2.1.19:

(i) The preservation of unions is obvious, as is the fact that $\Phi^{-1}$ preserves differences. To see that $\Phi$ may not preserve differences, consider the case when $E$ contains at least two elements and $\Phi$ is constant. In general, $\Phi(B \setminus A) \supseteq \Phi(B) \setminus \Phi(A)$. When $\Phi$ is one-to-one and $x \in B \setminus A$, $\Phi(x) \in B$ but $\Phi(x) \notin \Phi(A)$, since, if it were in $\Phi(A)$ then there would exist a $y \in A$ such that $\Phi(y) = \Phi(x)$, which, because $x \notin A$, would mean that $\Phi$ is not one-to-one.

(ii) Consider the set $\Sigma'$ of $\Gamma \subseteq E$ such that $\Phi^{-1}(\Gamma) \in B$. By (i), $\Sigma'$ is a $\sigma$-algebra, and, by assumption, $\Sigma' \supseteq \mathcal{C}'$. Hence, $\Sigma' \supseteq \sigma(\mathcal{C}')$. In the case of topological space when $\Phi$ is continuous, take $\mathcal{C}'$ to be the set of open subsets of $E$. When $\Phi$ is one-to-one, then, by the last part of (i), the set $\Sigma$ of $\Gamma \subseteq E$ such that $\Phi(\Gamma) \in B'$ is a $\sigma$-algebra, and therefore $\Sigma \supseteq \mathcal{C} \implies \Sigma \supseteq \sigma(\mathcal{C})$.

(iii) This part is an essentially immediate consequence of (i). Namely, since $\emptyset = \Phi^{-1}(\emptyset)$, $\mu'(\emptyset) = 0$. In addition, if $\{\Gamma'_n : n \geq 1\} \subseteq B'$ is a sequence of mutually disjoint sets, then $\{\Phi^{-1}(\Gamma'_n) : n \geq 1\}$ is a sequence of mutually disjoint elements of $B$, and therefore

$$\mu'(\bigcup_{n=1}^{\infty} \Gamma'_n) = \mu\left(\Phi^{-1}\left(\bigcup_{n=1}^{\infty} \Gamma'_n\right)\right) = \mu\left(\bigcup_{n=1}^{\infty} \Phi^{-1}(\Gamma'_n)\right) = \sum_{n=1}^{\infty} \mu'(\Gamma'_n).$$

In view of the second part of (i), the case when $\Phi$ is one-to-one and $\Phi$ takes elements of $B$ to elements of $B'$ is handled in exactly the same way.

2.1.21: When $n = 2$, (2.1.6) implies the result. Now assume the result for some $n \geq 2$. By (2.1.6),

$$\mu(\Gamma_1 \cup \cdots \cup \Gamma_{n+1}) = \mu(\Gamma_1 \cup \cdots \cup \Gamma_n) + \mu(\Gamma_{n+1}) - \mu((\Gamma_1 \cap \Gamma_{n+1}) \cup \cdots \cup (\Gamma_n \cap \Gamma_{n+1})),$$

and by induction hypothesis,

$$\mu(\Gamma_1 \cup \cdots \cup \Gamma_n) = -\sum_{F} (-1)^{\text{card} F} \mu(\Gamma_F)$$

and

$$-\mu((\Gamma_1 \cap \Gamma_{n+1}) \cup \cdots \cup (\Gamma_n \cap \Gamma_{n+1})) = \sum_{F} (-1)^{\text{card} F} \mu(\Gamma_F \cap \Gamma_{n+1}),$$

where in both cases $F$ runs over non-empty subsets of $\{1, \ldots, n\}$. Note that the sums in these last two can be written, respectively, as

$$-\sum_{F \supseteq (n+1)} (-1)^{\text{card} F} \mu(\Gamma_F) \quad \text{and} \quad -\sum_{F \not\supseteq (n+1)} (-1)^{\text{card} F} \mu(\Gamma_F),$$

where the sums now are over non-empty subsets of $\{1, \ldots, n+1\}$. Hence, after combining these, we see that the result holds for $n + 1$. 

Turning to the second part, note that if \( \text{card}(F) = m \) then \( \Gamma_F \) contains \( (n - m)! \) permutations and therefore that \( \mu(\Gamma_F) = \frac{(n-m)!}{n!} \). Furthermore, there are \( \binom{n}{m} \) \( F \)'s with \( m \) elements. Hence,

\[
\mu(\Gamma_1 \cup \cdots \cup \Gamma_n) = - \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{(n-m)!}{n!} = - \sum_{m=1}^{n} (-1)^m m!
\]

2.1.22: Obviously, if \( x \) is in all but finitely many \( B_n \)'s then it is in infinitely many \( B_n \)'s, and so \( \text{lim}_{n \to \infty} B_n \subseteq \text{lim}_{n \to \infty} B_n \). Moreover, because \( \text{lim}_{n \to \infty} B_n \) and \( \text{lim}_{n \to \infty} B_n \) are obtained from the \( B_n \)'s by taking countable unions and intersections, they are in \( B \) if the \( B_n \)'s are. Next, because \( \mu(\bigcap_{n=m}^{\infty} B_n) \leq \mu(B_m) \) for each \( m \) and \( \bigcap_{n=m}^{\infty} B_n \neq \text{lim}_{n \to \infty} B_n \), (2.1.23) follows from (2.1.10). Similarly, because \( \bigcup_{n=m}^{\infty} B_n \neq \text{lim}_{n \to \infty} B_n \), (2.1.24) follows from (2.1.11).

Finally, when \( \text{lim}_{n \to \infty} B_n \) exists and \( \mu(\bigcup_{n=1}^{\infty} B_n) \) is \( \infty \), the preceding leads to

\[
\mu \left( \text{lim}_{n \to \infty} B_n \right) \leq \text{lim}_{n \to \infty} \mu(B_n) \leq \text{lim}_{n \to \infty} \mu(B_n) \leq \mu \left( \text{lim}_{n \to \infty} B_n \right).
\]

2.1.27: To prove the first part, suppose not. Then there exists an \( \epsilon > 0 \) such that for each \( n \geq 1 \) there is a \( B_n \in B \) for which \( \nu(B_n) \leq 2^{-n} \) but \( \mu(B_n) \geq \epsilon \). Hence, if \( B = \text{lim}_{n \to \infty} B_n \), then, by Exercise 2.1.22, \( \nu(B) = 0 \), and by (2.1.24),

\[
\mu(B) = \lim_{m \to \infty} \mu \left( \bigcup_{n=m}^{\infty} B_n \right) \geq \text{lim}_{m \to \infty} \mu(B_m) \geq \epsilon,
\]

which contradicts \( \mu \ll \nu \). Turning to the second part, note that if \( \mu \ll \nu \) then the first part implies that for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( \nu(G) < \delta \Rightarrow \mu(G) < \epsilon \) for all \( G \in \mathcal{G}(E) \). To go the other direction, for each \( n \geq 1 \), use the hypothesis about \( \nu \) to choose \( \delta_n > 0 \) so that \( \nu(G) < \delta_n \Rightarrow \mu(G) \leq 2^{-n} \). Now suppose that \( \nu(B) = 0 \), and, for each \( n \geq 1 \), use regularity to choose \( B \in \mathcal{G}(E) \cap B \subseteq B \) so that \( \nu(G_n) < \delta_n \). Then \( \mu(B) \leq \mu(G_n) \leq 2^{-n} \) for all \( n \geq 1 \), and so \( \mu(B) = 0 \).

2.1.28: First assume that \( \mu \perp \nu \), and choose \( B \in \mathcal{B}_E \) so that \( \nu(B) = 0 = \mu(B^c) \). Given \( \delta > 0 \), use the regularity of \( \nu \) to find a \( \mathcal{G}(E) \cap G \supseteq B \) such that \( \nu(G) < \delta \), and note that \( \mu(G^c) \leq \mu(B^c) = 0 \). Conversely, if for each \( \delta > 0 \) there exists a \( G \in \mathcal{G}(E) \) such that \( \nu(G) < \delta \) and \( \mu(G^c) = 0 \), choose for each \( n \geq 1 \) a \( G_n \in \mathcal{G}(E) \) so that \( \nu(G_n) \leq 2^{-n} \) and \( \mu(G_n^c) = 0 \), and set \( B = \text{lim}_{n \to \infty} G_n \). By Exercise 2.1.22, \( \nu(B) = 0 \), and, since \( B^c = \lim_{n \to \infty} G_n^c \subseteq \bigcup_{n=1}^{\infty} \cup_1 = G_n^c \) and \( \mu(B^c) = 0 \).

2.2.32: As the intersection of closed sets, \( C \) is also closed. Moreover, for each \( n \geq 1 \), \( \lambda_R(C_n) = \frac{2}{3} \lambda_R(C_{n-1}) \), and therefore \( \lambda_R(C_n) = \left( \frac{2}{3} \right)^n \) and \( \lambda_R(C) = \lim_{n \to \infty} \lambda_R(C_n) = 0 \). To see that \( \Psi \) is one-to-one, suppose that \( \Psi(\omega) = \Psi(\omega') \) for some \( \omega, \omega' \in \Omega \). If \( \omega \neq \omega' \), then there is an \( \ell \geq 0 \) such that \( \omega(i) = \omega'(i) \) for \( 1 \leq i \leq \ell \) and \( \omega(\ell + 1) \neq \omega'(\ell + 1) \). Without loss in generality, assume that \( \omega(\ell + 1) = 1 \) and \( \omega'(\ell + 1) = 0 \), and derive the contradiction that

\[
\Psi(\omega) - \Psi(\omega') = \frac{2}{3^{\ell+1}} \left( 1 + \sum_{i=2}^{\infty} \frac{\omega(\ell + i) - \omega'(\ell + i)}{3^{i-1}} \right) > 0.
\]
Finally, it is clear that \([0,1] \setminus \mathcal{C}_n\) equals the union over \(1 \leq k \leq 2^n\) of the intervals \(\{(2k - 1)3^{-n}, 2k3^{-n}\}\). To check that \(\Psi(\omega) > (2k - 1)3^{-n} \implies \Psi(\omega) \geq 2k3^{-n}\), suppose that \(\Psi(\omega) < 2k3^{-n}\). Then \(2\sum_{m=1}^{n} 3^{n-i}\omega(i) < 2k\), and therefore, since both sides are even integers, \(2\sum_{m=1}^{n} 3^{n-i}\omega(i) \leq 2k - 2\), which means that
\[
\Psi(\omega) \leq (2k - 2)3^{-n} + \sum_{i=n+1}^{\infty} \frac{2}{3^i} \leq (2k - 1)3^{-n}.
\]

2.2.38 and 2.2.39: That absolute continuity implies uniform continuity is obvious, since, for \(x < y\), one can choose \(a_1 = x, b_1 = y\), and take \(a_n = 0 = b_n\) for \(n \geq 2\). Now assume that \(F\) is bounded and that \(\lim_{x \to -\infty} F(x) = 0\). If \(\mu_F \ll \lambda_{\mathbb{R}}\), then, by Exercise 2.1.17, for each \(\epsilon > 0\) there exists a \(\delta_\epsilon > 0\) such that \(\lambda_{\mathbb{R}}(\Gamma) < \delta_\epsilon \implies \mu_F(\Gamma) < \epsilon\). In particular, \(\mu_F\) is non-atomic and therefore \(F\) is continuous. Furthermore, if \(\{(a_n, b_n) : n \geq 1\}\) are mutually disjoint and \(\sum_{n=1}^{\infty} (b_n - a_n) < \delta_\epsilon\), then
\[
\sum_{n=1}^{\infty} (F(b_n) - F(a_n)) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n)) = \mu_F \left( \bigcup_{n=1}^{\infty} (a_n, b_n) \right) < \epsilon,
\]
and so \(F\) is absolutely continuous. Conversely, if \(F\) is absolutely continuous, for a given \(\epsilon > 0\) choose \(\delta > 0\) accordingly, and suppose that \(G \in \mathcal{G}(\mathbb{R})\) with \(\lambda_{\mathbb{R}}(G) < \delta\). Because there exists a sequence \(\{(a_n, b_n) : n \geq 1\}\) of mutually disjoint open intervals such that
\[
G = \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \lambda_{\mathbb{R}}(G) = \sum_{n=1}^{\infty} (b_n - a_n),
\]
and, because \(F\) is continuous,
\[
\mu_F(G) = \sum_{n=1}^{\infty} \mu_F((a_n, b_n)) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) < \epsilon.
\]
Hence, by the last part of Exercise 2.1.27, \(\mu_F \ll \lambda_{\mathbb{R}}\).

Turning to Exercise 2.2.39, first suppose that, for each \(\delta > 0\) there exist mutually disjoint intervals \(\{(a_n, b_n) : n \geq 1\}\) such that \(\sum_{n=1}^{\infty} (b_n - a_n) < \delta\) and \(F(\infty) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n))\). For a given \(\delta > 0\), choose \(\{(a_n, b_n) : n \geq 1\}\) accordingly and set \(G = \bigcup_{n=1}^{\infty} (a_n, b_n)\). Then \(\lambda_{\mathbb{R}}(G) < \delta\),
\[
\mu_F(\mathbb{R}) = F(\infty) = \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) = \mu_F(G),
\]
and so \(\mu_F(G_{\mathbb{C}}) = 0\). Hence, by Exercise 2.1.28, \(\mu_F \perp \lambda_{\mathbb{R}}\). Conversely, if \(\mu_F \perp \lambda_{\mathbb{R}}\), then, by that same exercise, for each \(\delta > 0\) there is a \(G \in \mathcal{G}(\mathbb{R})\) such that \(\lambda_{\mathbb{R}}(G) < \delta\) and \(\mu_F(G_{\mathbb{C}}) = 0\). By writing \(G\) as the countable union of disjoint open intervals, this quickly translates into a proof that \(F\) is singular.
(2.2.40): To check the measurability of $f$, first observe that both $\pi_0$ and $\pi_1$ are measurable and then that $\Phi_0 \equiv \Phi \circ \pi_0 \circ \hat{\Phi}^{-1}$ and $\Phi_1 \equiv \Phi \circ \pi_1 \circ \hat{\Phi}^{-1}$ are measurable. Next, use Exercise 2.1.19 to see that the set of $\Gamma \in B_{[0,1]^2}$ for which $f^{-1}(\Gamma) \in B_{[0,1]}$ is a $\sigma$-algebra and therefore, since $B_{\mathbb{R}^2}$ is generated by rectangles, that it suffices to note that

$$f^{-1}(\Gamma_1 \times \Gamma_2) = \Phi_0^{-1}(\Gamma_1) \cap \Phi_1^{-1}(\Gamma_2) \in B_{[0,1]}$$

for all $\Gamma_1, \Gamma_2 \in B_{[0,1]}$. Finally, to see that $\lambda_{[0,1]^2} = f_* \lambda_{[0,1]}$, use (2.2.27), (2.2.28), Theorem 2.2.25, and the preceding to justify the calculations

$$\lambda_{\mathbb{R}}(f^{-1}(\Gamma_1 \times \Gamma_2)) = \lambda_{\mathbb{R}}(\Phi_0^{-1}(\Gamma_1) \cap \Phi_1^{-1}(\Gamma_2))$$

$$= \lambda_{\mathbb{R}}(\{x \in [0,1] : \hat{\Phi}^{-1}(x) \in (\Phi \circ \pi_0)^{-1}(\Gamma_1) \cap (\Phi \circ \pi_1)^{-1}(\Gamma_2)\})$$

$$= \beta_2(\pi_0^{-1}(\Phi^{-1}(\Gamma_1)) \cap \pi_1^{-1}(\Phi^{-1}(\Gamma_2)))$$

$$= \beta_2(\Phi^{-1}(\Gamma_1)) \beta_2(\Phi^{-1}(\Gamma_2)) = \lambda_{[0,1]}(\Gamma_1) \lambda_{[0,1]}(\Gamma_2).$$

In particular, this proves that $f_* \lambda_{[0,1]}(I) = \text{vol}(I)$ for all rectangles $I \subseteq [0,1]^2$. Now apply the uniqueness assertion in Theorem 2.2.13.