Solution Set # 4

(3.1.9)

(**i**): Just follow the outline.

(**ii**): The fact that $f(-x) = \overline{f(x)}$ is a consequence of the general fact that if A is a bounded operator on a complex Hilbert space H and if $(\xi, A\xi)_H \in \mathbb{R}$ for all $\xi \in H$, then A is Hermitian (i.e., $A = A^*$). To prove this, note that

$$\mathbb{R} \ni \big((\xi+\eta), A(\xi+\eta)\big)_H - (\xi, A\xi)_H - (\eta, A\eta)_H = (\xi, A\eta)_H + (\eta, A\xi)_H,$$

and therefore that

$$0 = \mathfrak{Im}\big((\xi, A\eta)_H + (\eta, A\xi)_H\big) = (\xi, A\eta)_H - (A\eta, \xi)_H + (\eta, A\xi)_H - (A\xi, \eta)_H.$$

Applying this with ξ replaced by $\sqrt{-1}\,\xi$, one also has

$$(\xi, A\eta)_H + (A\eta, \xi)_H - (\eta, A\xi)_H - (A\xi, \eta)_H = 0.$$

Adding these gives $(\xi, A\eta)_H = (A\xi, \eta)_H$, which says that A is Hermitian. Now apply this to $H = \mathbb{C}^2$ and $A = \begin{pmatrix} 1 & f(x) \\ f(-x) & 1 \end{pmatrix}$.

To see that $|f(x)| \leq 1$, note that since $\begin{pmatrix} 1 & f(x) \\ f(-x) & 1 \end{pmatrix}$ is a non-negative, Hermitian matrix, its determinant is non-negative.

(**iii**) and (**iv**): There is nothing to do in (**iii**). As for (**iv**), note that, as non-negative definite function that is 1 at the origin, $f = \hat{\mu}$ for some $\mu \in \mathbf{M}_1(\mathbb{R}^N)$ if it is continuous, in which case, by Lemma 2.3.3 says that $\mu_n \implies \mu$. Conversely, if $\mu_n \implies \mu$, then $f = \hat{\mu}$ and is therefore continuous.

(3.1.11) To see that f is non-negative definite, let $\{h_1, \dots, h_n\} \subseteq H$ be given, and choose an orthonormal set $\{e_1, \dots, e_n\} \subseteq H$ whose span L contains $\{h_1, \dots, h_n\}$. Next, define $g(\xi) = f\left(\sum_{j=1}^n \xi_j e_j\right)$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Then $g(\xi) = e^{-\frac{1}{2}|\xi|^2} = \widehat{\gamma^n_{0,1}}(\xi)$, and therefore g is non-negative definite. Because $f(h) = g(\xi)$ when $h \in L$ and $\xi = \big((h, e_1)_H, \dots, (h, e_n)_H\big)$, it follows that $\big(\big(f(h_i - h_h)\big)\big)_{1 \leq i,j \leq n}$ is a non-negative definite matrix.

To complete the program, choose an orthonormal sequence $\{e_n : n \geq 1\}$ in H. By the preceding, we know that, for each $n \geq 1$, $\gamma^n_{0,1}$ is the distribution under μ of $h \in H \longmapsto \big((h, e_1)_H, \dots, (h, e_n)_H\big) \in \mathbb{R}^n$. Thus

$$\int_H e^{-\|h\|_H^2}\,\mu(dh) = \lim_{n\to\infty} \int_{\mathbb{R}^n} e^{-\frac{1}{2}|\xi|^2}\,\gamma^n_{0,1}(d\xi) = \lim_{n\to\infty} \left(\int_{\mathbb{R}} e^{-\xi^2}\,\gamma_{0,1}(d\xi)\right)^n = 0.$$

(3.1.12)

(**i**) Let A be the matrix in the hint and ξ the vector in $\mathbb{R}^n$ all of whose components equal 1. Then $(\xi, A\xi)_{\mathbb{C}^n} = n + 2\sum_{1 \leq k < \ell \leq n} f\big((\ell - k)R\big) < n - n(n-1)\alpha < 0$.

(**ii**) Suppose that μ has an atom, and set $\beta = \mu \star \breve{\mu}$, where $\breve{\mu}(\Gamma) = \mu(-\Gamma)$. Then β is symmetric and has an atom at 0. In addition, $\hat{\beta}(\xi) = |\hat{\mu}(\xi)|^2$, and so $\hat{\beta}(\xi) \longrightarrow 0$ as $|\xi| \to \infty$ if and only if $\hat{\mu}(\xi)$ does. Now set $p = \beta(\{0\})$ and $\nu = \frac{\beta - p\delta_0}{q}$. Then $p \in (0, 1]$, ν is symmetric, and $\hat{\beta}(\xi) = p + q\hat{\nu}(\xi)$. If $p = 1$, then it is clear that $\hat{\beta}(\xi) \equiv 1$. If $p \in (0, 1)$ and $\hat{\beta}(\xi) \longrightarrow 0$, then $\hat{\nu}(\xi) \longleftrightarrow -\frac{p}{q} < 0$, which contradicts (**i**).

(iii) For any $p \in (0,1)$, $\hat{\mu}_p(\xi) = \prod_{n=1}^\infty \big(pe^{\sqrt{-1}\,2^{-n}\xi} + q\big)$, and therefore $\mu_p(2^m\pi) = \mu_p(\pi)$ for all $m \in \mathbb{Z}^+$. Furthermore, $pe^{\sqrt{-1}\eta} + q \neq 0$ and for any $\nu \in \mathbb{R}$, and

$$\big|pe^{\sqrt{-1}2^{-n}\xi} + q - 1\big| \le 2^{-n}|\xi| \quad \text{for } \xi \in \mathbb{R}.$$

Hence, $\hat{\mu}_p$ never vanishes. Finally, $\hat{\mu}(\pi) = (q-p)\prod_{n=2}^\infty \big(pe^{\sqrt{-1}\,2^{-n}\pi} + q\big) = (q-p)\hat{\mu}_p\left(\frac{p}{2}\right)$. Thus, if $p \neq q$, $\mu_p(\xi)$ cannot tend to 0 at infinity.

<u>(3.2.23)</u> Let $f : \mathbb{C}^N \times \mathbb{R}^N \longrightarrow \mathbb{C}$ be a Borel measurable function with the properties that it bounded on compacts and $f(\,\cdot\,,\mathbf{x})$ is analytic for each $\mathbf{x} \in \mathbb{R}^N$. If $M \in \mathfrak{M}_2(\mathbb{R}^N)$, then, for each $n \ge 2$, $\boldsymbol{\xi} \rightsquigarrow F_n(\boldsymbol{\xi}) \equiv \int_{\frac{1}{n}<|\mathbf{x}|\le n} f(\boldsymbol{\xi},\mathbf{x})\,M(d\mathbf{x})$ is an analytic function on $\mathbb{C}^N$. To see this, simply check that derivatives of F_n are given by the integrals of the corresponding derivatives of f and therefore that F_n satisfies the Cauchy–Riemann equations. Next, assume that $\sup_{|\boldsymbol{\xi}|\le R}|f(\boldsymbol{\xi},\,\cdot\,)|$ is M-integrable of each $R > 0$, and set $F(\boldsymbol{\xi}) = \int f(\boldsymbol{\xi},\mathbf{x})\,M(d\mathbf{x})$. Then $F_n \longrightarrow F$ uniformly on compacts, and therefore F is analytic on $\mathbb{C}^N$.

Without loss in generality, take $\eta = \mathbf{1}_{\overline{B(\mathbf{0},1)}}$. Applying the preceding to

$$\begin{aligned}\boldsymbol{\xi} \rightsquigarrow F(\xi) &\equiv \int_{\mathbb{R}^N} \big(e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N} - 1 - (\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{\overline{B(\mathbf{0},1)}}(\mathbf{x})(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}\big)\,M(d\mathbf{x}) \\ &= \int_{\mathbb{R}^N}\int_{|\mathbf{x}|\le 1} \big(e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N} - 1 - (\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\big)\,M(d\mathbf{x}) + \int_{|\mathbf{x}|>1} \big(e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N} - 1} - 1\big)\,M(d\mathbf{x}),\end{aligned}$$

we see that F is analytic on $\mathbb{C}^N$. Hence, since $\ell_\mu(-\sqrt{-1}\boldsymbol{\xi}) = (\boldsymbol{\xi},\mathbf{m})_{\mathbb{R}^N} + \frac{1}{2}\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N} + F(\boldsymbol{\xi})$ when $\boldsymbol{\xi}$ is pure imaginary, it follows that $(\boldsymbol{\xi},\mathbf{m})_{\mathbb{R}^N} + \frac{1}{2}\big(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi}\big)_{\mathbb{R}^N} + F(\boldsymbol{\xi})$ is the necessarily unique analytic extension of $\boldsymbol{\xi} \rightsquigarrow \ell_\mu(-\sqrt{-1}\boldsymbol{\xi})$ to $\mathbb{C}^N$.

Because $\mu = \gamma_{\mathbf{m},\mathbf{C}} \star \nu$, where $\hat{\nu}(\boldsymbol{\xi}) = \exp\big(F(\sqrt{-1}\boldsymbol{\xi})$ and, at the same time,

$$\int_{\mathbb{R}^N} e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\gamma_{\mathbf{m},\mathbf{C}}(d\mathbf{x}) = e^{(\mathbf{m},\boldsymbol{\xi})_{\mathbb{R}^N} + \frac{1}{2}(\boldsymbol{\xi},\mathbf{C}\boldsymbol{\xi})_{\mathbb{R}^N}} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^N,$$

all that remains is to show that

$$\int_{\mathbb{R}^N} \sup_{|\boldsymbol{\xi}|\le R} \big|e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\big|\,\nu(d\mathbf{x}) < \infty \quad \text{for all } R > 0,$$

and obviously this reduces to checking that $\int_{\mathbb{R}^N} e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\nu(d\mathbf{x}) < \infty$ for each $\boldsymbol{\xi} \in \mathbb{R}^N$. To this end, set $M_r(d\mathbf{x}) = \mathbf{1}_{(r,\infty)}(|\mathbf{x}|)M(d\mathbf{x})$ and $\mathbf{m}_r = \int_{r<|\mathbf{x}|\le 1} \mathbf{x}\,M(d\mathbf{x})$. Then $\nu_r \equiv \delta_{-\mathbf{m}_r} \star \Pi_{M_r} \implies \mu$ as $r \searrow 0$, and therefore

$$\int_{\mathbb{R}^N} e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\nu(d\mathbf{x}) \le \sup_{r\in(0,1]} \int_{\mathbb{R}^N} e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\nu_r(d\mathbf{x}) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^N.$$

Finally,

$$\int e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\Pi_{M_r}(d\mathbf{x}) = e^{-M_r(\mathbb{R}^N)} \sum_{n=0}^\infty \int_{\mathbb{R}^N} e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,M_r^{*n}(d\mathbf{x}) = \exp\left(\int_{\mathbb{R}^N} \big(e^{(\boldsymbol{\xi},\mathbf{x})} - 1\big)\,M_r(d\mathbf{x})\right),$$

and so

$$\int e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}}\,\nu_r(d\mathbf{x}) = \exp\left(\int_{|\mathbf{x}|>r} \big(e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}} - 1 - \mathbf{1}_{\overline{B(\mathbf{0},1)}}(\mathbf{x})(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^N}\big)\,M(d\mathbf{x})\right),$$

which is bounded independent of $r \in (0,1]$ and $\boldsymbol{\xi}$ in bounded subsets of $\mathbb{R}^N$.

(3.2.25)

(i) Since $\delta_{m_\mu}\star\pi_{M^r}\big((-\infty,0)\big)=\pi_{M^r}\big((-\infty,-m_\mu)\big)\le\pi_{M^r}\big((-\infty,0)\big)$ and $\delta_{m_\mu}\star\pi_{M^r}\implies\mu$, it suffices to show that $\pi_{M^r}\big((-\infty,0)\big)=0$. But

$$\pi_{M^r}\big((-\infty,0)\big)=e^{-M^r(\mathbb{R})}\sum_{n=0}^{\infty}\frac{(M^r)^{\star n}\big((-\infty,0)\big)}{n!},$$

and it easy to see that $M^r\big((-\infty,0)\big)=0$ implies that $(M^r)^{\star n}\big((-\infty,0)\big)=0$ for all $n\ge 0$.

(ii) Because

$$\gamma_{0,\sigma^2}\star\nu\big((-\infty,0)\big)=\int_{\mathbb{R}}\nu\big(-\infty,-y)\big)\gamma_{0,\sigma^2}(dy)$$

and $\nu\big(-\infty,-y)\big)>0$ for sufficiently negative y's, it is clear that $\gamma_{0,\sigma^2}\star\nu\big((-\infty,0)\big)>0$ for all $\nu\in\mathbf{M}_1(\mathbb{R})$ and $\sigma>0$.

(iii) Because $\mu=\mu_{\frac{1}{n}}^{\star n}$ and

$$\nu_1\star\nu_2\big((-\infty,0)\big)\ge\int_{(-\infty,0)}\nu_1\big((-\infty,-y)\big)\,\nu_2(dy)\ge\nu_1\big((-\infty,0)\big)\nu_2\big((-\infty,0)\big)$$

for all $\nu_1,\ \nu_2\in\mathbf{M}_1(\mathbb{R})$, $\mu\big((-\infty,0)\big)\ge\mu_{\frac{1}{n}}\big((-\infty,0)\big)^n$. Thus, by Theorem 3.2.22,

$$\int_{\mathbb{R}}\varphi(y)\,M_\mu(dy)=\lim_{n\to\infty}n\int_{\mathbb{R}}\varphi(y)\,\mu_{\frac{1}{n}}(dy)=0$$

for any $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{C})$ which is compactly supported in $(-\infty,0)$, and from this it is clear that $M_\mu\big((-\infty,0)\big)=0$. Moreover, for any $\varphi\in\mathscr{S}(\mathbb{R};\mathbb{R})$ which is compactly supported in $(0,\infty)$ and satisfies $0\le\varphi(0)\le\eta_0(y)$ for $y\ge 0$, another application of Theorem 3.2.22 shows that $m_\mu^{\eta_0}\ge\int_{\mathbb{R}}\varphi(y)\,M(dy)$, from which it follows that $m_\mu^{\eta_0}\ge\int_R\eta_0(y)y\,M(dy)$ and therefore both that $m_\mu^{\eta_0}\ge 0$ and that $M\in\mathfrak{M}_1(\mathbb{R})$. Finally, one can now use Exercise 3.2.24 to also conclude that $m_\mu\ge 0$.

(iv) Let μ be the distribution of $|X|$, and suppose that $\mu\in\mathcal{I}(\mathbb{R})$. Clearly $\mu\big((-\infty,0)\big)=0$, and therefore, by the preceding, we would know that $\mu=\delta_m\star\pi_M$, where $m\ge 0$ and π_M is the generalized Poisson measure for some $M\in\mathfrak{M}_1(\mathbb{R})$ which is supported on $[0,\infty)$. Given $r\ge 1$, define M^r as in (**i**) and set $M_r=M-M^r$. Then $\mu=(\delta_m\star\pi_{M_r})\star\pi_{M^r}$, and so

$$\begin{aligned}\mu\big([r,\infty)\big)&=\int_{[0,\infty)}\pi_{M^r}\big([r-m-y,\infty)\big)\,\pi_{M_r}(dy)\ge\pi_{M^r}\big([r,\infty)\big)\\&=1-e^{-M([r,\infty))}\ge M\big([r,\infty)\big)e^{-M([1,\infty))},\end{aligned}$$

and therefore $M\big([r,\infty)\big)\le e^{M([1,\infty))}\mu\big([r,\infty)\big)\le Ce^{-\frac{r^2}{2}}$, where $C=2e^{M([1,\infty))}$. Thus, M satisfies the condition in Exercise 3.2.23, and therefore

$$2e^{\frac{\xi^2}{2}}\ge\int_{\mathbb{R}}e^{\xi y}\,\mu(dy)=\exp\left[\xi m+\int_{(0,\infty)}\left(e^{\xi y}-1\right)M(dy)\right]$$

for all $\xi\ge 0$, which is possible only if $M=0$. But if $M=0$, then $\mu=\delta_m$, which it certainly cannot.

(3.2.26) Only (**ii**) needs comment. To deal with (**ii**), note that

$$\ell_{\mu_t}(\xi) = t \lim_{s \searrow 0} \frac{\hat{\mu}_s(\xi) - 1}{s} = \lim_{s \searrow 0} t \lim_{s \searrow 0} \frac{1}{s\Gamma(s)} \int_0^\infty \left(e^{\sqrt{-1}\,\xi x} - 1\right) e^{-x} x^s \, \frac{dx}{x} = \int_0^\infty \left(e^{\sqrt{-1}\,\xi x} - 1\right) e^{-x} \frac{dz}{x},$$

since $s\Gamma(s) = \Gamma(s+1) \longrightarrow 1$ and $s \searrow 0$.

(3.3.13) Let β_t be the measure on the righthand side. Then

$$\widehat{\beta}_t(\boldsymbol{\xi}) = \int_{[0,\infty)} \widehat{\varphi_{\mathbf{0},2\tau\mathbf{I}}}(\boldsymbol{\xi}) \, \nu^{\frac{\alpha}{2}}(d\tau) = \widehat{\nu_t^{\frac{\alpha}{2}}}(|\boldsymbol{\xi}|^2) = e^{-t|\boldsymbol{\xi}|^\alpha}.$$

(3.3.14)

(**i**) Since

$$\int_0 \infty e^{\sqrt{-1}\xi\tau} h^\alpha(\tau)\, d\tau = \int_0^\infty e^{\sqrt{-1}\xi\tau} \nu_t^\alpha(d\tau) = \exp\left(-t\left(\frac{\xi^\alpha}{\sqrt{-1}}\right)\right) \quad \text{for } \xi \in \mathbb{R}$$

and both sides extend continuously to the open upper half of the complex plain as analytic functions, it follows that their extensions are equal. In particular, this shows that the Laplace transform of h_t^α is $e^{-t\lambda^\alpha}$. At the same time, a simple change of variables show that the Laplace transform of $\tau \rightsquigarrow t^{-\frac{1}{\alpha}} h_1(t^{-\frac{1}{\alpha}}\tau)$ is also equal to $e^{-t\lambda^\alpha}$, and therefore, $h_t^\alpha(\tau) = t^{-\frac{1}{\alpha}} h^\alpha(t^{-\frac{1}{\alpha}}\tau)$.

(**ii**) Follow the hint.