Solution Set # 8

(7.1.24) Just follow the outline.

(7.1.25) Set

$$F(\boldsymbol{\varphi},\boldsymbol{\psi})=\mathbf{1}_\Gamma\big(\boldsymbol{\varphi}(t-t\wedge\zeta(\boldsymbol{\psi}))+\boldsymbol{\psi}\big(t\wedge\zeta(\boldsymbol{\psi})\big)\big)\mathbf{1}_{[0,t]}\big(\zeta(\boldsymbol{\psi})\big).$$

Then F is $\mathcal{F}_{D(\mathbb{R}^N)}\times\mathcal{F}_\zeta$-measurable and $F\big(\delta_\zeta\boldsymbol{\psi},\boldsymbol{\psi}\big)=\mathbf{1}_\Gamma\big(\boldsymbol{\psi}(t)\big)\mathbf{1}_{[0,t]}\big(\zeta(\boldsymbol{\psi})\big)$. Hence, by Theorem 7.1.16,

$$\begin{aligned}\mathbb{Q}^\mu\big(\{\boldsymbol{\psi}:\,\boldsymbol{\psi}(t)\in\Gamma\ \&\ \zeta(\boldsymbol{\psi})\le t\}\big)&=E^{\mathbb{Q}^\mu}\big[F\big(\delta_\zeta\boldsymbol{\psi},\boldsymbol{\psi}\big)\big]\\&=\int\mathbf{1}_{[0,t]}\big(\zeta(\boldsymbol{\psi})\big)\left(\int\mathbf{1}_\Gamma\big(\boldsymbol{\varphi}(t-\zeta(\boldsymbol{\psi}))\big)\,\mathbb{Q}^\mu(d\boldsymbol{\varphi})\right)\mathbb{Q}^\mu(\boldsymbol{\psi})=\mathbb{E}^{\mathbb{Q}^\mu}\big[\mu_{t-\zeta}\big(\Gamma-\boldsymbol{\psi}(t)\big),\,\zeta\le t\big].\end{aligned}$$

(7.2.8) Begin by observing that if $F\in C^2_{\mathrm{b}}(\mathbb{R};\mathbb{R})$, then one can find a sequence $\{F_n:\,n\ge1\}\subseteq C_{\mathrm{c}}(\mathbb{R};\mathbb{R})$ such that, for each $\alpha\in\{0,1,2\}$, $\{\partial_x^\alpha F_n\,:\,n\ge1\}$ is a uniformly bounded sequence which tends uniformaly on compacts to $\partial_x^\alpha F$. Hence, it suffices to handle F's which are in $C^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{R})$.

Given $\epsilon>0$, define $\zeta_0=0$ and, for $n\ge1$, $\zeta_n=\inf\{t\ge\zeta_{n-1}\,:\,|X(t)-X(\zeta_{n-1})|\ge\epsilon\}$, $\Delta_n(t)=X(t\wedge\zeta_n)-X(t\wedge\zeta_{n-1})$, and $\tilde\Delta_n(t)=\Delta_n(t)-\langle X\rangle(t\wedge\zeta_n)+\langle X\rangle(t\wedge\zeta_{n-1})$. Then $\big(\Delta_n(t),\mathcal{F}_t,\mathbb{P}\big)$ and $\big(\tilde\Delta_n(t),\mathcal{F}_t,\mathbb{P}\big)$ are martingales. Next, given $F\in C^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{R})$, set

$$M(t)=\sum_{n=1}^\infty F'\big(X(\zeta_{n-1})\big)\Delta_n(t)+\frac12\sum_{n=1}^\infty F''\big(X(\zeta_{n-1})\big)\tilde\Delta_n(t),$$

and observe that all but a finite number to terms in each of these sums are 0. Further, notice that

$$\mathbb{E}^{\mathbb{P}}\left[\left|\sum_{n=1}^\infty F'\big(X(\zeta_{n-1})\big)\Delta_n(t)\right|^2\right]=\sum_{n=1}^\infty\mathbb{E}^{\mathbb{P}}\big[F'\big(X(\zeta_{n-1})\big)^2\Delta_n(t)^2\big]\le\|F'\|_{\mathrm{u}}\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)\big]<\infty$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\left|\sum_{n=1}^\infty F''\big(X(\zeta_{n-1})\big)\tilde\Delta_n(t)\right|\right]\le\|F''\|_{\mathrm{u}}\sum_{n=1}^\infty\mathbb{E}^{\mathbb{P}}\big[\Delta_n(t)^2+\langle X\rangle(t)\big]\le2\|F''\|_{\mathrm{u}}\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)\big]<\infty.$$

Thus, $(M(t),\mathcal{F}_t,\mathbb{P})$ is a martingale.

Clearly,

$$M(t)=F\big(X(t)\big)-F\big(X(0)\big)-\frac12\int_0^t F''\big(X(\tau)\big)\,\langle X\rangle(d\tau)+E(t)+\tilde E(t),$$

where

$$E(t)=\frac12\sum_{n=1}^\infty\Delta_n(t)^3\int_0^1(1-\xi)^2F'''\big(X(\zeta_{n-1})+\xi\Delta_n(t)\big)\,d\xi$$

and

$$\tilde E(t)=\frac12\sum_{n=1}^\infty\int_{t\wedge\zeta_{n-1}}^{t\wedge\zeta_n}\Big(F''\big(X(\zeta_{n-1})\big)-F''\big(X(\tau)\big)\Big)\,\langle X\rangle(d\tau).$$

Furthermore,

$$|E(t)|\le\tfrac12\|F'''\|_{\mathrm{u}}\epsilon\sum_{n=1}^\infty\Delta_n(t)^2\quad\text{and}\quad|\tilde E(t)|\le\tfrac12\|F'''\|_{\mathrm{u}}\epsilon\langle X\rangle(t).$$

Hence, if $0 \le s < t$ and $A \in \mathcal{F}_s$, then

$$\begin{aligned}
&\Bigg|\mathbb{E}^{\mathbb{P}}\left[F\big(X(t)\big) - F\big(X(0)\big) - \frac{1}{2}\int_0^t F''\big(X(\tau)\big)\,\langle X\rangle(d\tau),\, A\right] \\
&\qquad - \mathbb{E}^{\mathbb{P}}\left[F\big(X(s)\big) - F\big(X(0)\big) - \frac{1}{2}\int_0^t F''\big(X(\tau)\big)\,\langle X\rangle(d\tau),\, A\right]\Bigg| \\
&= \big|\mathbb{E}^{\mathbb{P}}\big[E(s) + \tilde{E}(s) - E(t) - \tilde{E}(t),\, A\big]\big| \le \|F'''\|_{\mathrm{u}}\epsilon\mathbb{E}^{\mathbb{P}}\big[\langle X\rangle(t)\big].
\end{aligned}$$

<u>(7.2.10)</u> By Doob's Stopping Time Theorem, $\big(Y(t)^2 - \langle X\rangle(t\wedge\zeta, \mathcal{F}_t, \mathbb{P}\big)$ is a martingale, and so, by uniqueness, $\langle Y\rangle(t) = \langle X\rangle(t\wedge\zeta)$.

<u>(7.2.11)</u> Applying Exercise 7.2.9 to $(\lambda X(t), \mathcal{F}_t, \mathbb{P})$, one sees that $\big(E_\lambda(t), \mathcal{F}_t, \mathbb{P}\big)$ is a martingale. Given this, the rest follows from the outline.

<u>(7.2.13)</u> The fact that $\big(X(t)Y(t) - \langle X, Y\rangle, \mathcal{F}_t, \mathbb{P}\big)$ is a martingale from

$$X(t)Y(t) = \frac{\big(X(t)+Y(t)\big)^2}{4} - \frac{\big(X(t)-Y(t)\big)^2}{4}.$$

As for the uniqueness, apply Theorem 7.1.19.

<u>(7.2.14)</u> Using Theorem 7.1.17, one knows that the expression in the hint is a martingale plus

$$\begin{aligned}
&f\big(t,\mathbf{B}(t)\big)^2 - 2\int_0^t X(\tau)\big(\partial_\tau + \tfrac{1}{2}\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau - \left(\int_0^t \big(\partial_\tau + \tfrac{1}{2}\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau\right)^2 \\
&\quad = f\big(t,\mathbf{B}(t)\big)^2 - 2\int_0^t f\big(\tau,\mathbf{B}(\tau)\big)\big(\partial_\tau + \tfrac{1}{2}\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau \\
&\qquad\quad - 2\int_0^t \left(\int_0^\tau \big(\partial_{\tau'} + \tfrac{1}{2}\Delta\big)f\big(\tau',\mathbf{B}(\tau')\big)\,d\tau'\right)\big(\partial_\tau + \tfrac{1}{2}\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau \\
&\qquad\qquad - \left(\int_0^t \big(\partial_\tau + \tfrac{1}{2}\Delta\big)f\big(\tau,\mathbf{B}(\tau)\big)\,d\tau\right)^2 \\
&\quad = f^2\big(t,\mathbf{B}(t)\big) - \int_0^t \big(\partial_\tau + \tfrac{1}{2}\Delta\big)f^2\big(\tau,\mathbf{B}(\tau)\big)\,d\tau + \int_0^t |\nabla f|^2\big(\tau,\mathbf{B}(\tau)\big)\,d\tau.
\end{aligned}$$

Since

$$\left(f^2\big(t,\mathbf{B}(t)\big) - \int_0^t \big(\partial_\tau + \tfrac{1}{2}\Delta\big)f^2\big(\tau,\mathbf{B}(\tau)\big)\,d\tau, \mathcal{F}_t, \mathbb{P}\right)$$

is a martingale, this completes the proof.