Solution Set # 1

(1.1.9) (**i**) This is really just an application of Fubini's Theorem. Namely, let μ_i be the distribution of X_i. Then $\mathbb{P}\big((X_1, X_2) \in \Gamma\big) = \mu_1 \times \mu_2(\Gamma)$ for $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$, and so, by Fubini's Theorem,

$$\mathbb{E}^{\mathbb{P}}\big[F(X_1, X_2)\big] = \int_{E_2} \left(\int_{E_2} F(x_1, x_2)\, \mu_1(dx_1) \right) \mu_2(dx_2) = \mathbb{E}^{\mathbb{P}}\big[f(X_2)\big]$$

when F is bounded. When F is non-negative, the same conclusion follows from Tonelli's Theorem. Thus the general case follows by linearity.

(**ii**) First, by writing $X_i = X_i^+ - X_i^-$, reduce to the case when the X_i's are non-negative. Next, assuming non-negativity, use induction and (**i**) to complete the proof.

(**iii**) Because E is second countable, the Borel field $\mathcal{B}_{E^2}$ equals $\mathcal{B}_E \times \mathcal{B}_E$. In particular, because it is closed in E^2, the diagonal $\Delta = \{(x, y) \in E^2 : x = y\} \in \mathcal{B}_E \times \mathcal{B}_E$. Thus, if μ and ν are Borel probability measures on E and if μ is non-atomic, then

$$\mu \times \nu(\Delta) = \int \left(\int \mathbf{1}_\Delta(x, y)\, \mu(dx) \right) \nu(dy) = \int \mu(\{y\})\, \nu(dy) = 0.$$

Applying this to the distributions of X_m and X_n for $m \neq n$, one sees that $\mathbb{P}(X_m = X_n) = 0$ and therefore that

$$\mathbb{P}\big(X_m = X_n \text{ for some } m \neq n\big) \le \sum_{m=1}^{\infty} \sum_{n \neq m} \mathbb{P}(X_m = X_n) = 0.$$

(1.1.12) The first part is Theorem 2.1.13 in my book *Essentials of Integration Theory for Analysis*. To prove the second part, let $n \ge 2$ and distinct $i_1, \dots, i_n$ be given, let $\mathcal{F}_1$ be the set of $A_1 \in \mathcal{F}$ such that (1.1.1) holds for all $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$. Show that $\mathcal{F}_1$ is a λ-system that contains $\mathcal{C}_1$, and conclude that $\mathcal{F}_1 \supseteq \sigma(\mathcal{C}_1)$. Next, given $2 \le m \le n$, assume that (1.1.1) holds if $A_i \in \sigma(\mathcal{C}_i)$ for $1 \le i < m$ and $A_i \in \mathcal{C}_i$ for $m \le i \le n$, and take $\mathcal{F}_m$ to be the set of $A_m \in \mathcal{F}$ such that (1.1.1) holds whenever $A_i \in \sigma(\mathcal{C}_i)$ for $1 \le i < m$ and $A_i \in \mathcal{C}_i$ for $m < i \le n$. Show that $\mathcal{F}_m$ is a λ-system that contains $\mathcal{C}_m$ and therefore $\sigma(\mathcal{C}_m)$. Now apply induction to complete the proof.

(1.1.13)

(**i**) The "only if" assertion is trivial. To prove the "if" assertion, first observe that

$$(*) \qquad \mathbb{E}^{\mathbb{P}}\big[f_1(X_1) \cdots f_n(X_n)\big] = \mathbb{E}^{\mathbb{P}}\big[f_1(X_1)\big] \cdots \mathbb{E}^{\mathbb{P}}\big[f_n(X_n)\big]$$

for all bounded, Borel measurable $f_1, \dots, f_n$ if it holds for continuous ones. Indeed, this follows from the fact that if $f_1, \dots, f_n$ are bounded, Borel measurable functions, then, for each $1 \le m \le n$ one can find a uniformly bounded sequences $\{\varphi_{m,k} : k \ge 1\} \subseteq C(\mathbb{R}; \mathbb{R})$ such that $\varphi_{m,k} \circ X_m \longrightarrow f_m \circ X_m$ $\mathbb{P}$-almost surely. Next, note that, by the hypothesis and linearity, (*) holds when the f_m are polynomials. Finally, assume that the f_m's are continuous, and apply Weirstrasse to find sequences $\{p_{m,k} : k \ge 1\}$ of polynomials such that $p_{m,k} \longrightarrow f_m$ uniformly on an interval in which the X_m's take their values, and conclude that (*) holds.

(**ii**) Just follow the hint.

(1.1.14) See § 8.2.3 in my book *Essential of Integration Theory for Analysis*.

(1.1.16) There is very little to do here. The X_i's are $\mathbb{P}$-indendent if and only if

$$\mathbf{X}_*\mathbb{P}\big(\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\big)=\prod_{m=1}^n\mathbb{P}\big(X_{i_m}\in\Gamma_{i_m}\big)$$

for all $n\geq 2$, distinct $i_1,\dots,i_n\in\mathcal{I}$, and $\Gamma_{i_1}\in\mathcal{B}_{i_1},\dots,\Gamma_{i_n}\in\mathcal{B}_{i_n}$.

(1.2.12) & (1.2.13) Because

$$\mathbb{P}(S_n=k)=e^{-n}\sum_{k_1+\cdots+k_n=k}\frac{t^k}{k_1!\cdots k_n!}=\frac{t^ke^{-n}}{k!}=\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1\dots k_n}=\frac{(tn)^ke^{-n}}{k!},$$

S_n is a Poisson random variable with mean value nt. Hence, $e^{-nt}\sum_{0\leq k\leq nT}=\mathbb{P}(\overline{S_n}\leq T)$. By the Weak Law of Large Numbers, for each T

$$\mathbb{P}(\overline{S_n}\leq T)\longrightarrow\begin{cases}1 & \text{if } T>t\\ 0 & \text{if } T<t,\end{cases}$$

and this completes (1.2.12). To handle (1.2.13), first remember that a function of bounded variation is the difference of two non-decreasing funcions. Thus, it is enough to treat the case when F is non-decreasing. Given a non-decreasing F, let μ be the Borel measure on $[0,\infty)$ determined by $\mu([0,t])=F(t)$. Next, set

$$p_n^T(t)=e^{-nt}\sum_{0\leq k\leq nT}\frac{(nt)^k}{k!}.$$

Then $0\leq p_n^T(t)\leq 1$ and, by (1.2.12), $\lim_{n\to\infty}p_n^T(t)$ equals 1 if $t<T$ and 0 if $t>T$. Finally, suppose that F is continuous at T. Then $\mu(\{T\})=0$, and therefore

$$\sum_{k\leq nT}\frac{(-nt)^k}{k!}[D^k\varphi](n)=\int_{(0,\infty)}p_n^T(t)\,\mu(dt)=\int_{[0,T)}p_n^T(t)\,\mu(dt)+\int_{(T,\infty)}p_n^T(t)\,\mu(dt).$$

By Lebesgue's Dominated Convergence Theorem

$$\int_{(0,T)}p_n^T(t)\,\mu(dt)\longrightarrow\mu\big([0,T))\big)=\mu\big([0,T]\big)=F(T)$$

and $\int_{(T,\infty)}p_n^T(t)\,\mu(dt)\longrightarrow 0$ as $n\to\infty$.