Solution Set # 7

<u>(5.2.36)</u> Just follow the outline.

<u>(5.2.38)</u> First note that $\mathbb{E}^{\mathbb{P}}[A_\infty] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[X_n] - \mathbb{E}^{\mathbb{P}}[X_0] < \infty$. Thus, since $0 \le A_n \le A_{n+1}$, $\{A_n : n \ge 0\}$ is uniformly integrable. Next, because the sequence formed by summing the members of two uniformly integrable sequences of functions is again uniformly integrable, it follows that $\{M_n : n \ge 0\}$ is also uniformly integrable. Thus, if $M_\infty = \lim_{n\to\infty} M_n$ (a.s., $\mathbb{P}$), $M_n = \mathbb{E}^{\mathbb{P}}[M_\infty \,|\, \mathcal{F}_n]$, and so $X_n = \mathbb{E}^{\mathbb{P}}[M_\infty \,|\, \mathcal{F}_n] + A_n$ (a.s., $\mathbb{P}$) for all $n \ge 0$.

Now let ζ be a stopping time. Given $A \in \mathcal{F}_\zeta$, $B \cap \{\zeta \le n\} \in \mathcal{F}_{n\wedge\zeta}$ and therefore Hunt's theorem implies that

$$\mathbb{E}^{\mathbb{P}}[M_{n\wedge\zeta},\, B\cap\{\zeta \le n\}] = \mathbb{E}^{\mathbb{P}}[M_\infty,\, B\cap\{\zeta \le n\}] \text{ and } \mathbb{E}^{\mathbb{P}}[|M_{n\wedge\zeta}|,\, B\cap\{\zeta \le n\}] \le \mathbb{E}^{\mathbb{P}}[|M_\infty|,\, B\cap\{\zeta \le n\}].$$

In particular, as

$$\sup_{m\ge 0} \mathbb{E}^{\mathbb{P}}\big[|M_{m\wedge\zeta}|,\, |M_{m\wedge\zeta}| > R\big] = \sup_{m\ge 0} \mathbb{E}^{\mathbb{P}}\big[|M_\infty|,\, |M_{m\wedge\zeta}| > R\big] \le \mathbb{E}^{\mathbb{P}}\left[|M_\infty|,\, \sup_{n\ge 0}|M_n| \ge R\right] \longrightarrow 0$$

as $R \to \infty$, since, by Doob's inequality, $\mathbb{P}\big(\sup_{n\ge 0}|M_n| \ge R\big) \le R^{-1}\mathbb{E}^{\mathbb{P}}\big[|M_\infty|\big]$. Hence, $\{M_{n\wedge\zeta} : n \ge 0\}$ is uniformly integrable, and so

$$\mathbb{E}^{\mathbb{P}}\big[M_\zeta,\, B \cap \{\zeta < \infty\}\big] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[M_{n\wedge\zeta},\, B \cap \{\zeta \le n\}] = \mathbb{E}^{\mathbb{P}}[M_\infty,\, B \cap \{\zeta < \infty\}]$$

for $B \in \mathcal{F}_\zeta$, which means that $\mathbb{E}^{\mathbb{P}}\big[M_\zeta,\, B\big] = \mathbb{E}^{\mathbb{P}}\big[M_\infty,\, B\big]$ for all $B \in \mathcal{F}_\zeta$ and therefore that $M_\zeta = \mathbb{E}^{\mathbb{P}}[M_\infty \,|\, \mathcal{F}_\zeta]$ (a.s, $\mathbb{P}$).

Knowing the preceding, we have $X_\zeta = \mathbb{E}^{\mathbb{P}}[M_\infty \,|\, \mathcal{F}_\zeta] + A_\zeta$ (a.s., $\mathbb{P}$) for all stopping times ζ. Thus, if $\zeta \le \zeta'$ and $B \in \mathcal{F}_\zeta$, then, since $\mathcal{F}_\zeta \subseteq \mathcal{F}_{\zeta'}$,

$$\mathbb{E}^{\mathbb{P}}\big[X_\zeta,\, B\big] = \mathbb{E}^{\mathbb{P}}\big[M_\infty,\, A\big] + \mathbb{E}^{\mathbb{P}}[A_\zeta,\, B] = \mathbb{E}^{\mathbb{P}}\big[M_{\zeta'},\, B\big] + \mathbb{E}^{\mathbb{P}}\big[A_\zeta,\, B\big],$$

and so $\mathbb{E}^{\mathbb{P}}[X_{\zeta'} \,|\, \mathcal{F}_\zeta] = X_\zeta + \mathbb{E}^{\mathbb{P}}[A_{\zeta'} - A_\zeta \,|\, \mathcal{F}_\zeta]$ (a.s., $\mathbb{P}$).

<u>(5.2.41)</u> Just follow the outline.

<u>(5.2.42)</u>

(**i**) Let $Z = \frac{\mathbb{P}\restriction\Sigma + \mathbb{Q}\restriction\Sigma}{d\mu}$. Then $\frac{\mathbb{P}\restriction\Sigma}{d\mu} = X_\Sigma Z$, $\frac{\mathbb{Q}\restriction\Sigma}{d\mu} = Y_\Sigma Z$, and so

$$\int \left(\frac{\mathbb{P}\restriction\Sigma}{d\mu}\right)^{\frac12} \left(\frac{\mathbb{Q}\restriction\Sigma}{d\mu}\right)^{\frac12} d\mu = \int X_\Sigma^{\frac12} Y_\Sigma^{\frac12} Z\, d\mu = \int\int X_\Sigma^{\frac12} Y_\Sigma^{\frac12}\, d(\mathbb{P} + \mathbb{Q}).$$

Finally, since $\mathbb{P}\restriction\Sigma \perp \mathbb{Q}\restriction\Sigma \iff X_\Sigma Y_\Sigma = 0$ (a.s., $\mathbb{P} + \mathbb{Q}$), it is obvious that $\mathbb{P}\restriction\Sigma \perp \mathbb{Q}\restriction\Sigma \iff (\mathbb{P}, \mathbb{Q})_\Sigma = 0$.

(**ii**) Assume, without loss in generality, that $\mathcal{F} = \bigvee_{n=0}^\infty \mathcal{F}_n$, and set $\mu = \frac{\mathbb{P}+\mathbb{Q}}{2}$, $X = \frac{d\mathbb{P}}{d\mu}$, and $Y = \frac{d\mathbb{Q}}{d\mu}$. Then it is an easy matter to check that $X_n \equiv \mathbb{E}^\mu[X \,|\, \mathcal{F}_n] = \frac{d\mathbb{P}\restriction\mathcal{F}_n}{d\mu\restriction\mathcal{F}_n}$ and $Y_n \equiv \mathbb{E}^{\mathbb{P}}[Y \,|\, \mathcal{F}_n] = \frac{d\mathbb{Q}\restriction\mathcal{F}_n}{d\mu\restriction\mathcal{F}_n}$ (a.s., μ). Further, without loss in generality, take X, Y, the X_n's, and Y_n's all to be non-negative. By Corollary 5.2.4, $X_n \longrightarrow X$ and $Y_n \longrightarrow Y$ in $L^1(\mu;\mathbb{R})$, and so, since $(b^{\frac12} - a^{\frac12})^2 \le |b - a|$ for any $a, b \in [0,\infty)$, $X_n^{\frac12} \longrightarrow X^{\frac12}$ and $Y_n \longrightarrow Y^{\frac12}$ in $L^2(\mu;\mathbb{R})$. Hence, $X_n^{\frac12} Y_n^{\frac12} \longrightarrow X^{\frac12} Y^{\frac12}$ in $L^1(\mu;\mathbb{R})$, and therefore

$$(\mathbb{P}, \mathbb{Q})_{\mathcal{F}_n} = \mathbb{E}^\mu[X_n^{\frac12} Y_n^{\frac12}] \longrightarrow \mathbb{E}^\mu[X^{\frac12} Y^{\frac12}] = (\mathbb{P}, Q)_{\mathcal{F}}.$$

(**iii**) Simply take $\mu = \mathbb{P}$ and note that $(\mathbb{P}, \mathbb{Q})_{\mathcal{F}_n} = \mathbb{E}^{\mathbb{P}}[\sqrt{X_n}]$.

(**iv**) Set

$$h_{\sigma,a,b}(x)=\frac{d\gamma_{b,\sigma^2}}{d\gamma_{a,\sigma^2}}=e^{\frac{a^2-b^2}{2\sigma^2}}e^{\frac{(b-s)x}{\sigma^2}}.$$

Then, for any $p\in(0,\infty)$,

$$\int h^p_{\sigma,a,b}\,d\gamma_{a,\sigma^2}=\int h^p_{\sigma,a,b}(x+a)\,\gamma_{0,\sigma^2}(dx)=e^{\frac{p(b-a)^2}{2\sigma^2}}\int e^{\frac{p(b-a)x}{\sigma^2}}\,\gamma_{0,\sigma^2}(dx)=e^{\frac{p(p-1)(b-a)^2}{2\sigma^2}}.$$

Now set $\Omega=\mathbb{R}^{\mathbb{N}}$, $\mathcal{F}_n=\sigma\big(\{\omega:\omega_m\in\Gamma\}:0\le m\le n\ \&\ \Gamma\in\mathcal{B}_{\mathbb{R}}\big\}$, $\mathcal{F}=\bigvee_{n+0}^{\infty}\mathcal{F}_n$, $\mathbb{P}=\prod_{n=0}^{\infty}\gamma_{a_n,\sigma^2}$, and $\mathbb{Q}=\prod_{n=0}^{\infty}\gamma_{b_n,\sigma^2}$. Then $X_n(\omega)\equiv\frac{d\mathbb{Q}\restriction\mathcal{F}_n}{d\mathbb{Q}\restriction\mathcal{F}_n}(\omega)=\prod_{m=0}^{n}h_{\sigma_m,a_m,b_m}(\omega_m)$, and so $\mathbb{E}^{\mathbb{P}}[X_n^p]=\exp\Big(\frac{p(p-1)}{2}\sum_{m=0}^{n}\frac{(b_m-a_m)^2}{\sigma_m^2}\Big)$. Hence, $\sum_{m=0}^{\infty}\frac{(b_m-a_m)^2}{\sigma_m^2}=\infty\implies\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}}[\sqrt{X_n}]\longrightarrow 0$ and $\sum_{m=0}^{\infty}\frac{(b_m-a_m)^2}{\sigma_m^2}<\infty\implies\sup_{n\ge0}\mathbb{E}^{\mathbb{P}}[X_n^2]<\infty$. In the first case $\mathbb{Q}\perp\mathbb{P}$ and in the second $\mathbb{Q}\ll\mathbb{P}$. Reversing the roles of the a_n's and b_n's, one sees that, in the second case $\mathbb{P}\ll\mathbb{Q}$ as well.