Solution Set # 5

<u>(4.1.9)</u> Let Σ be the collection of subsets of the form $\{\omega:\,(X_{i_1}(\omega),\dots,X_{i_m},\dots)\in\Gamma\}$, where $\{i_m:\,m\ge 1\}\subseteq\mathcal{I}$ and $\Gamma\in\mathcal{B}^{\mathbb{Z}^+}$. Clearly Σ is a σ-algebra. Moreover, because

$$\left\{\omega:\,(X_{i_1}(\omega),\dots,X_{i_m},\dots)\in\prod_{m=1}^{\infty}\Gamma_m\right\}=\bigcap_{m=1}^{\infty}\{\omega:\,X_{i_m}(\omega)\in\Gamma_m\}\in\mathcal{F}$$

for any $\{\Gamma_m:\,m\ge 1\}\subseteq\mathcal{B}$ and $\mathcal{B}^{\mathbb{Z}^+}$ is generated by sets of the form $\prod_{m=1}^{\infty}\Gamma_m$, it follows that $\Sigma\subseteq\mathcal{F}$. Conversely, for every $i\in\mathcal{I}$ and $\Gamma\in\mathcal{B}$, $\{\omega:\,X_i(\omega)\in\Gamma\}\in\Sigma$, and so $\mathcal{F}\subseteq\Sigma$.

Next, from the preceding, we know that, for any $\{i_m:\,m\ge 1\}\subseteq\mathcal{I}$, $\omega\rightsquigarrow\big(X_{i_1}(\omega),\dots,X_{i_m}(\omega),\dots\big)$ is $\mathcal{F}$-measurable into $(E^{\mathbb{Z}^+},\mathcal{B}^{\mathbb{Z}^+})$. Hence, if $f:\,E^{\mathbb{Z}^+}\longrightarrow\mathbb{R}$ is $\mathcal{B}^{\mathbb{Z}^+}$-measurable, then the function $\omega\rightsquigarrow f\big(X_{i_1}(\omega),\dots,X_{i_m}(\omega),\dots\big)$ is $\mathcal{F}$-measurable. Conversely, if $A\in\mathcal{F}$, then, by the preceding, $\mathbf{1}_A(\omega)=\mathbf{1}_\Gamma\big(X_{i_1}(\omega),\dots,X_{i_m}(\omega),\dots\big)$ for some $\{i_m:\,m\ge 1\}\subseteq\mathcal{I}$ and $\Gamma\in\mathcal{B}^{\mathbb{Z}^+}$. Thus, every $\mathcal{F}$-measurable, simple function has the required form, and therefore, since every $\mathcal{F}$-measurable function is the pointwise limit of simple ones, every $\mathcal{F}$-measurable functions does also.

<u>(4.1.12)</u> Just follow the hint.

<u>(4.2.19)</u> Because $N(n)=\sum_{m=1}^{n}\big(N(m)-N(m-1)\big)$ and $\{N(m)-N(m-1):\,m\ge 1\}$ is a sequence of mutually independent, identically distributed random integrable variables with mean value 1, the strong law implies that $\lim_{n\to\infty}\frac{N(n)}{n}=1$ (a.s., $\mathbb{P}$). Next, because $t\rightsquigarrow N(t)$ is non-decreasing, $\sup_{t\in[n,n+1]}\frac{N(t)-N(n)}{t}\le\frac{N(n+1)-N(n)}{n}$, and therefore

$$\mathbb{P}\left(\sup_{t\in[n,n+1]}\frac{N(t)-N(n)}{t}\ge\epsilon\right)\le\mathbb{P}\left(\frac{N(n+1)-N(n)}{n}\ge\epsilon\right)=\mathbb{P}\big(N(1)\ge n\epsilon\big)\le\frac{\mathbb{E}^{\mathbb{P}}[N(1)^2]}{n^2\epsilon^2}=\frac{2}{n^2\epsilon^2}.$$

Hence, $\mathbb{P}$-almost surely, $\sup_{t\in[n,n+1]}\frac{N(t)-N(n)}{t}\le\epsilon$ for all sufficiently large n's, and so, $\mathbb{P}$-almost surely,

$$\varlimsup_{t\to\infty}\left|\frac{N(t)}{t}-\frac{N(\lfloor t\rfloor)}{\lfloor t\rfloor}\right|\le\epsilon+\varlimsup_{t\to\infty}\frac{N(\lfloor t\rfloor)}{t\lfloor t\rfloor}=\epsilon.$$

<u>(4.2.20)</u> If $t\rightsquigarrow Z(t)$ is $\mathbb{P}$-almost surely non-decreasing, then $Z(1)\ge 0$ (a.s., $\mathbb{P}$) and therefore, by Exercise 3.2.25, $M\in\mathfrak{M}_1(\mathbb{R}^N)$, $M\big((-\infty,0)\big)=0$, and $m\ge\int_{[-1,1]}y\,M(dy)$. Conversely, if $M\in\mathfrak{M}_1(\mathbb{R}^N)$, $M\big((-\infty,0)\big)=0$, and $m\ge\int_{[-1,1]}y\,M(dy)$, then, by Exercise 3.2.25, for each $0\le s\le t<\infty$, $Z(t)-Z(s)\ge 0$ (a.s., $\mathbb{P}$). Hence, by right continuity, $\mathbb{P}$-almost surely, $Z(t)-Z(s)\ge 0$ for all $0\le s\le t<\infty$.

<u>(4.2.23)</u> The reduction, described in (**iv**), to the case when $\nu(\{\mathbf{0}\})=0$ is easy. Thus, assume that $\nu(\{\mathbf{0}\})=0$ throughout. Then, $M(\{\mathbf{0}\})=\nu(\{\mathbf{0}\})=0$, and, for each $r>0$, $M\big(B(\mathbf{0},r)\complement\big)=\nu\big(\overline{B(\mathbf{0},r)}\big)<\infty$. Hence, $M\in\mathfrak{M}_\infty(\mathbb{R}^N)$. Next, if $\{\Gamma_m:\,1\le m\le n\}$ are mutually disjoint Borel sets, then so are $\{F(\Gamma_m):\,1\le m\le n\}$, and therefore $P(\Gamma_1,\,\cdot\,)=j\big(1,F(\Gamma_1),\,\cdot\,\big),\dots,P(\Gamma_n,\,\cdot\,)=j\big(1,F(\Gamma_n),\,\cdot\,\big)$ are mutually independent random variables. Finally, for any $\Gamma\in\mathcal{B}_{\mathbb{R}^N}$, $P(\Gamma,\,\cdot\,)=j\big(1,F(\Gamma),\,\cdot\,\big)$ is a Poisson random variable with rate $M\big(F(\Gamma)\big)=\nu(\Gamma)$.