**Solution Set # 2**

(1.3.17) Because

$$M_\nu(\xi) = \int e^{\xi \frac{x-m}{\sigma}}\,\mu(dx) = e^{-\xi\frac{m}{\sigma}} M_\mu\big(\tfrac{\xi}{\sigma}\big),$$

$\Lambda_\nu(\xi) = -\frac{\xi m}{\sigma} + \Lambda_\mu\left(\frac{\xi}{\sigma}\right)$. Therefore $\Lambda_\mu(\xi) = \xi m + \Lambda_\nu(\sigma\xi)$, Image$(\Lambda'_\mu) = m + \sigma$Image$(\Lambda')$, and

$$I_\mu(x) = \sup_\xi\big(\xi(x-m) - \Lambda_\nu(\sigma\xi)\big) = \sup_\xi\left(\frac{\xi(x-m)}{\sigma} - \Lambda_\nu(\xi)\right) = I_\nu\left(\frac{x-m}{\sigma}\right).$$

Finally, if $x \in$ Image$(\Lambda'_\mu)$,

$$I_\mu(x) = \xi \iff \Lambda'_\mu(\xi) = x \iff m + \sigma\Lambda_\nu(\sigma\xi) = x \iff \Lambda_\nu(\sigma\xi) = \frac{x-m}{\sigma} \iff \sigma^{-1} I_\nu\left(\frac{x-m}{\sigma}\right) = \xi,$$

and therefore $I_\mu(x) = \sigma^{-1} I_\nu\left(\frac{x-m}{\sigma}\right)$.

(1.3.18)

(**i**) Since $\Lambda_\mu \le \Lambda_\nu$, $\xi x - \Lambda_\nu(\xi) \le \xi x - \Lambda_\mu(\xi)$ and therefore $I_\nu(x) \le I_\mu(x)$.

(**ii**) By Exercise 1.3.17, it suffices to handle the case when $m = 0$ and $\sigma = 1$. If $\mu$ is the Gaussian measure with mean 0 and variance 1, then $M_\mu(\xi) = e^{\frac{\xi^2}{2}}$ and therefore $I_\mu(x) = \sup_\xi\left(\xi x - \frac{x^2}{2}\right) = \frac{x^2}{2}$. If $\mu(\{a\}) = p \in (0,1)$ and $\mu(\{b\}) = q \equiv 1-p$, then $M_\mu(\xi) = pe^{\xi a} + qe^{\xi b}$. If $x \in (a,b)$, then

$$I_\mu(x) = -\log\frac{M_\mu(\xi)}{e^{\xi x}} \quad \text{where } M'_\mu(\xi) = xM_\mu(\xi).$$

Since

$$M'_\mu(\xi) = xM_\mu(\xi) \iff pae^{\xi a} + qbe^{\xi b} = pxe^{\xi a} + qxe^{\xi b} \iff e^\xi = \left(\frac{p(x-a)}{q(b-x)}\right)^{\frac{1}{b-a}},$$

$$\begin{aligned}\frac{M_\mu(\xi)}{e^{\xi x}} &= p\left(\frac{p(x-a)}{q(b-x)}\right)^{\frac{a-x}{b-a}} + q\left(\frac{p(x-a)}{q(b-x)}\right)^{\frac{b-x}{b-a}} \\ &= p^{\frac{b-x}{b-a}} q^{\frac{x-a}{b-a}}\left[\left(\frac{b-x}{x-a}\right)^{\frac{x-a}{b-a}} + \left(\frac{x-a}{b-x}\right)^{\frac{b-x}{b-a}}\right] = \left(\frac{q(b-a)}{x-a}\right)^{\frac{x-a}{b-a}}\left(\frac{p(b-a)}{b-x}\right)^{\frac{b-x}{b-a}},\end{aligned}$$

and so

$$I_\mu(x) = \frac{x-a}{b-a}\log\frac{x-a}{q(b-a)} + \frac{b-x}{b-a}\log\frac{b-x}{p(b-a)},$$

(1.3.29) Just follow the steps in the corrected version.

<u>(1.4.18)</u> Recall (cf. Theorem 5.1.6 in my *Essentails of Integration Theory for Analysis*) that for any Borel measure $\mu$ on $[0,\infty)$ and any non-decreasing $\varphi \in C^1\big([0,\infty);[0,\infty)br\big)$ with $\varphi(0)=0$,

$$\int \varphi(x)\,\mu(dx) = \int_{(0,\infty)} \varphi'(t)\mu\big(\varphi > t\big)\,dt.$$

Next, use the Monotone Convergence Theorem to see that

$$\mathbb{P}(X>t) = \lim_{s\searrow t}\mathbb{P}(X\ge s) = \lim_{s\searrow t}\mathbb{E}^{\mathbb{P}}\big[Y,\,X\ge s\big] = \mathbb{E}^{\mathbb{P}}\big[Y,\,X>t\big].$$

for $t>0$. Hence

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X^p\big] &= p\int_{[0,\infty)} t^{p-1}\mathbb{P}(X>t)\,dt \le p\int_{[0,\infty)} t^{p-2}\mathbb{E}^{\mathbb{P}}\big[Y,\,X>t\big]\,dt \\ &= \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}\big[X^{p-1}Y\big] \le \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}\big[X^p\big]^{1-\frac{1}{p}}\mathbb{E}^{\mathbb{P}}\big[Y^p\big]^{\frac{1}{p}},\end{aligned}$$

where the last inequality is Hölder's. If $\mathbb{E}^{\mathbb{P}}[X^p]<\infty$, this completes the proof. In general, let $R\in(0,\infty)$ and replace $X$ by $X\wedge R$. Since $\mathbb{P}(X\wedge R\ge t)\le \mathbb{E}^{\mathbb{P}}\big[Y,\,X\wedge R\ge t\big]$, the Monotone Convergence Theorem plus the preceding shows that

$$\mathbb{E}^{\mathbb{P}}\big[X^p\big]^{\frac{1}{p}} = \lim_{R\nearrow\infty}\mathbb{E}^{\mathbb{P}}\big[(X\wedge R)^p\big]^{\frac{1}{p}} \le \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}\big[Y^p\big]^{\frac{1}{p}}.$$

<u>(1.4.23)</u> For any $c\in\mathbb{R}$,

$$\mathbb{E}^{\mathbb{P}}\big[X-c,\,X\ge c\big] = \int_{(0,\infty)}\mathbb{P}(X-c>t)\,dt = \int_{(c,\infty)}\mathbb{P}(X>t)\,dt = \int_{(c,\infty)}\mathbb{P}(X\ge t)\,dt,$$

since $\mathbb{P}(X>t)=\mathbb{P}(X\ge t)$ for Lebesgue a.e. $t\in\mathbb{R}$. Similarly,

$$\mathbb{E}^{\mathbb{P}}\big[c-X,\,X\le c\big] = \int_{(-\infty,c)}\mathbb{P}(X\le t)\,dt = .$$

Thus,

$$\mathbb{E}^{\mathbb{P}}\big[|X-c|\big] = \int_{(c,\infty)}\mathbb{P}(X\ge t)\,dt + \int_{(-\infty c)}\mathbb{P}(X\le t)\,dt.$$

Now suppose that $a<b$. Then

$$\mathbb{E}^{\mathbb{P}}\big[|X-b|\big]-\mathbb{E}^{\mathbb{P}}\big[|X-a|\big] = \int_{[a,b)}\mathbb{P}(X\le t)\,dt - \int_{(a,b]}\mathbb{P}(X\ge t)\,dt = \int_a^b\big[\mathbb{P}(X\le t)-\mathbb{P}(X\ge t)\big]\,dt,$$

where final integral is a Riemann integral. By reversing the roles of $a$ and $b$, one sees that

$$\mathbb{E}^{\mathbb{P}}\big[|X-b|\big]-\mathbb{E}^{\mathbb{P}}\big[|X-a|\big] = \int_a^b\big[\mathbb{P}(X\le t)-\mathbb{P}(X\ge t)\big]\,dt$$

for all $a,b\in\mathbb{R}$. Finally, if $\alpha\in\text{med}(X)$ and $c\ge\alpha$, then $\mathbb{P}(X\le t)\ge\frac{1}{2}\le\mathbb{P}(X\ge t)$ for $t\in[\alpha,c]$, and so $\mathbb{E}^{\mathbb{P}}[|X-c|]\ge\mathbb{E}^{\mathbb{P}}[|X-\alpha|]$. Similarly, if $c<\alpha$, then the same inequality holds, and therefore $\mathbb{E}^{\mathbb{P}}[|X-\alpha|]\le\mathbb{E}^{\mathbb{P}}[|X-c|]$ for all $c\in\mathbb{R}$. Conversely, if $\mathbb{E}^{\mathbb{P}}[|X-\alpha|]\le\mathbb{E}^{\mathbb{P}}[|X-c|]$ for all $c\in\mathbb{R}$, then

$$\int_\alpha^c\mathbb{P}(X\le t)\,dt \ge \int_\alpha^c\mathbb{P}(X\ge t)\,dt \quad\text{for all } c>\alpha,$$

Thus there exists $\{t_n:\,n\ge1\}\subseteq(\alpha,\infty)$ such that $t_n\searrow\alpha$ and $\mathbb{P}(X\le t_n)\ge\mathbb{P}(X\ge t_n)$ for all $n\ge1$, and so

$$\mathbb{P}(X\le\alpha) = \lim_{n\to\infty}\mathbb{P}(X\le t_n) \ge \lim_{n\to\infty}\mathbb{P}(X\ge t_n) = \mathbb{P}(X>\alpha),$$

which implies that $\mathbb{P}(X\le\alpha)\ge\frac{1}{2}$. Similarly, by considering $c<\alpha$, one concludes that there exists $\{t_n:\,n\ge1\}\subseteq(-\infty,\alpha)$ such that $t_n\nearrow\alpha$ and $\mathbb{P}(X\le t_n)\le\mathbb{P}(X\ge t_n)$ for all $n\ge1$. Hence, after passing to the limit as $n\to\infty$, $\mathbb{P}(X<\alpha)\le\mathbb{P}(X\ge\alpha)$, and so $\mathbb{P}(X\ge\alpha)\ge\frac{1}{2}$.

(1.4.25) For each $n \geq 1$, set $A_n = \{|S_n| \geq 2t \text{ and } |S_m| < 2t \text{ for } 1 \leq m < n\}$. Then the $A_n$'s are mutually disjoint and $\{\max_{1\leq m\leq n} |S_m| \geq 2t\} = \bigcup_{m=1}^n A_m$. In addition, for $1 \leq m \leq n$, $A_m$ is independent of $S_n - S_m$, $\{|S_n| \geq t\} \supseteq A_m \cap \{|S_n - S_m| \leq t\}$, and therefore

$$\begin{aligned}
&\left(1 - \max_{1\leq m\leq n} \mathbb{P}\big(|S_n - S_m| > t\big)\right) \mathbb{P}\left(\max_{1\leq m\leq n} |S_m| \geq 2t\right) \\
&\quad = \min_{1\leq m\leq n} \mathbb{P}\big(|S_n - S_m| \leq t\big) \sum_{m=1} \mathbb{P}(A_m) \\
&\quad \leq \sum_{m=1}^{n} \mathbb{P}\big(A_m \cap \{|S_n - S_m| \leq t\big) \leq \mathbb{P}(|S_n| \geq t).
\end{aligned}$$

(1.4.27)

(**i**) Only the initial estimate needs comment. To prove it, note that, for any $t \geq 0$,

$$\mathbb{P}(Y \geq t) \geq \mathbb{P}(X \geq t + \alpha \;\&\; X' \leq \alpha) \geq \tfrac{1}{2}\mathbb{P}(X \geq t + \alpha)$$

and

$$\mathbb{P}(Y \leq -t) \geq \mathbb{P}(X \leq -t + \alpha \;\&\; X' \geq \alpha) \geq \tfrac{1}{2}\mathbb{P}(X \geq -t + \alpha).$$

(1.4.28) Just follow the steps given.