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi} \\ &= \int_0^1 \left(\int_{\mathbb{R}^N} e^{\frac{t}{n}\overline{\ell_\mu(\boldsymbol{\xi})}}\,\overline{\ell_\mu(\boldsymbol{\xi})}\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}\right)dt \longrightarrow \int_{\mathbb{R}^N}\overline{\ell_\mu(\boldsymbol{\xi})}\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi},\end{aligned}$$

where (keeping in mind that $|e^{\frac{1}{n}\ell_\mu}| = |\hat{\mu}_{\frac{1}{n}}(\boldsymbol{\xi})| \le 1$, $\ell_\mu(\boldsymbol{\xi})$ has a most quadratic growth, and $\hat{\varphi}(\boldsymbol{\xi})$ is rapidly decreasing) the passage to the second line is justified by Fubini's Theorem and the limit is an application of Lebesgue's Dominated Convergence Theorem. □

Lemma 3.2.9, especially (3.2.10), provides us with two critical pieces of information which will enable us to characterize ℓ_μ. The first of these is the **minimum principle** for the linear functional A_μ. Namely, it is clear from the definition of A_μ that when $A = A_\mu$

$$A\mathbf{1} = 0 \quad \text{and} \quad A\varphi \geq 0 \text{ if } \varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{R}) \text{ and } \varphi(\mathbf{0}) = \min_{\mathbf{x}\in\mathbb{R}^N} \varphi(\mathbf{x}). \tag{3.2.11}$$

Secondly, it tells us that if $\varphi_R(\mathbf{x}) = \varphi\left(\frac{\mathbf{x}}{R}\right)$ for $R > 0$, then

$$\lim_{R\to\infty} A\varphi_R = 0 \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C}). \tag{3.2.12}$$

To see this, note that $\widehat{\varphi_R}(\boldsymbol{\xi}) = R^N\hat{\varphi}(R\boldsymbol{\xi})$, and therefore that

$$(2\pi)^N A_\mu \varphi_R = \int_{\mathbb{R}^N} \overline{\ell_\mu\big(R^{-1}\boldsymbol{\xi}\big)}\hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \longrightarrow 0,$$

since $\ell_\mu(\mathbf{0}) = 0$ and $\sup_{R\geq 1} |\ell_\mu(\boldsymbol{\xi})||\hat{\varphi}(\boldsymbol{\xi})|$ is rapidly decreasing. It turns out that these two properties allow us to say a great deal about A_μ.

For $\alpha \in [0,\infty)$, let $\mathfrak{M}_\alpha(\mathbb{R}^N)$ be the space of non-negative, Borel measures on $\mathbb{R}^N$ satisfying

$$M(\{\mathbf{0}\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|\mathbf{y}|^\alpha}{1+|\mathbf{y}|^\alpha}\, M(d\mathbf{y}) < \infty.$$

The class $\mathfrak{M}_2(\mathbb{R}^N)$, whose elements are called **Lévy measures**, will play a particularly important role in what follows.

It should be clear that if $M \in \mathfrak{M}_\alpha(\mathbb{R}^N)$, then for every Borel measurable $\varphi : \mathbb{R}^N \longrightarrow \mathbb{C}$

$$\sup_{\mathbf{y}\in\mathbb{R}^N\setminus\{\mathbf{0}\}} \frac{|\varphi(\mathbf{y})|}{1\wedge|\mathbf{y}|^\alpha} < \infty \implies \varphi \in L^1(M;\mathbb{C}). \tag{3.2.13}$$

Using (3.2.13), one can easily check that if $\varphi \in C^2_{\mathrm{b}}(\mathbb{R}^N;\mathbb{C})$ and $\eta \in \mathscr{S}(\mathbb{R}^N;\mathbb{R})$ equals 1 in a neighborhood of $\mathbf{0}$, then

$$\mathbf{y} \rightsquigarrow \varphi(y) - \varphi(0) - \eta(\mathbf{y})\big(\mathbf{y}, \nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}$$

is M-integrable for every $M \in \mathfrak{M}_2(\mathbb{R}^N)$.

Finally, given $n \in \mathbb{Z}^+$ and $\varphi \in C^n(\mathbb{R}^N;\mathbb{C})$, define $\nabla^n\varphi(\mathbf{x})$ to be the multilinear map on $(\mathbb{R}^N)^n$ into $\mathbb{C}$ by

$$\big[\nabla^n\varphi(x)\big](\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_n) = \frac{\partial^n}{\partial_{t_1}\cdots\partial t_n}\varphi\left(\mathbf{x} + \sum_{m=1}^n t_m\boldsymbol{\xi}_m\right)\bigg|_{t_1=\cdots=t_n=0}.$$

Obviously, $\nabla\varphi$ and $\nabla^2\varphi$ can be identified as the gradient of φ and Hessian of φ.

Lemma 3.2.14. *Let* $\mathbf{D}$ *be the space of functions* $\varphi\in C^\infty(\mathbb{R}^N;\mathbb{R})$ *such that* $\varphi-\varphi(\infty)\mathbf{1}\in\mathscr{S}(\mathbb{R}^N;\mathbb{R})$ *for some* $\varphi(\infty)\in\mathbb{R}$. *If* $A:\mathbf{D}\longrightarrow\mathbb{R}$ *is a linear functional on* $\mathbf{D}$ *which satisfies* (*3.2.11*) *and* (*3.2.12*), *then there is a unique* $M\in\mathfrak{M}_2(\mathbb{R}^N)$ *such that* $A\varphi=\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,M(d\mathbf{y})$ *for* $\varphi\in\mathscr{S}(\mathbb{R}^N;\mathbb{R})$ *which satisfy* $\varphi(\mathbf{0})=0$, $\nabla\varphi(\mathbf{0})=\mathbf{0}$, *and* $\nabla^2\varphi(\mathbf{0})=\mathbf{0}$. *Next, given* $\eta\in C^\infty_{\mathrm{c}}\big(\mathbb{R}^N;[0,1]\big)$ *satisfying* $\eta=1$ *in a neighborhood of* $\mathbf{0}$, *set* $\eta_{\boldsymbol{\xi}}(\mathbf{y})=(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\eta(\mathbf{y})$ *for* $\boldsymbol{\xi}\in\mathbb{R}^N$, *and define* $\mathbf{m}^\eta\in\mathbb{R}^N$ *and* $\mathbf{C}\in\mathrm{Hom}(\mathbb{R}^N;\mathbb{R}^N)$ *by*

$$(\boldsymbol{\xi},\mathbf{m}^\eta)=A\eta_{\boldsymbol{\xi}}\quad\textit{and}\quad\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}'\big)_{\mathbb{R}^N}=A\big(\eta_{\boldsymbol{\xi}}\eta_{\boldsymbol{\xi}'}\big)-\int_{\mathbb{R}^N}(\eta_{\boldsymbol{\xi}}\eta_{\boldsymbol{\xi}'})(\mathbf{y})\,M(d\mathbf{y}).\tag{3.2.15}$$

Then $\mathbf{C}$ *is symmetric, non-negative definite, and independent of the choice of* η. *Finally, for any* $\varphi\in\mathbf{D}$,

$$\begin{aligned}A\varphi&=\tfrac12\mathrm{Trace}\big(\mathbf{C}\nabla^2\varphi(\mathbf{0})\big)+\big(\mathbf{m}^\eta,\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\\&\quad+\int_{\mathbb{R}^N}\Big(\varphi(\mathbf{y})-\varphi(\mathbf{0})-\eta(\mathbf{y})\big(\mathbf{y},\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\Big)\,M(d\mathbf{y}).\end{aligned}\tag{3.2.16}$$

Proof: Choose $\psi\in C^\infty\big(\mathbb{R}^N;[0,1]\big)$ so that ψ has compact support in $B(\mathbf{0},2)\setminus\overline{B\big(\mathbf{0},\frac14\big)}$ and $\psi(\mathbf{y})=1$ when $\frac12\le|\mathbf{y}|\le1$, and set $\psi_m(\mathbf{y})=\psi(2^m\mathbf{y})$ for $m\in\mathbb{Z}$. Then, if $\mathbf{y}\in\mathbb{R}^N\setminus\{\mathbf{0}\}$ and $2^{-m-1}\le|\mathbf{y}|\le2^{-m}$, $\psi_m(\mathbf{y})=1$ and $\psi_n(\mathbf{y})=0$ unless $-m-2\le n\le-m+1$. Hence, if $\Psi(\mathbf{y})=\sum_{m\in\mathbb{Z}}\psi_m(\mathbf{y})$ for $\mathbf{y}\in\mathbb{R}^N\setminus\{\mathbf{0}\}$, then Ψ is a smooth function with values in $[1,4]$; and therefore, for each $m\in\mathbb{Z}$, the function χ_m given by $\chi_m(\mathbf{0})=0$ and $\chi_m(\mathbf{y})=\frac{\psi_m(\mathbf{y})}{\Psi(\mathbf{y})}$ for $\mathbf{y}\in\mathbb{R}^N\setminus\{\mathbf{0}\}$ is a smooth, $[0,1]$-valued function which vanishes off of $B(\mathbf{0},2^{-m+1})\setminus\overline{B(\mathbf{0},2^{-m-2})}$. In addition, $\sum_{m\in\mathbb{Z}}\chi_m(\mathbf{y})=1$, and, for each $\mathbf{y}\in\mathbb{R}^N\setminus\{0\}$, $\chi_n(\mathbf{y})=0$ unless $2^{-n-2}\le|\mathbf{y}|\le2^{-n+1}$

Next, define $\Lambda_m\varphi=A(\chi_m\varphi)$ for $\varphi\in C^\infty\big(\overline{B(\mathbf{0},2^{-m+1})}\setminus B(\mathbf{0},2^{-m-2};\mathbb{R}\big)$, where

$$\chi_m\varphi(\mathbf{y})=\begin{cases}\chi_m(\mathbf{y})\varphi(\mathbf{y}) & \text{if } 2^{-m-2}\le|\mathbf{y}|\le2^{-m+1}\\ 0 & \text{otherwise.}\end{cases}$$

Clearly Λ_m is linear. In addition, if $\varphi\ge0$, then $\chi_m\varphi\ge0=\chi_m\varphi(\mathbf{0})$, and so, by (3.2.11), $\Lambda_m\varphi\ge0$. Similarly, for any $\varphi\in C^\infty\big(\overline{B(\mathbf{0},2^{-m+1})}\setminus B(\mathbf{0},2^{-m-2});\mathbb{R}\big)$, $\|\varphi\|_{\mathrm{u}}\chi_m\pm\chi_m\varphi\ge0=\big(\|\varphi\|_{\mathrm{u}}\chi_m\pm\chi_m\varphi\big)(\mathbf{0})$, and therefore $|\Lambda_m\varphi|\le K_m\|\varphi\|_{\mathrm{u}}$, where $K_m=A\chi_m$. Hence, Λ_m admits a unique extension as a continuous linear functional on $C\big(\overline{B(\mathbf{0},2^{-m+1})}\setminus B(\mathbf{0},2^{-m-2});\mathbb{R}\big)$ which is non-negativity preserving and has norm K_m; and so, by the Riesz Representation Theorem, we now know that there is a unique non-negative Borel measure M_m on $\mathbb{R}^N$ such that M_m is supported on $\overline{B(\mathbf{0},2^{-m+1})}\setminus B(\mathbf{0},2^{-m-2})$, $K_m=M_m(\mathbb{R}^N)$, and $A(\chi_m\varphi)=\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,M_m(d\mathbf{y})$ for all $\varphi\in\mathscr{S}(\mathbb{R}^N;\mathbb{R})$.

Now define the non-negative, Borel measure M on $\mathbb{R}^N$ by $M=\sum_{m\in\mathbb{Z}}M_m$. Clearly, $M(\{\mathbf{0}\})=0$. In addition, if $\varphi\in C^\infty_{\mathrm{c}}\big(\mathbb{R}^N\setminus\{\mathbf{0}\};\mathbb{R}\big)$, then there is an

$n \in \mathbb{Z}^+$ such that $\chi_m\varphi \equiv 0$ unless $|m| \le n$. Thus,

$$\begin{aligned} A\varphi = \sum_{m=-n}^{n} A(\chi_m\varphi) &= \sum_{m=-n}^{n} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M_m(d\mathbf{y}) \\ &= \int_{\mathbb{R}^N} \left(\sum_{m=-n}^{n} \chi_m(\mathbf{y})\varphi(\mathbf{y}) \right) M(d\mathbf{y}) = \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}) \end{aligned}$$

and therefore

$$A\varphi = \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}) \tag{3.2.17}$$

for $\varphi \in C_c^\infty\big(\mathbb{R}^N \setminus \{\mathbf{0}\};\mathbb{R}\big)$.

Before taking the next step, observe that, as an application of (3.2.11), if $\varphi_1, \varphi_2 \in \mathbf{D}$, then

$$\varphi_1 \le \varphi_2 \text{ and } \varphi_1(\mathbf{0}) = \varphi_2(\mathbf{0}) \implies A\varphi_1 \le A\varphi_2. \tag{*}$$

Indeed, by linearity, this reduce to checking that if $\varphi \in \mathbf{D}$ is non-negative and $\varphi(\mathbf{0}) = 0$, then $A\varphi \ge 0$. But, for such a φ, $\big(\varphi - \varphi(\infty)\mathbf{1}\big)$ is an element of $\mathscr{S}(\mathbb{R}^N;\mathbb{R})$ which achieves its minimum value at $\mathbf{0}$, and therefore $A\varphi = \varphi(\infty)A\mathbf{1} + A\big(\varphi - \varphi(\infty)\mathbf{1}\big) \ge 0$.

With these preparations, we can show that, for any $\varphi \in \mathbf{D}$,

$$\varphi \ge 0 = \varphi(\mathbf{0}) \implies \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}) \le A\varphi. \tag{**}$$

To check this, apply (*) to $\varphi_n = \sum_{m=-n}^{n} \chi_m\varphi$ and φ, and use (3.2.17) together with the Monotone Converence Theorem to conclude that

$$\int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}) = \lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi_n(\mathbf{y})\, M(d\mathbf{y}) = \lim_{n\to\infty} A\varphi_n \le A\varphi.$$

Now let η be as in the statement, and set $\eta_R(\mathbf{y}) = \eta(R^{-1}\mathbf{y})$ for $R > 0$. By (**) with $\varphi(\mathbf{y}) = |\mathbf{y}|^2\eta(\mathbf{y})$ we know that

$$\int_{\mathbb{R}^N} |\mathbf{y}|^2\eta(\mathbf{y})\, M(d\mathbf{y}) \le A\varphi < \infty.$$

At the same time, by (3.2.17) and (*),

$$\int_{\mathbb{R}^N} \big(1 - \eta(\mathbf{y})\big)\eta_R(\mathbf{y})\, M(d\mathbf{y}) \le A(\mathbf{1} - \eta)$$

for all $R > 0$, and therefore, by Fatou's Lemma,

$$\int_{\mathbb{R}^N} \bigl(1 - \eta(\mathbf{y})\bigr)\, M(d\mathbf{y}) \le A(\mathbf{1} - \eta) < \infty.$$

Hence, we have proved that $M \in \mathfrak{M}_2(\mathbb{R}^N)$.

We are now in a position to show that (3.2.17) continuous to hold for any $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{R})$ which vanishes along with its first and second order derivatives at $\mathbf{0}$. To this end, first suppose that φ vanishes in a neighborhood of $\mathbf{0}$. Then, for each $R > 0$, (3.2.17) applies to $\eta_R\varphi$, and so

$$\int_{\mathbb{R}^N} \eta_R(\mathbf{y})\varphi(\mathbf{y})\, M(d\mathbf{y}) = A(\eta_R\varphi) = A\varphi + A\bigl((\mathbf{1} - \eta_R)\varphi\bigr).$$

By (*) applied to $\pm(\mathbf{1} - \eta_R)\varphi$ and $(\mathbf{1} - \eta_R)\|\varphi\|_{\mathrm{u}}$,

$$\bigl|A\bigl((\mathbf{1} - \eta_R)\varphi\bigr)\bigr| \le \|\varphi\|_{\mathrm{u}} A(\mathbf{1} - \eta_R) = -\|\varphi\|_{\mathrm{u}} A\eta_R \longrightarrow 0 \quad \text{as } R \to \infty,$$

where, we used first (3.2.11) and then (3.2.12). Thus,

$$A\varphi = \lim_{R\to\infty} \int_{\mathbb{R}^N} \eta_R(\mathbf{y})\varphi(\mathbf{y})\, M(d\mathbf{y}) = \int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}),$$

because, since M is finite on the support of φ and therefore φ is M-integrable, Lebesgue's Dominated Convergence Theorem applies. We still have to replace the assumption that φ vanishes in a neighborhood of $\mathbf{0}$ by the original assumption. For this purpose, first note that, by (3.2.13), φ is certainly M-integrable, and therefore

$$\int_{\mathbb{R}^N} \varphi(\mathbf{y})\, M(d\mathbf{y}) = \lim_{R\searrow 0} A\bigl((\mathbf{1} - \eta_R)\varphi\bigr) = A\varphi - \lim_{R\searrow 0} A(\eta_R\varphi).$$

By our assumptions about φ at $\mathbf{0}$, we can find a $C < \infty$ such that $|\eta_R\varphi(\mathbf{y})| \le CR|\mathbf{y}|^2\eta(\mathbf{y})$ for all sufficiently small $R > 0$. Hence, by (*), $|A(\eta_R\varphi)| \le C'R$ for small $R > 0$, and therefore $A(\eta_R\varphi) \longrightarrow 0$ as $R \searrow 0$.

To complete the proof from here, let $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{R})$ be given, and set

$$\tilde\varphi(\mathbf{x}) = \varphi(\mathbf{x}) - \varphi(\mathbf{0}) - \eta(\mathbf{x})\bigl(\mathbf{y}, \nabla\varphi(\mathbf{0})\bigr)_{\mathbb{R}^N} - \tfrac{1}{2}\eta(\mathbf{x})^2\bigl(\mathbf{x}, \nabla^2\varphi(\mathbf{0})\mathbf{x}\bigr)_{\mathbb{R}^N}.$$

Then (3.2.17) holds for $\tilde\varphi$ and, after one re-arranges terms, says that (3.2.16) holds. Thus, all the remains is to prove the properties of $\mathbf{C}$. That $\mathbf{C}$ is symmetric requires no comment. In addition, from (*), it is clearly non-negative definite. Finally, to see that it is independent of the η chosen, let η' be a second choice, note that $\eta'_{\boldsymbol{\xi}} - \eta_{\boldsymbol{\xi}}$ vanishes in a neighborhood of $\mathbf{0}$, and apply (3.2.17). $\square$

Let $\mu \in \mathcal{I}(\mathbb{R}^N)$, and define A_μ accordingly, as in (3.2.10). By applying Lemma 3.2.14 to A_μ, we see that there is a unique $M_\mu \in \mathfrak{M}_2(\mathbb{R}^N)$, symmetric, non-negative definite $\mathbf{C}_\mu \in \operatorname{Hom}(\mathbb{R}^N;\mathbb{R}^N)$, and $\mathbf{m}_\mu^\eta \in \mathbb{R}^N$ such that (3.2.16) holds with $A = A_\mu$. In order to use this to compute ℓ_μ, we need to show that, for any $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$,

$$\begin{aligned}(2\pi)^N \operatorname{Trace}\big(\mathbf{C}_\mu \nabla^2\varphi(\mathbf{0})\big) &= -\int_{\mathbb{R}^N} \big(\boldsymbol{\xi}, \mathbf{C}_\mu\boldsymbol{\xi}\big)_{\mathbb{R}^N}\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi},\\ (2\pi)^N\big(\mathbf{m}_\mu^\eta, \nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N} &= -\sqrt{-1}\int_{\mathbb{R}^N}\big(\mathbf{m}_\mu^\eta,\boldsymbol{\xi}\big)_{\mathbb{R}^N}\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi},\end{aligned}$$

and

$$\begin{aligned}&(2\pi)^N\int_{\mathbb{R}^N}\Big(\varphi(\mathbf{y}) - \varphi(\mathbf{0}) - \eta(\mathbf{y})\big(\mathbf{y},\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\Big)\,M_\mu(d\mathbf{y})\\ &\qquad = \int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N}\Big(e^{-\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big)\,M_\mu(d\mathbf{y})\bigg)\,\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}.\end{aligned}$$

The first two of these are completely familiar computations based on Parseval's identity and the relationship between the Fourier transform and derivatives. To prove the third, we must learn more about the inner integral on the right hand side. Specifically, it will be useful to know that

$$\lim_{|\boldsymbol{\xi}|\to\infty}|\boldsymbol{\xi}|^{-2}\int_{\mathbb{R}^N}\Big|e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big|\,M(d\mathbf{y}) = 0 \tag{3.2.18}$$

for any $M \in \mathfrak{M}_2(\mathbb{R}^N)$. To see this, choose $R \in (0,\infty)$ so that $\eta = 1$ on $\overline{B(\mathbf{0},R)}$. Then, for $r \in (0,R]$,

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big|e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big|\,M(d\mathbf{y})\\ &\qquad \le \frac{|\boldsymbol{\xi}|^2}{2}\int_{B(\mathbf{0},r)}|\mathbf{y}|^2\,M(d\mathbf{y}) + C(1+|\boldsymbol{\xi}|)M\big(B(\mathbf{0},r)\complement\big)\end{aligned}$$

for some $C < \infty$ depending only on η. Hence, the limit in (3.2.18) is dominated by $\frac{1}{2}\int_{B(\mathbf{0},r)}|\mathbf{y}|^2\,M(d\mathbf{y})$ for every $r \in (0,R]$, and this quantity tends to 0 as $r \searrow 0$. Once one has (3.2.18), one knows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big|e^{-\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big|\,|\hat{\varphi}(\boldsymbol{\xi})|\,M_\mu(d\mathbf{y})\,d\boldsymbol{\xi} < \infty,$$

and therefore, by Fubini's Theorem, that

$$\begin{aligned}&\int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N}\Big(e^{-\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big)\,M_\mu(d\mathbf{y})\bigg)\,\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}\\ &\qquad = \int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N}\Big(e^{-\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big)\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}\bigg)\,M_\mu(d\mathbf{y}).\end{aligned}$$

Finally, by elementary Fourier analysis

$$(2\pi)^N \int_{\mathbb{R}^N} \Big(e^{-\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 + \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\Big)\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$= \varphi(\mathbf{y}) - \varphi(\mathbf{0}) - \eta(\mathbf{y})\big(\mathbf{y},\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}$$

for each $\mathbf{y}\in\mathbb{R}^N$.

We are now ready to prove the result toward which we have been working. In the following, $\mathcal{L}(\mathbb{R}^N)$ is the set of **Lévy systems** $(\mathbf{m},\mathbf{C},M)$, where, $\mathbf{m}\in\mathbb{R}^N$, $\mathbf{C}$ is a symmetric, non-negative definite transformation on $\mathbb{R}^N$, and $M\in\mathfrak{M}_2(\mathbb{R}^N)$ is a Lévy measure. Given a Lévy system $(\mathbf{m},\mathbf{C},M)$ and a bounded, Borel measurable $\eta:\mathbb{R}^N\longrightarrow\mathbb{R}$ satisfying

$$\left(\sup_{\mathbf{y}\in B(\mathbf{0},1)\setminus\{\mathbf{0}\}} |\mathbf{y}|^{-2}|1-\eta(\mathbf{y})|\right)\vee\left(\sup_{\mathbf{y}\notin B(\mathbf{0},1)}|\mathbf{y}||\eta(\mathbf{y})|\right)<\infty, \tag{3.2.19}$$

define

$$\begin{aligned}\ell^{\eta}_{(\mathbf{m},\mathbf{C},M)}(\boldsymbol{\xi}) =& \sqrt{-1}\big(\mathbf{m}^{\eta},\boldsymbol{\xi}\big)_{\mathbb{R}^N} - \tfrac{1}{2}\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N}\\ &+\int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}\,(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big)\,M(d\mathbf{y}).\end{aligned} \tag{3.2.20}$$

THEOREM 3.2.21 (**Lévy–Khinchine).** *For each $\mu\in\mathcal{I}(\mathbb{R}^N)$, there is a unique $\ell_\mu\in C(\mathbb{R}^N;\mathbb{C})$ such that $\ell_\mu(\mathbf{0})=0$ and $\hat{\mu}=e^{\ell_\mu}$, and, for each $n\in\mathbb{Z}^+$, $e^{\frac{1}{n}\ell_\mu}$ is the Fourier transform of the unique $\mu_{\frac{1}{n}}\in\mathbf{M}_1(\mathbb{R}^N)$ satisfying $\mu=\mu_{\frac{1}{n}}^{\star n}$. Next, let be a $\eta:\mathbb{R}^N\longrightarrow\mathbb{R}$ be a bounded, measurable function which satisfies (3.2.19). Then, for each $\mu\in\mathcal{I}(\mathbb{R}^N)$, there is a unique $(\mathbf{m}^{\eta}_{\mu},\mathbf{C}_\mu,M_\mu)\in\mathcal{L}(\mathbb{R}^N)$ such that $\ell_\mu=\ell^{\eta}_{(\mathbf{m}^{\eta}_{\mu},\mathbf{C}_\mu,M_\mu)}$, and, for each $(\mathbf{m},\mathbf{C},M)\in\mathcal{L}(\mathbb{R}^N)$, there is a unique $\mu\in\mathcal{I}(\mathbb{R}^N)$ such that $\ell_\mu=\ell^{\eta}_{(\mathbf{m},\mathbf{C},M)}$. In fact, if $\eta_0\in C^\infty_{\mathrm{c}}\big(\mathbb{R}^N;[0,1]\big)$ satisfies $\eta_0=1$ in a neighborhood of $\mathbf{0}$ and $\mu\in\mathcal{I}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,M_\mu(d\mathbf{y}) = \lim_{n\to\infty} n\langle\varphi,\mu_{\frac{1}{n}}\rangle$$

for all $\varphi\in\mathscr{S}(\mathbb{R}^N;\mathbb{C})$ which satisfy $\lim_{|\mathbf{y}|\searrow 0}|\mathbf{y}|^{-2}|\varphi(y)|=0$,

$$\mathbf{C}_\mu = \lim_{n\to\infty} n\int_{\mathbb{R}^N}\eta_0(\mathbf{y})^2\mathbf{y}\otimes\mathbf{y}\,\mu_{\frac{1}{n}}(d\mathbf{y}) - \int_{\mathbb{R}^N}\eta_0(\mathbf{y})^2\mathbf{y}\otimes\mathbf{y}\,M_\mu(d\mathbf{y}),$$

and, for any Borel measurable η satisfying (3.2.19),

$$\mathbf{m}^{\eta}_{\mu} = \mathbf{m}^{\eta_0}_{\mu} + \int_{\mathbb{R}^N}\big(\eta(\mathbf{y})-\eta_0(\mathbf{y})\big)\,M_\mu(d\mathbf{y})$$

where $\mathbf{m}^{\eta_0}_{\mu} = \lim_{n\to\infty} n\int_{\mathbb{R}^N}\eta_0(\mathbf{y})\mathbf{y}\,\mu_{\frac{1}{n}}(d\mathbf{y})$.

PROOF: We have already proved the initial assertion. To prove the second assertion, we begin by showing that $\ell^{\eta}_{(\mathbf{m},\mathbf{C},M)}$ is continuous for all measurable η satisfying (3.2.19) and $(\mathbf{m},\mathbf{C},M) \in \mathcal{L}(\mathbb{R}^N)$. Since, for each $r \in (0,\infty)$, it is obvious that $\ell^{\eta}_{(\mathbf{m},\mathbf{C},M^r)} \in C(\mathbb{R}^N;\mathbb{C})$ where $M^r(\Gamma) = M\big(\Gamma \cap B(\mathbf{0},r)\complement\big)$, it suffices to check that

$$\lim_{r\searrow 0}\int_{B(\mathbf{0},r)}\Big|e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \sqrt{-1}\eta(\mathbf{y})\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\Big|\,M(d\mathbf{y}) = 0$$

uniformly for $\boldsymbol{\xi}$ in compacts. But the integrand in the above is dominated by a constant times $|\boldsymbol{\xi}|^2|\mathbf{y}|^2$, and so there is no problem.

Now let η_0 be given. For each $\mu \in \mathcal{I}(\mathbb{R}^N)$, we know from Lemma 3.2.14 that the quantities $\mathbf{m}^{\eta_0}_{\mu}$, $\mathbf{C}_\mu$, and M_μ described in the final part of the statement exist, that $(\mathbf{m}^{\eta_0}_{\mu},\mathbf{C}_\mu,M_\mu)$ is a Lévy system, and that

$$\begin{aligned}\lim_{n\to\infty} n\big(\langle\varphi,\mu_{\frac{1}{n}}\rangle - \varphi(\mathbf{0})\big) =& \big(\mathbf{m}^{\eta_0}_{\mu},\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N} + \tfrac{1}{2}\mathrm{Trace}\big(\mathbf{C}_\mu\nabla^2\varphi(\mathbf{0})\big)\\ &+ \int_{\mathbb{R}^N}\Big(\varphi(\mathbf{y}) - \varphi(\mathbf{0}) - \eta_0(\mathbf{y})\big(\mathbf{y},\nabla\varphi(\mathbf{0})\big)_{\mathbb{R}^N}\Big)\,M_\mu(d\mathbf{y})\end{aligned}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$. Thus, when we combine this with (3.2.10) and the calculations preceding the statement of the present theorem, we conclude that

$$\int_{\mathbb{R}^N}\overline{\ell^{\eta_0}_{(\mathbf{m}^{\eta_0}_{\mu},\mathbf{C}_\mu,M_\mu)}(\boldsymbol{\xi})}\,\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int_{\mathbb{R}^N}\overline{\ell_\mu(\boldsymbol{\xi})}\,\hat{\varphi}(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$

for every $\varphi \in \mathscr{S}(\mathbb{R}^N;\mathbb{C})$; and from this, together with the continuity of both $\ell^{\eta_0}_{(\mathbf{m}^{\eta_0}_{\mu},\mathbf{C}_\mu,M)}$ and ℓ_μ, it is clear that $\ell^{\eta_0}_{(\mathbf{m}^{\eta_0}_{\mu},\mathbf{C}_\mu,M)} = \ell_\mu$.

Conversely, given $(\mathbf{m},\mathbf{C},M) \in \mathcal{L}(\mathbb{R}^N)$, we must check that $e^{\ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M)}}$ is the Fourier transform of a $\mu \in \mathcal{I}(\mathbb{R}^N)$. But we have already shown this when $M \in \mathfrak{M}_0(\mathbb{R}^N)$, and so, if M^r is as above, then we know that, for each $r > 0$, $e^{\ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M^r)}} = \widehat{\mu^r}$ for some $\mu^r \in \mathcal{I}(\mathbb{R}^N)$. Furthermore, by the argument given to prove the continuity of $\ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M)}$, we know that $\ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M^r)} \longrightarrow \ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M)}$ uniformly on compacts. Hence, by Lévy's Continuity Theorem, $\mu^r \Longrightarrow \mu$ and $\hat{\mu} = e^{\ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M)}}$ as $r \searrow 0$. Thus, because $\mathcal{I}(\mathbb{R}^N)$ is closed, $\mu \in \mathcal{I}(\mathbb{R}^N)$.

Finally, suppose that η is any other bounded, measurable function which satisfies (3.2.19). Then it is clear that

$$\ell^{\eta}_{(\mathbf{m},\mathbf{C},M)}(\boldsymbol{\xi}) = \ell^{\eta_0}_{(\mathbf{m},\mathbf{C},M)}(\boldsymbol{\xi}) + \sqrt{-1}\int_{\mathbb{R}^N}\big(\eta(\mathbf{y}) - \eta_0(\mathbf{y})\big)\big(\boldsymbol{\xi},\mathbf{y}\big)_{\mathbb{R}^N}\,M(d\mathbf{y}),$$

and so the relationship between $\mathbf{m}^{\eta}_{\mu}$ and $\mathbf{m}^{\eta_0}_{\mu}$ is obvious. □

The formula for ℓ_μ given in Theorem 3.2.21 is known as the **Lévy–Khinchine formula**.

Exercises for § 3.2

EXERCISE 3.2.22. Given $M \in \mathfrak{M}_2(\mathbb{R}^N)$ and (3.2.18), show that

$$\big(\boldsymbol{\xi}, \mathbf{C}_\mu \boldsymbol{\xi}\big)_{\mathbb{R}^N} \equiv -2 \lim_{t\to\infty} t^{-2} \ell_\mu(t\boldsymbol{\xi}) \quad \text{for all } \mu \in \mathcal{I}(\mathbb{R}^N) \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N.$$

Similarly, when $M_\mu \in \mathfrak{M}_1(\mathbb{R}^N)$, show that

$$\mathbf{m}_\mu \equiv \mathbf{m}_\mu^\eta - \int_{\mathbb{R}^N} \eta(\mathbf{y})\mathbf{y}\, M_\mu(d\mathbf{y})$$

is independent of the choice of η satisfying (3.2.19) and, for each $\boldsymbol{\xi} \in \mathbb{R}^N$,

$$\begin{gathered}\big(\boldsymbol{\xi}, \mathbf{m}_\mu\big) = -\sqrt{-1} \lim_{t\to\infty} t^{-1}\Big(\ell_\mu(t\boldsymbol{\xi}) + \tfrac{t^2}{2}\big(\boldsymbol{\xi}, \mathbf{C}_\mu\boldsymbol{\xi}\big)_{\mathbb{R}^N}\Big) \quad \text{and} \\ \ell_\mu(\boldsymbol{\xi}) = -\tfrac{1}{2}\big(\boldsymbol{\xi}, \mathbf{C}_\mu\boldsymbol{\xi}\big)_{\mathbb{R}^N} + \sqrt{-1}\big(\boldsymbol{\xi}, \mathbf{m}_\mu\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N}\Big(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\Big)\, M_\mu(d\mathbf{y}).\end{gathered}$$

Finally, if $\mu \in \mathcal{I}(\mathbb{R}^N)$ is symmetric, show that M_μ is also symmetric and that

$$\ell_\mu(\boldsymbol{\xi}) = -\frac{1}{2}\big(\boldsymbol{\xi}, \mathbf{C}_\mu\boldsymbol{\xi}\big) + \int_{\mathbb{R}^N}\Big(\cos\big(\boldsymbol{\xi}, \mathbf{y}\big)_{\mathbb{R}^N} - 1\Big)\, M_\mu(d\mathbf{y}).$$

EXERCISE 3.2.23. Given $\mu \in \mathcal{I}(\mathbb{R})$, show that $\mu\big((-\infty, 0)\big) = 0$ if and only if $C_\mu = 0$, $M_\mu \in \mathfrak{M}_1(\mathbb{R})$, $M_\mu\big((-\infty,0)\big) = 0$, and (cf. the preceding exercise) $m_\mu \ge 0$. The following are steps which you might follow.

(**i**) To prove the "if" assertion, set $M^r(dy) = \mathbf{1}_{[r,\infty)}(y)\, M_\mu(dy)$ for $r > 0$, and show that $\delta_{m_\mu} \star \pi_{M^r}\big((-\infty, 0]\big) = 0$ for all $r > 0$ and $\delta_{m_\mu} \star \pi_{M^r} \Longrightarrow \mu$ as $r \searrow 0$. Conclude from these that $\mu\big((-\infty,0)\big) = 0$.

(**ii**) Now assume that $\mu\big((-\infty,0)\big) = 0$. To see that $C_\mu = 0$, show that if $\sigma > 0$, then $\gamma_{0,\sigma^2} \star \nu\big((-\infty,0)\big) > 0$ for any $\nu \in \mathbf{M}_1(\mathbb{R})$.

(**iii**) Continuing (**ii**), show that $\mu\big((-\infty,0)\big) \ge \mu_{\frac{1}{n}}\big((-\infty,0]\big)^n$, and conclude first that $\mu_{\frac{1}{n}}\big((-\infty,0)\big) = 0$ for all $n \in \mathbb{Z}^+$ and then that

$$M_\mu\big((-\infty,0)\big) = 0 \text{ and } m_\mu^\eta \ge \int_{\mathbb{R}^N} \eta(y) y\, M_\mu(dy).$$

Finally, deduce from these that $M_\mu \in \mathfrak{M}_1(\mathbb{R})$ and that $m_\mu \ge 0$.

(**iv**) Suppose that $X \in \mathcal{N}(0,1)$, and show that the distribution of $|X|$ cannot be infinitely divisible.

EXERCISE 3.2.24. Let $\mu \in \mathbf{M}_1(\mathbb{R}^N)$, and suppose that for each $m \in \mathbb{Z}^+$ there is an $n > m$ and a $\mu_{\frac{1}{n}} \in \mathbf{M}_1(\mathbb{R}^N)$ such that $\mu = \mu_{\frac{1}{n}}^{\star n}$. Show $\mu \in \mathcal{I}(\mathbb{R}^N)$.

Hint: Check that the arguments used to prove Lemmas 3.2.4 and the final part of Theorem 3.2.7 work equally well for subsequences.

EXERCISE 3.2.25. In this exercise, we will be reconsidering the topic in Exercise 2.2.25. Given $\alpha \in (0,\infty)$, define $T_\alpha : \mathbf{M}_1(\mathbb{R}^N) \longrightarrow \mathbf{M}_1(\mathbb{R}^N)$ by the prescription

$$T_\alpha\mu(\Gamma) = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \mathbf{1}_\Gamma\left(\frac{\mathbf{x}+\mathbf{y}}{2^{\frac{1}{\alpha}}}\right)\mu(d\mathbf{x})\mu(d\mathbf{y}),$$

and let $F_\alpha(\mathbb{R}^N)$ be the set of non-trivial fixed points of T_α. That is, $F_\alpha(\mathbb{R}^N) = \bigl\{\mu \in \mathbf{M}_1(\mathbb{R}^N)\setminus\{\delta_{\mathbf{0}}\} : \mu = T_\alpha\mu\bigr\}$. We can now say much more about $F_\alpha(\mathbb{R}^N)$ than we could earlier.

(**i**) As an application of the preceding exercise, show that $F_\alpha(\mathbb{R}^N) \subseteq \mathcal{I}(\mathbb{R}^N)$, and conclude that $\mu \in F_\alpha(\mathbb{R}^N)$ if and only if $\mu \in \mathcal{I}(\mathbb{R}^N)$ and $\ell_\mu(\boldsymbol{\xi}) = 2\ell_\mu\bigl(2^{-\frac{1}{\alpha}}\boldsymbol{\xi}\bigr)$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$. Next, using this and Exercise 3.2.22, show that if $\mu \in F_\alpha(\mathbb{R}^N)$, then

$$\ell_\mu(\boldsymbol{\xi}) = 2^{-n}\ell_\mu(2^{\frac{n}{\alpha}}\boldsymbol{\xi}) = 2^{n(\frac{2}{\alpha}-1)}2^{-\frac{2n}{\alpha}}\ell_\mu(2^{\frac{n}{\alpha}}\boldsymbol{\xi}) \longrightarrow \begin{cases} 0 & \text{if } \alpha > 2 \\ -\frac{1}{2}(\boldsymbol{\xi},\mathbf{C}_\mu\boldsymbol{\xi})_{\mathbb{R}^N} & \text{if } \alpha = 2. \end{cases}$$

Thus, we have already recovered the result in Exercise 2.2.25 and extended it to $\mathbf{M}_1(\mathbb{R}^N)$.

(**ii**) Define $\hat{T}_\alpha M$ for $M \in \mathfrak{M}_2(\mathbb{R}^N)$ to be the Borel measure determined by

$$\int_{\mathbb{R}^N}\varphi(\mathbf{y})\,\hat{T}_\alpha M(d\mathbf{y}) = 2\int_{\mathbb{R}^N}\varphi(2^{-\frac{1}{\alpha}}\mathbf{y})\,M(d\mathbf{y})$$

for Borel measurable $\varphi : \mathbb{R}^N \longrightarrow [0,\infty)$. Show first that T_α maps $\mathcal{I}(\mathbb{R}^N)$ into itself and that $\hat{T}_\alpha$ maps $\mathfrak{M}_2(\mathbb{R}^N)$ into itself. Second, show that if $\mu \in \mathcal{I}(\mathbb{R}^N)$, then $M_{T_\alpha\mu} = \hat{T}_\alpha M_\mu$, $\mathbf{C}_{T_\alpha\mu} = 2^{1-\frac{2}{\alpha}}\mathbf{C}_\mu$, and

$$\mathbf{m}^\eta_{T_\alpha\mu} = 2^{1-\frac{1}{\alpha}}\mathbf{m}^\eta_\mu + \int_{\mathbb{R}^N}\bigl(\eta(\mathbf{y}) - \eta(2^{\frac{1}{\alpha}}\mathbf{y})\bigr)\mathbf{y}\,\hat{T}_\alpha M_\mu(d\mathbf{y}).$$

Conclude that if $\mu \in F_\alpha(\mathbb{R}^N)$, then $M_\mu = \hat{T}_\alpha M_\mu$, $\mathbf{C}_\mu = 2^{1-\frac{2}{\alpha}}\mathbf{C}_\mu$, and

$$(*)\qquad (1 - 2^{1-\frac{1}{\alpha}})\mathbf{m}^\eta_\mu = \int_{\mathbb{R}^N}\bigl(\eta(\mathbf{y}) - \eta(2^{\frac{1}{\alpha}}\mathbf{y})\bigr)\mathbf{y}\,M_\mu(d\mathbf{y}).$$

Further, show that for $\mu \in F_\alpha(\mathbb{R}^N)$, the equality in (*) holds for all η satisfying (3.2.19) if it does for any one of them.

(iii) Based on the results in **(i)** and **(ii)**, show that if $\alpha \in (0,2)$, then

$$\mu \in F_\alpha(\mathbb{R}^N) \iff \begin{cases} \mathbf{C}_\mu = \mathbf{0}, \quad M_\mu = \hat{T}_\alpha M_\mu, \text{and} \\ (1-2^{\frac{1}{\alpha}})\mathbf{m}_\mu^\eta = \int_{\mathbb{R}^N} \big(\eta(\mathbf{y}) - \eta(2^{1-\frac{1}{\alpha}}\mathbf{y})\mathbf{y}\, M_\mu(d\mathbf{y}). \end{cases}$$

(iv) Set

$$\beta_\alpha = \int_{\mathbb{R}} \frac{1-\cos y}{y^{1+\alpha}}\, dy \quad \text{for } \alpha \in (0,2),$$

and let $\omega_{N-1} = \frac{(2\pi)^N}{\Gamma(\frac{N}{2}-1)}$ denote the surface area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Given $\alpha \in (0,2)$, show that, for each $t > 0$,

$$\exp\left(\frac{t}{\beta_\alpha \omega_{N-1}} \int_{\mathbb{R}^N} \big(\cos(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N} - 1\big)|\mathbf{y}|^{-N-\alpha}\, d\mathbf{y}\right) = e^{-t|\boldsymbol{\xi}|^\alpha}$$

is the Fourier transform of an element of an element μ_t^α of $F_\alpha(\mathbb{R}^N)$. The measures μ_t^α, $t \in (0,\infty)$, are called the **symmetric α-stable laws**. See part **(v)** of Exercise 3.2.28 below for a computation of the numbers β_α.

(v) Show that, for any $t > 0$, $e^{-t|\boldsymbol{\xi}|^\alpha}$ is the Fourier transform of a $\mu \in \mathbf{M}_1(\mathbb{R}^N) \setminus \{\delta_{\mathbf{0}}\}$ if and only if $\alpha \in (0,2]$.

EXERCISE 3.2.26. We continue in the setting of Exercise 3.2.25.

(i) Show that, for any $M \in \mathfrak{M}_2(\mathbb{R}^N) \setminus \{0\}$ and $\alpha \in (0,2)$ satisfying $M = \hat{T}_\alpha M$, $M \in \mathfrak{M}_\beta(\mathbb{R}^N)$ all $\beta \in (\alpha, 2]$ but that $M \notin \mathfrak{M}_\alpha(\mathbb{R}^N)$.

(ii) If $\alpha \in (1,2) \setminus \{1\}$, show that $\mu \in F_\alpha(\mathbb{R}^N)$ if and only if there exists an $M \in \left(\bigcap_{\beta\in(\alpha,2]} \mathfrak{M}_\beta(\mathbb{R}^N)\right) \setminus \mathfrak{M}_\alpha(\mathbb{R}^N)$ satisfying $M = \hat{T}_\alpha M$ such that $\ell_\mu(\boldsymbol{\xi})$ equals

$$\frac{\sqrt{-1}}{1-2^{1-\frac{1}{\alpha}}} \int_{2^{-\frac{1}{\alpha}} < |\mathbf{y}| \le 1} (\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}\, M(d\mathbf{y}) + \int_{\mathbb{R}^N} \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\right) M(d\mathbf{y}).$$

Hint: Take $\eta(\mathbf{y}) = \mathbf{1}_{[0,1]}(|\mathbf{y}|)$.

(iii) Show that $\mu \in F_1(\mathbb{R})$ if and only if

$$\ell_\mu(\boldsymbol{\xi}) = \big(\mathbf{m}, \boldsymbol{\xi}\big)_{\mathbb{R}^N} + \int_{\mathbb{R}^N} \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{[0,1]}\big(|\mathbf{y}|\big)\right) M(d\mathbf{y})$$

for some $\mathbf{m} \in \mathbb{R}^N$ and $M \in \left(\bigcap_{\beta\in(1,2]} \mathfrak{M}_\beta(\mathbb{R}^N)\right) \setminus \mathfrak{M}_1(\mathbb{R}^N)$ satisfying $M = \hat{T}_\alpha M$ and

$$\int_{\frac{1}{2} < |\mathbf{y}| \le 1} \mathbf{y}\, M(d\mathbf{y}) = \mathbf{0}.$$

(iv) If $\alpha \in (0,1)$, show that $\mu \in F_\alpha(\mathbb{R}^N)$ if and only if there exists an $M \in \left(\bigcap_{\beta\in(\alpha,2]} \mathfrak{M}_\beta(\mathbb{R}^N)\right) \setminus \mathfrak{M}_\alpha(\mathbb{R}^N)$ such that

$$\ell_\mu(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} \left(e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{y})_{\mathbb{R}^N}} - 1\right) M(d\mathbf{y}).$$

Hint: Show that if $\alpha \in (0,1)$ and $M = \hat{T}_\alpha M$, then

$$\left(1 - 2^{1-\frac{1}{\alpha}}\right) \int_{\overline{B(\mathbf{0},1)}} \mathbf{y}\, M(dy) = \int_{\{2^{-\frac{1}{\alpha}} < |\mathbf{y}| \le 1\}} \mathbf{y}\, M(d\mathbf{y}).$$

(v) Given $\alpha \in (0,2)$, let $\mathbf{M}^\alpha(\mathbb{R}^N)$ be the set of finite, non-negative, Borel measures ν on $\mathbb{R}^N$ which are supported on $\overline{B(\mathbf{0},1)}\setminus\overline{B(\mathbf{0},2^{-\frac{1}{\alpha}})}$, and, for $\nu \in \mathbf{M}^\alpha(\mathbb{R}^N)$, define the Borel measure $M^{\alpha,\nu}$ by

$$M^{\alpha,\nu}(\Gamma) = \sum_{m\in\mathbb{Z}} 2^{-m} \int_{\mathbb{R}^N} \mathbf{1}_\Gamma(2^{\frac{m}{\alpha}}\mathbf{y})\, \nu(d\mathbf{y}).$$

Show that $\nu \in \mathbf{M}^\alpha(\mathbb{R}^N) \longmapsto M^{\alpha,\nu}$ is a one to one map onto the set of $M \in \mathfrak{M}_1(\mathbb{R}^N)$ such that $M = \hat{T}_\alpha M$. Thus, for each $\alpha \in (0,2]$, $F_\alpha(\mathbb{R}^N)$ contains lots of elements!

Exercise 3.2.27. Here are a few further properties of elements of $F_\alpha(\mathbb{R}^N)$.

(i) Show that there is $\mu \in F_\alpha(\mathbb{R}^N)$ such that $\mu\left(\{\mathbf{y} : (\mathbf{e},\mathbf{y})_{\mathbb{R}^N} < 0\}\right) = 0$ for some $\mathbf{e} \in \mathbb{S}^{N-1}$ if and only if $\alpha \in (0,1)$.

Hint: Reduce to the case when $N = 1$, and look at Exercise 3.2.23.

(ii) If $\mu \in F_1(\mathbb{R}^N)$, show that $\mu\left(\{\mathbf{y} : (\mathbf{e},\mathbf{y})_{\mathbb{R}^N} < 0\}\right) = \infty$ for every $\mathbf{e} \in \mathbb{S}^{N-1}$.

(iii) If $\alpha \in (1,2)$, show that for each $\epsilon > 0$ there is a $\mu \in F_\alpha(\mathbb{R})$ such that $\mu\left((-\infty,-\epsilon]\right) = 0$.

Exercise 3.2.28. Given $\alpha \in (0,1)$ and $t \in (0,\infty)$, show that there is a measure $\nu_t^\alpha \in F_\alpha(\mathbb{R})$ whose Fourier transform is given by

$$\exp\left(\frac{\alpha t}{\Gamma(1-\alpha)} \int_{(0,\infty)} \frac{e^{\sqrt{-1}\,\xi y} - 1}{y^{1+\alpha}}\, dy\right),$$

and that $\nu_t^\alpha\left((-\infty,0)\right) = 0$. A measure of this sort is called a **one-sided α-stable law**.

In the remainder of this exercise, we will develop another, more explicit, expression for $\widehat{\nu_t^\alpha}$, and clearly it suffices to deal with the case when $t = 1$.

(**i**) Set

$$f_\alpha(\zeta) = \int_{(0,\infty)} \frac{e^{\sqrt{-1}\zeta y} - 1}{y^{1+\alpha}}\, dy$$

for $\alpha \in (0,1)$ and $\zeta \in \mathbb{C}$ with $\mathfrak{Im}(\zeta) \geq 0$. Clearly f_α is analytic in the open upper half $\mathbb{C}_+ \equiv \{\zeta \in \mathbb{C} : \mathfrak{Im}(\zeta) > 0\}$ of the complex plane and continuous on $\overline{\mathbb{C}_+}$. Next, argue that $f_\alpha(1) = -c_\alpha e^{\sqrt{-1}\theta_\alpha}$ for some $c_\alpha > 0$ and $\theta_\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and conclude that $f_\alpha(\xi) = -c_\alpha e^{\sqrt{-1}\theta_\alpha}\xi^\alpha$ for $\xi \in (0,\infty)$.

(**ii**) Note that the only analytic extensions of $\xi \in (0,\infty) \longmapsto \xi^\alpha \in \mathbb{C}$ to $\overline{\mathbb{C}_+}$ are of the form

$$\zeta \in \mathbb{C}_+ \longrightarrow |\zeta|^\alpha e^{\sqrt{-1}\alpha(\arg(\zeta)+2\pi n)}$$

for some $n \in \mathbb{Z}$, and conclude that

$$f_\alpha(\zeta) = -c_\alpha e^{\sqrt{-1}\theta_\alpha}\zeta^\alpha, \quad \text{where } \zeta^\alpha \equiv |\zeta|^\alpha e^{\sqrt{-1}\alpha\arg(\zeta)}, \quad \text{for } \zeta \in \overline{\mathbb{C}_+}.$$

(**iii**) Show that

$$f_\alpha\left(\sqrt{-1}\right) = -e^{-\sqrt{-1}\frac{\alpha\pi}{2}}\left(\sqrt{-1}\right)^\alpha \frac{\Gamma(1-\alpha)}{\alpha},$$

and conclude that

$$f_\alpha(\zeta) = -\frac{\Gamma(1-\alpha)}{\alpha} e^{-\sqrt{-1}\frac{\alpha\pi}{2}}\zeta^\alpha \quad \text{for all } \zeta \in \overline{\mathbb{C}_+}.$$

In particular, this means that

$$\ell_{\nu_1^\alpha}(\xi) = -e^{-\sqrt{-1}\frac{\alpha\pi}{2}}\xi^\alpha, \quad \xi \in \mathbb{R}.$$

(**iv**) Show that

$$\int_{[0,\infty)} e^{-\eta y}\, \nu_t^\alpha(dy) = \exp\left(-t\eta^\alpha\right) \quad \text{for } t \in (0,\infty) \text{ and } \eta \in [0,\infty).$$

(**v**) Recall the numbers β_α in part (**iv**) in Exercise 3.2.25, and show that

$$\beta_\alpha = \begin{cases} \frac{2\Gamma(2-\alpha)}{\alpha|\alpha-1|}\sin\left(\frac{\pi}{2}|\alpha-1|\right) & \text{if } \alpha \in (0,2)\setminus\{1\} \\ \pi & \text{if } \alpha = 1. \end{cases} \tag{3.2.29}$$

Hint: When $\alpha \in (0,1)$, simply observe that $\beta_\alpha = -2\mathfrak{Re}\left(f_\alpha(1)\right)$. To handle $\alpha \in (1,2)$, use

$$\frac{\beta_\alpha}{2}\xi^\alpha = \int_{(0,\infty)} \frac{1-\cos(\xi y)}{y^{1+\alpha}}\, dy$$

to obtain

$$\frac{\alpha\beta_\alpha}{2} = \int_{(0,\infty)} \frac{\sin y}{y^\alpha}\, dy = -\mathfrak{Im}\left(f_{\alpha-1}(1)\right).$$

Finally, when $\alpha = 1$, note that $\beta_1 = \lim_{\alpha\searrow 1}\beta_\alpha$.

EXERCISE 3.2.30. Because their Fourier transforms are rapidly decreasing, we know that each of the measures ν_t^α admits a smooth density with respect to Lebesgue measure $\lambda_{\mathbb{R}}$ on $\mathbb{R}$. In this exercise, we examine these densities.

(**i**) For $\alpha \in (0,1)$, set

$$h_t^\alpha = \frac{d\nu_t^\alpha}{d\lambda_{\mathbb{R}}} \quad \text{for } t \in (0,\infty), \tag{3.2.31}$$

and show that

$$\int_0^\infty e^{-\lambda\tau} h_t^\alpha(\tau)\, d\tau = e^{-t\lambda^\alpha}, \quad \lambda \in [0,\infty),$$

and that

$$h_t^\alpha(\tau) \equiv t^{-\frac{1}{\alpha}} h_1^\alpha(t^{-\frac{1}{\alpha}}\tau).$$

(**ii**) Only when $\alpha = \frac{1}{2}$ is an explicit expression for h_1^α readily available. To find this expression, first note that, by the uniqueness of the Laplace transform (cf. Exercise 1.2.12) and (**i**), $h_1^{\frac{1}{2}}$ is uniquely determined by

$$\int_0^\infty e^{-\lambda^2\tau} h_1^{\frac{1}{2}}(\tau)\, d\tau = e^{-\lambda}, \quad \lambda > 0.$$

Next, show that

$$\int_0^\infty \tau^{-\frac{1}{2}} e^{-(\frac{a^2}{\tau}+b^2\tau)}\, d\tau = \frac{\pi^{\frac{1}{2}} e^{-2ab}}{b} \quad \text{and} \quad \int_0^\infty \tau^{-\frac{3}{2}} e^{-(\frac{a^2}{\tau}+b^2\tau)}\, d\tau = \frac{\pi^{\frac{1}{2}} e^{-2ab}}{a}$$

for all $(a,b) \in (0,\infty)^2$, and conclude from the second of these that

$$h_1^{\frac{1}{2}}(\tau) = \frac{\mathbf{1}_{(0,\infty)}(\tau) e^{-\frac{1}{4\tau}}}{\sqrt{4\pi}\tau^{\frac{3}{2}}}. \tag{3.2.32}$$

Hint: To prove the first identity, try the change of variables $x = a\tau^{-\frac{1}{2}} - b\tau^{\frac{1}{2}}$, and get the second by differentiating the first with respect to a.

EXERCISE 3.2.33. In this exercise we will discuss the densities of the symmetric stable laws μ_t^α for $\alpha \in (0,2)$ (cf. part (**iv**) of Exercise 3.2.25). Once again, we know that each μ_t^α admits a smooth density with respect to Lebesgue measure $\lambda_{\mathbb{R}^N}$ on $\mathbb{R}^N$. Further, it is clear that this density is symmetric and that

$$\frac{d\mu_t^\alpha}{d\lambda_{\mathbb{R}^N}}(\mathbf{x}) = t^{-\frac{1}{\alpha}} \frac{d\mu_1^\alpha}{d\lambda_{\mathbb{R}^N}}(t^{-\frac{1}{\alpha}}\mathbf{x}) \quad \text{for } t \in (0,\infty).$$

(**i**) Referring to Exercise 3.2.30, show that

$$\int_{(0,\infty)} \widehat{\gamma_{\mathbf{0},2\tau\mathbf{I}}}(\boldsymbol{\xi}) h^{\frac{\alpha}{2}}(\tau)\, d\tau = e^{-|\xi|^\alpha},$$

and therefore that

$$\frac{d\mu_1^\alpha}{d\lambda_{\mathbb{R}^N}}(\mathbf{x}) = \frac{1}{(4\pi)^{\frac{N}{2}}}\int_0^\infty \tau^{-\frac{N}{2}} e^{-\frac{|\mathbf{x}|^2}{4\tau}} h^{\frac{\alpha}{2}}(\tau)\,d\tau. \tag{3.2.34}$$

(**ii**) Because we have an explicit expression for $h_1^{\frac{1}{2}}$, we can use (3.2.34) to get an explicit expression for $\frac{d\mu_1^1}{d\lambda_{\mathbb{R}^N}}$. In fact, show that

$$\frac{d\mu_t^1}{d\lambda_{\mathbb{R}^N}}(\mathbf{x}) = \Pi_t^{\mathbb{R}^N}(\mathbf{x}) \equiv \frac{\Gamma\left(\frac{N+1}{2}\right)t^N}{\left(\pi(t^2+|\mathbf{x}|^2)\right)^{\frac{N+1}{2}}}, \quad (t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^N. \tag{3.2.35}$$

The function $\Pi_1^{\mathbb{R}}$ is the density for what probabilists call the **Cauchy distribution**. For general N's, $(t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^N \longmapsto \Pi_t^{\mathbb{R}^N}(\mathbf{x})$ is the **Poisson kernel** for the right half space in $\mathbb{R}^{N+1}$. That is, if $f \in C_{\mathrm{b}}(\mathbb{R}^N;\mathbb{R})$, then

$$(t,\mathbf{x}) \rightsquigarrow u_f(t,\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{x}-\mathbf{y})\,\Pi_t^{\mathbb{R}^N}(\mathbf{y})\,d\mathbf{y}$$

is the unique, bounded harmonic extension of f to the right half space.

(**iii**) Given $\alpha \in (0,2)$, show that

$$\|f\|_\alpha^2 \equiv \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} e^{-|\mathbf{y}-\mathbf{x}|^\alpha} f(\mathbf{x})\overline{f(\mathbf{y})}\,d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^N} |\hat{f}(\boldsymbol{\xi})|^2\,\mu_1^\alpha(d\boldsymbol{\xi})$$

for $f \in L^1(\mathbb{R}^N;\mathbb{C})$. This can be used to prove that $\|\cdot\|_\alpha$ determines a Hilbert norm on $C_{\mathrm{c}}(\mathbb{R}^N;\mathbb{C})$.

EXERCISE 3.2.36. Another famous source of infinitely divisible laws is provided by the **Gamma distributions**. Namely, consider the family $\{\mu_t : t \in (0,\infty)\} \subseteq \mathbf{M}_1(\mathbb{R})$ given by

$$\mu_t(dx) = \mathbf{1}_{(0,\infty)}(x)\frac{x^{t-1}e^{-x}}{\Gamma(t)}\,dx.$$

(**i**) Show by direct computation that

$$\mu_s \star \mu_t(dx) = \frac{B(s,t)}{\Gamma(s)\Gamma(t)}\mathbf{1}_{(0,\infty)}(x)x^{s+t-1}e^{-x}\,dx,$$

where

$$B(s,t) \equiv \int_{(0,1)} \xi^{s-1}(1-\xi)^{t-1}\,d\xi$$

is Euler's **Beta function**, and conclude that $\mu_{s+t} = \mu_s \star \mu_t$. In particular, one gets, as a dividend, the famous identity $B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$.

(**ii**) As a consequence of (**i**), we know that the μ_t's are infinitely divisible. Show that their Lévy–Khinchine representation is

$$\widehat{\mu_t}(\xi) = \exp\left[t\int_{(0,\infty)}\left(e^{\sqrt{-1}\,\xi y}-1\right)e^{-y}\,\frac{dy}{y}\right].$$