Equivalence of Norms in Finite Dimensions

**Theorem.** Let $X$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\{b_1, \ldots, b_N\}$ a basis for $X$. For $1 \leq m \leq N$, define $a_m : X \to \mathbb{C}$ so that
\[
x = \sum_{n=1}^{N} a_n(x)b_n \text{ for all } x \in X.
\]
Then for any norm $\| \cdot \|$ on $X$ there exist $0 < c < C < \infty$ such that
\[
c \sum_{n=1}^{N} |a_n(x)| \leq \|x\| \leq C \sum_{n=1}^{N} |a_n(x)|.
\]
Hence the norm topology is the same for all norms.

**Proof.** Define $\|x\| = \sum_{n=1}^{N} |a_n(x)|$. Then $\| \cdot \|$ is a norm and, by the triangle inequality,
\[
\|x\| \leq \sum_{n=1}^{N} |a_n(x)||b_n| \leq C\|x\|,
\]
where $C = \max\{\|b_n\| : 1 \leq n \leq N\}$. Hence, by the triangle inequality, $\|x\| - \|y\| \leq C\|x - y\|$, and so $x \mapsto \|x\|$ is continuous with respect of the $\| \cdot \|$-topology. Next set $K = \{x : \|x\| = 1\}$. If $\{x_k : k \geq 1\} \subseteq K$, then $|a_n(x_k)| \leq 1$ for all $1 \leq n \leq N$, and so there exists a subsequence $\{x_{k_j} : j \geq 1\}$ and $a_n \in \mathbb{C}$ such that $a_n(x_{k_j}) \to a_n$ for each $1 \leq n \leq N$. Hence, if $x = \sum_{n=1}^{N} a_nb_n$, then $x \in K$ and $\|x_{k_j} - x\| \to 0$, and so $K$ is compact in the $\| \cdot \|$-topology. Since $\| \cdot \|$ is continuous and positive on $K$, this proves that there is a $c > 0$ such that $\|x\| \geq c$ for all $x \in K$. Hence, for any $x \neq 0$,
\[
\|x\| \geq c. \quad \Box
\]

Here is another proof based on the following lemma. This one doesn’t require compactness and therefore works even if the vector space is over a field like the rational numbers.

**Lemma.** If $(X, \| \cdot \|)$ is a normed space and $\Lambda$ is a linear functional on $X$, then $\Lambda$ is continuous if and only if its null space is closed.

**Proof.** If $\Lambda$ is continuous, then it is obvious that its null space is closed. Now suppose that $L \equiv \text{Null}(\Lambda)$ is closed. If $L = X$, then there is nothing to do. Thus, assume that $b \in X \setminus L$. Then $x - \frac{\Lambda(x)}{\Lambda(b)}b \in L$ for every $x \in X$. Hence, if $\{x_k : k \geq 1\} \subseteq B(0,1)$ and $1 \leq |\Lambda(x_k)| \to \infty$, then
\[
- \frac{b}{\Lambda(b)} = \lim_{k \to \infty} x_k - \frac{\Lambda(x_k)}{\Lambda(b)}b \in L,
\]
and therefore we would have the contradiction that $b \in L$. \(\Box\)

**Second Proof of Theorem.** First observe that the right hand inequality in (1) is a trivial consequence of the triangle inequality.

To prove the left hand inequality, we work by induction on the dimension $N$ of $X$. If $N = 1$ and $b \in X \setminus \{0\}$, then $x = a(x)b$, and so $\|x\| = |a(x)||b|$. Now assume assume that $N \geq 2$ and that the result holds for $(N-1)$-dimensional spaces. Let $\{b_1, \ldots, b_N\}$ be a basis for $X$. For each $1 \leq m \leq N$, $x \mapsto a_m(x)$ is linear and $L_m \equiv \text{Null}(a_m) = \text{span}(\{b_n : 1 \leq n \neq m \leq N\})$. Hence (1) with $N-1$ replacing $N$ holds for the restriction $a_m$ to $L_m$, and therefore $L_m$ is closed. By the preceding lemma, this proves that $a_m$ is a continuous and therefore the existence of a $c > 0$ for which the left hand inequality of (1) holds. \(\Box\)

**Corollary.** If $X$ is a normed vector space over $\mathbb{C}$, then every finite dimensional subspace is closed and locally compact.

**Proof.** Let $L$ be a subspace with basis $\{b_1, \ldots, b_N\}$. Then, by (1) applied to $L$, the map $(\xi_1, \ldots, \xi_N) \in \mathbb{C}^N \mapsto \sum_{m=1}^{N} \xi_mb_m \in L$ is a homeomorphism from $\mathbb{C}^N$ onto $L$. \(\Box\)