Solution Set # 6

(4.3.10)

(**i**): If $\{\mathbf{B}(t) : t \geq 0\}$ is a Brownian motion, $0 = t_0 < \cdots < t_\ell$, and $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_\ell \in \mathbb{R}^N$, then

$$\sum_{k=1}^{\ell} \big(\boldsymbol{\xi}_k, \mathbf{B}(t_k)\big)_{\mathbb{R}^N} = \sum_{k=1}^{\ell} \big(\boldsymbol{\Xi}_k, \mathbf{B}(t_k) - \mathbf{B}(t_{k-1})\big)_{\mathbb{R}^N},$$

where $\boldsymbol{\Xi}_k = \sum_{j=k}^{\ell} \boldsymbol{\xi}_j$. Therefore, $\sum_{k=1}^{\ell}\big(\boldsymbol{\xi}_k, \mathbf{B}(t_k)\big)_{\mathbb{R}^N}$ is a sum of mutually independent, centered Gaussian random variable and, as such, is a centered Gaussian random variable. Thus, the span of $\{(\boldsymbol{\xi}, \mathbf{B}(t))_{\mathbb{R}^N} : t \geq 0 \ \&\ \boldsymbol{\xi} \in \mathbb{R}^N\}$ is a Gaussian family. In addition, if $t' \geq t$, then because $\mathbf{B}(t') - \mathbf{B}(t)$ is independent of $\mathbf{B}(t)$,

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}', \mathbf{B}(t')\big)_{\mathbb{R}^N}\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big] \\ &\quad = \mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}', \mathbf{B}(t') - \mathbf{B}(t)\big)_{\mathbb{R}^N}\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big] + \mathbb{E}^{\mathbb{P}}\big[\big(\boldsymbol{\xi}', \mathbf{B}(t)\big)_{\mathbb{R}^N}\big(\boldsymbol{\xi}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big] = t(\boldsymbol{\xi}', \boldsymbol{\xi})_{\mathbb{R}^N}. \end{aligned}$$

Conversely, if $\{\mathbf{X}(t) : t \geq 0\}$ has the stated properties, then, for any $0 \leq s \leq t$, $\tau \in [0, s]$, and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$,

$$\mathbb{E}^{\mathbb{P}}\big[(\boldsymbol{\xi}, \mathbf{X}(t) - \mathbf{X}(s))_{\mathbb{R}^N}\big(\boldsymbol{\eta}, \mathbf{X}(\tau)\big)_{\mathbb{R}^N} = \tau(\boldsymbol{\xi}, \boldsymbol{\eta})_{\mathbb{R}^N} - \tau(\boldsymbol{\xi}, \boldsymbol{\eta})_{\mathbb{R}^N} = 0,$$

and so $\mathbf{X}(t) - \mathbf{X}(s)$ is independent of $\sigma\big(\{\mathbf{X}(\tau) : \tau \in [0, s]\}\big)$. In addition,

$$\mathbb{E}^{\mathbb{P}}\big[(\boldsymbol{\xi}, \mathbf{X}(t) - \mathbf{X}(s))^2_{\mathbb{R}^N}\big] = (t - 2s + s)|\boldsymbol{\xi}|^2 = (t - s)|\boldsymbol{\xi}|^2,$$

and so $\mathbf{X}(t) - \mathbf{X}(s)$ is a centered Gaussian random variable with covariance $(t - s)\mathbf{I}$. Hence, $\{\mathbf{X}(t) : t \geq 0\}$ is a Brownain motion.

(**ii**) & (**iii**): The first of these follows from (**i**) by taking $\mathbf{X}(t) = \mathcal{O}\mathbf{B}(t)$, and second by taking $\mathbf{X}(t) = \lambda^{-\frac{1}{2}}\mathbf{B}(\lambda t)$.

(4.3.11)

(**i**): By (**i**) in Exercise 4.3.10, $\{\mathbf{X}(t) : t > 0\}$ is a Gaussian family satisfying

$$\mathbb{E}^{\mathbb{P}}\big[(\boldsymbol{\xi}', \mathbf{X}(t'))_{\mathbb{R}^N}(\boldsymbol{\xi}, \mathbf{X}(t))_{\mathbb{R}^N}\big] = t \wedge t'(\boldsymbol{\xi}, \boldsymbol{\xi}')_{\mathbb{R}^N} \quad \text{for all } t, t' \geq 0 \text{ and } \boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^N.$$

Thus, $\{\mathbf{X}(t) : t > 0\}$ has the same distribution as a Brownian motion restricted to time interval $(0, \infty)$, and therefore, with probability 1, $\lim_{t \to 0} \mathbf{X}(t) = \mathbf{0}$. Knowing this, the rest is just an application of (**i**) in Exercise 4.3.10.

(**ii**): This is a trivial application of (**i**).

(4.3.12)

(**i**): Since

$$\mathbb{P}\left(\sup_{t \in [0,T]} |\mathbf{B}(t)| \geq R\right) \leq N \max_{\mathbf{e} \in \mathbb{S}^{N-1}} \mathbb{P}\left(\sup_{t \in [0,T]} \big|\big(\mathbf{e}, \mathbf{B}(t)\big)_{\mathbb{R}^N}\big| \geq N^{-\frac{1}{2}} R\right),$$

it suffices to treat the case when $N = 1$, and, by Brownian scaling, when $T = 1$. Thus, let $\{B(t) : t \geq 0\}$ be an $\mathbb{R}$-valued Brownian motion. Since $B(m2^{-n}) = \sum_{k=1}^{m}\big(B(k2^{-n}) - B((k-1)2^{-n})\big)$, Theorem 1.4.13 says that

$$\mathbb{P}\left(\max_{1 \leq m \leq 2^n} |B(k2^{-n})| \geq R\right) \leq 2\mathbb{P}\big(|B(1)| \geq R\big).$$

Hence, since

$$\mathbb{P}\big(\|B\|_{[0,1]} \ge R\big) = \lim_{r\nearrow R} \mathbb{P}\big(\|B\|_{[0,1]} > r\big) \text{ and } \mathbb{P}\big(\|B\|_{[0,1]} > R\big) \le \lim_{n\to\infty} \mathbb{P}\left(\max_{1\le m\le 2^n} |B(k2^{-n})| \ge R\right),$$

all that remains is the show that $\mathbb{P}\big(B(1) \ge R\big) \le \frac{1}{2}e^{-\frac{R^2}{2}}$. Because $\int_R^\infty e^{-\frac{x^2}{2}}\,dx \le R^{-1}e^{-\frac{R^2}{2}}$, this is obvious when $R \ge \sqrt{\frac{2}{\pi}}$. To prove it when $0 < R \le \sqrt{\frac{2}{\pi}}$, note that $\mathbb{P}\big(B(1) \ge R\big) = \frac{1}{2} - \mathbb{P}\big(0 \le B(1) \le R\big)$ and therefore that it suffices to check that

$$1 \le e^{-\frac{R^2}{2}} + \sqrt{\frac{2}{\pi}} \int_0^R e^{-\frac{x^2}{2}}\,dx \quad \text{for } 0 \le R \le \sqrt{\frac{2}{\pi}}.$$

Finally, observe that the righthand side is a non-decreasing functin of $R \in \left[0, \sqrt{\frac{2}{\pi}}\right]$.

(ii) & **(iii)**: Part **(iii)** follows immediately from **(ii)**, and the verification of **(ii)** can be done by the outlined argument.

(4.3.15) Just follow the steps suggested.

(4.3.19) Because

$$\begin{aligned} B(1)^2 &= \sum_{m=1}^n \Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big)\Big(B\big(\tfrac{m}{n}\big) + B\big(\tfrac{m-1}{n}\big)\Big) \\ &= \sum_{m=1}^n \Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big)^2 + 2\sum_{m=1}^n B\big(\tfrac{m-1}{n}\big)\Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big) \\ &= \sum_{m=1}^n \Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big)^2 - 2\sum_{m=1}^n B\big(\tfrac{m}{n}\big)\Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big), \end{aligned}$$

the first two equations follow. To prove the third, start in the same way as above and thereby arrive at

$$\begin{aligned} B(1)^2 &- 2\sum_{m=1}^n B\big(\tfrac{2m-1}{2n}\big)\Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big) \\ &= \sum_{m=1}^n \Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{2m-1}{2n}\big)\Big)\Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big) \\ &\qquad - \sum_{m=1}^n \Big(B\big(\tfrac{2m-1}{2n}\big) - B\big(\tfrac{m-1}{n}\big)\Big)\Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{m-1}{n}\big)\Big) \\ &= \sum_{m=1}^n \Big(B\big(\tfrac{m}{n}\big) - B\big(\tfrac{2m-1}{2n}\big)\Big)^2 - \sum_{m=1}^n \Big(B\big(\tfrac{2m-1}{2n}\big) - B\big(\tfrac{m-1}{n}\big)\Big)^2 \end{aligned}$$

Finally, proceed as in the proof of Theorem 4.3.5 to see that each of the sums in the final line tend to $\frac{1}{2}$ $\mathbb{P}$-almost surely.

(5.1.16) Because $X - \Pi_L X \perp L$, $X - \Pi_L X$ is independent of Σ_L. Thus, $\mathbb{E}^{\mathbb{P}}\big[X, A\big] = \mathbb{E}^{\mathbb{P}}\big[\Pi_L X, A\big]$ for all $A \in \Sigma_L$, and so, since $\Pi_L X$ is Σ_L-measurable, $\Pi_L X = \mathbb{E}^{\mathbb{P}}\big[X \,\big|\, \Sigma_L\big]$.