Products of uniquely completable partial latin squares

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Abstract

We introduce a new type of forcing of entries into partial latin squares and extend the class of partial latin squares for which Gower's conjecture on completable products is true.

1 Introduction

A *latin square* of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs exactly once in each row and column. Without loss of generality we may take $N = \{0, 1, \ldots, n-1\}$ and assume that rows and columns are indexed by N. We may also represent a latin square by a set of n^2 triples (i, j, k) such that element k appears in row i and column j.

A *partial latin square* of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs at most once in each row and column. We shall also use the corresponding set of triples to represent a partial latin square.

If $\{a, b, c\} = \{1, 2, 3\}$, then the (a, b, c)-*conjugate* of P is denoted and defined by $P_{(a,b,c)} = \{(x_a, x_b, x_c) : (x_1, x_2, x_3) \in P\}$. For $\theta \in S_3$, the symmetric group on $\{1, 2, 3\}$, we define $\theta(x_1, x_2, x_3) = (x_{\theta(1)}, x_{\theta(2)}, x_{\theta(3)})$.

A set of triples defining a partial latin square, P, of order n is *uniquely completable (UC)* if there is only one latin square, L, of order n that contains P. It is immediate that P is UC to L if and only if $P_{(a,b,c)}$ is UC to $L_{(a,b,c)}$.

The addition of a triple (i, j, k) to a partial latin square, P, is said to be *forced* if either:
1. $\forall h \neq k, \exists z$ such that $(i, z, h) \in P$ or $(z, j, h) \in P$; or
2. $\theta(i, j, k)$ satisfies 1 in $P_{\theta(1,2,3)}$ for some $\theta \in S_3$.

A UC set, U, is *strong*, if we can find a sequence of sets of triples $U = S_1 \subset S_2 \subset \ldots \subset S_f = L$ such that each triple $t \in S_{v+1} \setminus S_v$ is forced in S_v. A UC set which is not strong is called *weak*.

Let P be a partial latin square of order n defined on N, then A_P is an *array of alternatives* for P if:

1. A_P is an $n \times n$ array;
2. whenever the $(i,j)^{th}$ cell of P is filled, the $(i,j)^{th}$ cell of A_P is empty;
3. whenever the $(i,j)^{th}$ cell of P is empty, the $(i,j)^{th}$ cell of A_P contains all the elements of N which do not appear in the i^{th} row or j^{th} column of P.

We denote the entry in row i and column j of A_P by $A_P(i,j)$.

A partial latin square P has no forced additional triple if and only if its array of alternatives, A_P, is such that no element occurs in A_P in exactly one row or column and no cell of A_P contains exactly one element.

2 A special class of partial latin squares

In this section we introduce a certain class of uniquely completable partial latin squares which includes certain weak uniquely completable ones. We begin with an example of a uniquely completable partial latin square.

Example 1 *In [4] Keedwell showed that the partial latin square P is weak UC to M, see Figure 1. We will begin by showing why P is uniquely completable.*

$$P: \begin{array}{|ccccc} \hline 0 & 1 & 2 & \bullet & \bullet \\ 1 & 0 & \bullet & \bullet & 2 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 3 & 2 & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet & 0 \end{array} \qquad M: \begin{array}{|ccccc} \hline 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 & 2 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 2 & 4 & 0 & 1 \\ 4 & 3 & 1 & 2 & 0 \end{array}$$

Figure 1: Partial latin square P with unique completion to M

Consider the array of alternatives for P, A_P in Figure 2. Let us consider row 2 of A_P; if we can complete P to a latin square, then the entries 3 and 4 in this row must occur in (2,1) and (2,4) and hence nowhere else in this row. In

$$A_P: \begin{array}{|c|c|c|c|c|} \hline & & & 3,4 & 3,4 \\ \hline & & 3,4 & 3,4 & \\ \hline 2,4 & 3,4 & 0,1,3,4 & 0,1,2,3,4 & 3,4 \\ \hline & & 0,4 & 0,4 & \\ \hline 2,4 & 3,4 & 1,3,4 & 1,2,3,4 & \\ \hline \end{array}$$

Figure 2: Array of alternatives for P

particular we cannot have 4 in position (2,0). Thus from A_P we deduce that the entry in position (2,0) must be 2. The resulting latin square is strongly UC to M.

We know from [4] that P is weak UC to M. In Example 1 we showed that we could remove 4 from $A_P(2,0)$ without reducing the number of possible completions for P. We formalise this idea as follows. For a given partial latin square P, we will say that a triple $(i,j,k') \in A_p$ is *forced out* of A_P if either:
1. there exists $r > 0$ and $i_1, \dots, i_r$ (all $\neq i$) with $k' \in A_P(i_1,j) \cup \dots \cup A_P(i_r,j)$ and $|A_P(i_1,j) \cup \dots \cup A_P(i_r,j)| = r$; or
2. $\theta(i,j,k')$ is satisfies 1 in $A_{P_{\theta(1,2,3)}}$ for some $\theta \in S_3$.

Definition 1 *Let P be a partial latin square, the* reduced array of alternatives, *RA_P, is the array obtained from A_P by successively removing elements which are forced out until no more are present. The addition of a triple (i,j,k) to P is said to be semi-forced if either:*
1. k is the only element in $RA_P(i,j)$; or
2. k occurs exactly once in either the i^{th} row or j^{th} column of $RA_P(i,j)$.

Note that if a triple is forced then it is also semi-forced.

Definition 2 *A UC set U is near-strong UC to the latin square L if we can find a sequence of sets of triples $U = S_1 \subset \dots \subset S_f = L$ such that each triple $t \in S_{v+1} \setminus S_v$ is semi-forced in S_v.*

So, P in Example 1 is near-strong UC. Other examples of near-strong UC partial latin squares can be found in the literature. For example in [2] the weak UC partial latin squares in C_n are near-strong UC, and [1] contains examples of weak UC partial latin squares some of which are near-strong and some not.

3 Products of latin squares

Given two latin squares we may take their direct product to produce, in general, a larger latin square. We recall the following definition from [3].

Definition 3 *Let M and N be latin squares of order m and n respectively with entries from the sets $\{0,1,\dots,m-1\}$ and $\{0,1,\dots,n-1\}$ respectively. Define N^r to be the array obtained from N by adding rn to each entry of N, for $r = 0,1,\dots,m-1$. The direct product of M with N is the latin square of order mn constructed by replacing the entry r in M by the array N^r. This is denoted by $M \times N$.*

Example 2 *Let M and N be the latin squares in Figure 3. Then by taking their direct product we obtain the latin square $M \times N$.*

$$M : \begin{array}{|cc} \hline 0 & 1 \\ 1 & 0 \end{array} \quad N : \begin{array}{|ccc} \hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array} \quad M \times N : \begin{array}{|cccccc} \hline 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 0 & 4 & 5 & 3 \\ 2 & 0 & 1 & 5 & 3 & 4 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 3 & 1 & 2 & 0 \\ 5 & 3 & 4 & 2 & 0 & 1 \end{array}$$

Figure 3: The direct product of latin squares M and N

In [3] Gower introduced the concept of the completable product of two UC partial latin squares.

Definition 4 *Let P be a partial latin square of order m with entries taken from the set $\{0, 1, \ldots, m-1\}$ such that P completes uniquely to the latin square M and let Q be a partial latin square, but of order n, with entries taken from the set $\{0, 1, \ldots, n-1\}$ such that Q completes uniquely to N. Let Q^r be the array obtained from Q by adding rn to each non-empty cell of Q, for $r = 0, 1, \ldots, m-1$. Similarly let N^r be the array obtained from N by adding rn to the entry in each cell of N, for $r = 0, 1, \ldots, m-1$. Define the completable product of P with Q, written $P \times Q$, to be the partial latin square R of order mn which is the array obtained by replacing each entry r of P with the array N^r and each entry r of $M \setminus P$ with the array Q^r.*

It is clear that the completable product of P with Q is contained in the direct product of M with N. In [3] Gower proved the following result.

Theorem 1 *Let P and Q be as in Definition 4. Suppose at least one of P or Q is strongly uniquely completable. Then the partial latin square $P \times Q$ has unique completion to the latin square $M \times N$.*

In [3] Gower offers the following conjecture as an extension of Theorem 1.

Conjecture 1 *The completable product of any two partial latin squares with unique completion also has unique completion.*

We will extend Theorem 1 to the case where one of the partial latin squares is near-strong UC. Before we do that consider the following example.

Example 3 *Let P be as in Figure 1 with unique completion to the latin square M. Further, let Q be a partial latin square of order n with unique completion to the latin square N. Consider the completable product of P with Q, see Figure 4.*

For $0 \leq x \leq 5n-1$ there exists a unique integer t with $0 \leq t \leq 4$ and $tn \leq x < (t+1)n$ and we say x is of type t. Recall the array of alternatives for P

$$P \times Q : \begin{array}{|ccccc} \hline N^0 & N^1 & N^2 & Q^3 & Q^4 \\ N^1 & N^0 & Q^3 & Q^4 & N^2 \\ Q^2 & Q^4 & Q^0 & Q^1 & Q^3 \\ N^3 & N^2 & Q^4 & Q^0 & N^1 \\ Q^4 & Q^3 & Q^1 & Q^2 & N^0 \end{array}$$

Figure 4: Completable product of P with Q

in Figure 2. From A_P we deduce that if we are to complete $P \times Q$ to a latin square then the entries in block position $(0,2)$ must be of types 2 or 4. Let $0 \leq r \leq n-1$ and consider row $2n+r$ of $P \times Q$, from A_P we see that in completing $P \times Q$ to a latin square all the entries of types 3 or 4 in row $2n+r$ must lie in columns indexed $n, n+1, \ldots, 2n-1$ and $4n, 4n+1, \ldots, 5n-1$ Hence, in particular, there are no entries of type 4 in positions $(2n+r, 0), (2n+r, 1), \ldots, (2n+r, n-1)$ Thus when completing $P \times Q$ to a latin square all the entries in row $2n+r$ and columns 0 to $n-1$ must be of type 2. Since r was arbitrary we deduce that in completing $P \times Q$ to a latin square all the entries in block position $(2,0)$ must be of type 2. Now since Q is UC to N we have Q^2 is UC to N^2 when restricted to the set of entries of type 2. Thus if $P \times Q$ completes to the latin square L then L must have a copy of N^2 in block position (2,0). That is, $P \cup (0,2,2) \times Q \subset L$, and, since $P \cup (0,2,2)$ is strongly completable to M, from Theorem 1 we deduce that $P \times Q$ is uniquely completable.

The above argument is a special case of the following.

Lemma 1 *Let P be a partial latin square of order m that is UC to the latin square M and let Q be a partial latin square of order n that is UC to N and let L be a latin square to which $P \times Q$ completes. Suppose that the addition of the triple (i,j,k) is semi-forced in P then L must contain a copy of N^k in block position (i,j).*

Proof: For $0 \leq x \leq mn-1$ there exists a unique t with $0 \leq t \leq m-1$ and $tn \leq x < (t+1)n$ and we say that x is of type t. The fact that the triple (i,j,k) is semi-forced in P implies that the entries in block position (i,j) of L must all be of type k. However block position (i,j) already contains a copy of Q^k which is uniquely completable to N^k when restricted to the entries of type k, so this $n \times n$ subarray of $P \times Q$ must complete to N^k.□

Theorem 2 *Let P be a partial latin square of order m which is near-strong UC to the latin square M and let Q be a partial latin square of order n which is UC to the latin square N. Then the completable product of P with Q has unique completion to the latin square $M \times N$.*

Proof: Since P is near-strong UC to M there is a sequence of partial latin squares $\{P_1, P_2, \ldots, P_f\}$ satisfying Definition 2. We have $P_1 = P$ and define $R_1 = P_1 \times Q$. Now, $P_2 = P_1 \cup \{(i_1, j_1, k_1)\}$, where the triple (i_1, j_1, k_1) is semi-forced in P_1 and so, by Lemma 1, R_1 must complete to $P_2 \times Q = R_2$. Hence we obtain a sequence of partial latin squares $R_1, R_2, \ldots, R_f$ with $R_f = P_f \times Q = M \times Q = M \times N$ and so $P \times Q$ is uniquely completable to $M \times N$.□

The following two results are both from [3].

Lemma 2 *If P is a partial latin square which is UC to the latin square L and (ϕ, ψ, χ) is an isotopism from P to $\mathcal{P}$ then $\mathcal{P}$ is UC to the latin square $\mathcal{L}$ and $\mathcal{L}$ is isotopic to L.*

Lemma 3 *Let P and Q be partial latin squares with unique completion to the latin squares M and N respectively. Then the completable products $P \times Q$ and $Q \times P$ are isomorphic.*

We may now prove the following corollary to Theorem 2.

Corollary 1 *The result still holds if Q, instead of P, is the partial latin square which is near-strong UC.*

Proof: By Lemmas 2 and 3 the completable product of P with Q must have unique completion.□

Bringing Theorem 2 and Corollary 1 together we have the following theorem.

Theorem 3 *Let P be a partial latin square which is uniquely completable to the latin square M and let Q be a partial latin square which is uniquely completable to the latin square N. Suppose that at least one of the partial latin squares P and Q is near-strong uniquely completable. Then the completable product of P with Q, $P \times Q$, has unique completion to $M \times N$.*

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