Unramified Hilbert Modular Forms, with Examples Relating to Elliptic Curves

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Let F be a totally real number field.

Goal. Compute the space, $S_2(F)$, of unramified Hilbert cusp forms of parallel weight 2 for F.

By compute we mean, decompose this space into Hecke eigenspaces and compute the Hecke eigenvalues to any degree.

Use Jacquet-Langlands correspondence to turn computation into one involving quaternion algebras. Analogous to work of Pizer for $F = \mathbf{Q}$.

Assume that $h^+(F) = 1$ and that F has even degree over $\mathbf{Q}$.

We take

- R: ring of integers in F

- $\mathbf{B}$: quaternion algebra over F ramified only at the infinite primes of F

- $\mathcal{O}$: maximal order in $\mathbf{B}$

- X: set of right equivalent classes of left $\mathcal{O}$ ideals

Adelically if we set $\mathbf{G} = \mathbf{B}^{\times}$, an algebraic group over F, then after making appropriate identifications we can identify X as

$$X = K \setminus \mathbf{G}(\mathbf{A}_F^f)/\mathbf{G}(F)$$

where $K = \prod_{\mathfrak{p}} \mathrm{GL}(2, R_{\mathfrak{p}})$ and $\mathbf{A}_F^f$ is the ring of finite adeles.

Let S denote the space

$$S = \{f : X \to \mathbf{C}\}/\{\text{constant functions}\}.$$

There is a natural definition of Hecke operators on the space of functions on X which descends to this space. The action of $T_{\mathfrak{p}}$ is given by summing f over the left K cosets in $K g_{\mathfrak{p}} K$ where

$$g_{\mathfrak{p}} = \begin{pmatrix} \pi_{\mathfrak{p}} & \\ & 1 \end{pmatrix} \in \mathbf{G}(F_{\mathfrak{p}}) \hookrightarrow \mathbf{G}(\mathbf{A}_F^f).$$

By work of Shimizu and Jacquet-Langlands there is a Hecke equivarient isomorphism of $S_2(F)$ with the space S.

Our goal is now to compute the action of the Hecke operators on S.

Let $\mathbf{nr} : \mathbf{B} \to F$ denote the reduced norm.

For a left $\mathcal{O}$ ideal J let $\mathbf{nr}(J)_+ \in F$ be a totally positive generator for the fractional ideal of F generated by $\mathbf{nr}(J)$. For $x \in J$ define

$$\mathcal{N}_J(x) = \mathbf{nr}(x)/\mathbf{nr}(J)_+.$$

Let $H = \#X$ be the class number of $\mathcal{O}$ and let $\{J_1, \ldots, J_H\}$ be a set of representatives for X.

For each i, let e_i denote the number of elements of norm one in $\mathcal{O}_r(J_i)$, the right order of J_i.

For $\beta \in R^+$ we define, for $1 \leq i, j \leq H$,

$$b_{ij}(\beta) = \frac{1}{e_j} c_{\beta, J_j^{-1} J_i}$$

where, for any ideal J in $\mathbf{B}$, $c_{\beta,J}$ denotes the cardinality of

$$\{x \in J : \mathcal{N}_J(x) = \beta\}.$$

We define the $H \times H$ matrix $B(\beta) = (b_{ij}(\beta))$.

Proposition. Let $\mathfrak{p} = (\pi)$ be a prime ideal in R. Then the action of $T_{\mathfrak{p}}$ on the space of functions $\{f : X \to \mathbf{C}\}$ is given by $B(\pi)$.

In order to determine the action of $T_{\mathfrak{p}}$ on the space

$$S = \{f : X \to \mathbf{C}\}/\{\text{constant functions}\}$$

and hence on $S_2(F)$, we need to "isolate" the action on the constant functions.

If one takes

$$A = \begin{pmatrix} 1 & e_1/e_2 & e_1/e_3 & \dots & e_1/e_H \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

then for $\beta \in R^+$ we have

$$AB(\beta)A^{-1} = \begin{pmatrix} b(\beta) & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B'(\beta) & \\ 0 & & & \end{pmatrix}.$$

Tying everything together gives.

Theorem. Let $\{v_i\}$ be a basis for $\mathbf{C}^{H-1}$ of simultaneous eigenvectors for the family of matrices $\{B'(\beta)\}$. Then each v_i corresponds to an unramified Hilbert modular cuspidal eigenform f_i of parallel weight 2 whose eigenvalue with respect to the $\mathfrak{p}^{th}$ Hecke operator is the eigenvalue of v_i with respect to $B'(\pi)$ where $\pi \in R^+$ is a generator of $\mathfrak{p}$.

To compute the space $S_2(F)$ one needs to do the following.

- Find defining relations for $\mathbf{B}$ and a maximal order $\mathcal{O}$ in $\mathbf{B}$.

- Compute the class number of $\mathcal{O}$.

- Find ideal class representatives $\{J_i\}$.

Given $M \in \mathbf{R}$ one can then compute the matrices $B(\beta)$ for all $\beta \in R^+$ with $\mathrm{Tr}(\beta) \leq M$. Computing the matrices $B(\beta)$ amounts to computing the number of elements in $J_j^{-1} J_i$ of a given norm.

We say a few words about finding ideal class representatives. It is easy to manufacture left ideals of $\mathcal{O}$ when they are of a particular form.

Let $\alpha \in \mathbf{B} \setminus F$ then $K = F(\alpha)$ is a quadratic extension of F contained in $\mathbf{B}$.

Let I be an ideal in K then $J = \mathcal{O}I$ is a left $\mathcal{O}$ ideal. If I and I' are in the same ideal class in K then J and J' represent the same class in X. Moreover every element of X can be represented by such an ideal.

To check if J and J' represent the same ideal class one computes the first few terms of the sequences $(c_{\beta,J})$ and $(c_{\beta,J'})$.

For now assume that $F = \mathbf{Q}(\sqrt{m})$. If $h^+(F) = 1$ then $m = 2$ or m is a prime congruent to $1 \bmod 4$.

Proposition. Assume that $m > 5$ then

$$H = \frac{1}{48m}\sum_{u=1}^{m}\chi(u)u^2 + \frac{h(\mathbf{Q}(\sqrt{-m}))}{8} + \frac{h(\mathbf{Q}(\sqrt{-3m}))}{6}$$

where $\chi = \left(\frac{\cdot}{m}\right)$.

Proposition. Assume that $m \not\equiv 1 \bmod 8$ then $\mathbf{B} = (-1,-1)$ (i.e. $\mathbf{B} = F \oplus Fi \oplus Fj \oplus Fk$ with $k = ij$, $i^2 = j^2 = -1$, $ij = -ji$). If $m \equiv 5 \bmod 8$ then a maximal order is given by

$$\mathcal{O} = R\left[\frac{1+i+j+k}{2}, \frac{i+\theta j+(1+\theta)k}{2}, j, k\right]$$

where $\theta = (1+\sqrt{m})/2$.

Let f be a Hilbert cuspidal eigenform of parallel weight 2 with rational Hecke eigenvalues.

In most case one knows the existence of an elliptic curve attached to f. This is known if F has odd degree or if the automorphic representation associated to f is discrete series at some prime.

Conjecture. Let F be a totally real number field of even degree over $\mathbf{Q}$. Then to each unramified Hilbert modular eigenform f, over F and of parallel weight 2, which has rational Hecke eigenvalues one can attach an elliptic curve E_f defined over F with good reduction everywhere, such that the L-functions of E_f and f agree at each place of F.

We verify this conjecture for $F = \mathbf{Q}(\sqrt{509})$. As we shall see, there exists forms in $S_2(F)$ that do not arise via base change from $\mathbf{Q}$. It is not know how to attach curves to such forms.

We take $\mathbf{B} = (-1, -1)$ and $\mathcal{O}$ to be the maximal order in $\mathbf{B}$ as above. We compute $H = 24$, hence $\dim S_2(F) = 23$.

One finds $\mathcal{O}$ ideal class representatives using ideals from $F(\sqrt{-1})$, $F(\sqrt{-359})$ and $F(\sqrt{-383})$ which can all be found in $\mathbf{B}$.

We find that the characteristic polynomial of $B'(19+\theta)$ has three distinct rational roots and an irreducible factor of degree 20. Therefore there are three eigenforms in $S_2(F)$ which have rational Hecke eigenvalues.

π	$\pi \mid p$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
3	3	-4	-4	1
7	7	-6	-6	9
$11+\theta$	5	3	-2	-2
$12-\theta$	5	-2	3	-2
$12+\theta$	29	0	10	-5
$13-\theta$	29	10	0	-5
13	13	1	1	26
$14+\theta$	83	14	9	14
$15-\theta$	83	9	14	14
$15+\theta$	113	11	6	11
$16-\theta$	113	6	11	11
$17+\theta$	179	0	25	10
$18-\theta$	179	25	0	10
19	19	-12	-12	38
$20+\theta$	293	16	26	-9
$21-\theta$	293	26	16	-9
$22+\theta$	379	-20	20	-10
$23-\theta$	379	20	-20	-10

Gal$(F/\mathbf{Q})$ acts on the space of forms, permuting the Hecke eigenvalues via its action on primes of F. Using this we see that $\mathbf{v}_3$ is fixed under Galois and hence corresponds to an eigenform which arises via base change. Whereas $\mathbf{v}_1$ and $\mathbf{v}_2$ correspond to a pair of conjugate eigenforms which do not arise via base change from $\mathbf{Q}$.

One knows, by Cremona, of the existence of a pair of classical forms in $S_2(\Gamma_0(509), \chi)$ defined over $\mathbf{Q}(\sqrt{509})$ which give rise to an elliptic curve E with everywhere good reduction over F by the construction of Shimura. These forms base change to the form corresponding to $\mathbf{v}_3$ and E will be attached to this form.

By work of Pinch, there is an elliptic curve E/F with everywhere good reduction which is not a $\mathbf{Q}$-curve. Explicitly E is given by

$$y^2 - xy - \theta y = x^3 + (2+2\theta)x^2 + (162+3\theta)x + 71 + 34\theta.$$

Let f denote the form corresponding to the vector $\mathbf{v}_1$. One can readily compute that the first few Hecke eigenvalues of f agree with the corresponding numbers for E.

In order to show that E is attached to f we use Galois representations attached to both objects.

Fix a rational prime ℓ.

We have a representation $\sigma_E : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbf{Q}_\ell)$ given by the action of Galois on the ℓ-adic Tate module of E.

On the other hand we have a representation $\sigma_f : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbf{Q}_\ell)$ by work of Taylor, and independently of Blasius and Rogawski.

In order to conclude that E is attached to f we need to show that for some prime ℓ these representations are equivalent. For this one can take $\ell = 2$ and use a result due to Faltings and Serre.

Theorem. Let K be a global field, S a finite set of primes of K, and E a finite extension of $\mathbf{Q}_2$. Denote the maximal ideal in the ring of integers of E by $\mathfrak{p}$ and the compositum of all quadratic extensions of K unramified outside S by K_S. Suppose that

$$\rho_1, \rho_2 : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$$

are continuous representations, unramified outside S, and furthermore satisfying

1. $\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 \bmod \mathfrak{p}$
2. $\det \rho_1 \equiv \det \rho_2 \bmod \mathfrak{p}$
3. There exists a set T of primes of K, disjoint from S, for which

(i) The image of the set $\{\mathrm{Fr}_t : t \in T\}$ in the $\mathbf{Z}/2\mathbf{Z}$-vector space $\mathrm{Gal}(K_S/K)$ is non-cubic.
(ii) $\mathrm{Tr}\,\rho_1(\mathrm{Fr}_t) = \mathrm{Tr}\,\rho_2(\mathrm{Fr}_t)$ and $\det \rho_1(\mathrm{Fr}_t) = \det \rho_2(\mathrm{Fr}_t)$ for all $t \in T$.

Then ρ_1^{ss} and ρ_2^{ss} are isomorphic.

We cannot apply this result directly since condition 1 is not satisfied.

Using class field theory we can identify the extensions of F cut out by the mod 2 representations, and then we can apply this result. This required one to compute the Hecke eigenvalues of f for primes generated by a totally positive element of the form $a + b\theta$ with $a \leq 183$.