IAS TALK: ON THE FUNDAMENTAL LEMMA FOR WEIGHTED ORBITAL INTEGRALS

DAVID WHITEHOUSE

Abstract. The first half of this talk will be devoted to describing Arthurs variant of the fundamental lemma for weighted orbital integrals. The second half will describe, in detail, a proof of the weighted fundamental lemma for the group Sp(4).

This talk will consist of two parts.

(1) Statement of the weighted fundamental lemma.
(2) Proof of the weighted fundamental lemma for Sp(4).

In the first we will give the statement of the weighted fundamental lemma. The previous lectures have been devoted to the (unweighted) fundamental lemma which is required to stabilize the elliptic part of the trace formula. The weighted variant of it is conjectured by Arthur and is required for the stabilization of the full trace formula.

In the second part of this talk we will present a proof of the weighted fundamental lemma for Sp(4). We note that Sp(4) is one of the first groups for which the extension of the fundamental lemma to weighted orbital integrals is non-trivial.

Notes for this talk can be found online at `www.math.ias.edu/~dw`.

1. The weighted fundamental lemma

We begin by fixing some notation.

- F local nonarchimedean field
- $\mathcal{O}_F$ ring of integers in F
- $|\ |$ the multiplicative valuation on F
- q the cardinality of the residue field of F
- G/F an (unramified) connected reductive group
- K a hyperspecial maximal compact subgroup of $G(F)$

1.1. **Weight functions.** We begin with the definition of Arthur's weight functions. We fix a Levi subgroup M of G which is in "good position" relative to K; see [Art81, Section 1] for the definition. Arthur defines a weight function

$$v_M : G(F) \to \mathbf{R}$$

which we will now describe.

We denote by $\mathcal{P}(M)$ for the (finite) set of parabolic subgroups P of G with Levi component M.

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Let A_M denote the split component of the center of M. Let $X(M)_F$ denote the group of characters of M defined over F and set

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R}),$$

a real vector space of dimension equal to the dimension of A_M. By restriction we have

$$\mathfrak{a}_M \hookrightarrow \mathfrak{a}_{M_0}$$

We fix a Weyl invariant Euclidean metric on $\mathfrak{a}_{M_0}$, the restriction of this metric provides a Euclidean metric on any subspace.

By restriction to A_M we have a canonical identification

$$\mathfrak{a}_M = \mathrm{Hom}(X(A_M), \mathbf{R}).$$

We set $\mathfrak{a}_M^* = X(M)_F \otimes_{\mathbf{Z}} \mathbf{R} = X(A_M) \otimes_{\mathbf{Z}} \mathbf{R}$. Then $\mathfrak{a}_M^*$ is the dual vector space of $\mathfrak{a}_M$ in the natural way.

We note that we have $A_G \subset A_M$ and by restriction we have a map $X(A_M) \twoheadrightarrow X(A_G)$, which yields a surjection $\mathfrak{a}_M^* \twoheadrightarrow \mathfrak{a}_G^*$ and hence an embedding $\mathfrak{a}_G \hookrightarrow \mathfrak{a}_M$. On the other hand, since $M \subset G$ we have by restriction an injection $\mathfrak{a}_G^* \hookrightarrow \mathfrak{a}_M^*$. This gives rise to canonical splittings

$$\mathfrak{a}_M = \mathfrak{a}_G \oplus \mathfrak{a}_M^G$$

and

$$\mathfrak{a}_M^* = \mathfrak{a}_G^* \oplus (\mathfrak{a}_M^G)^*.$$

We have a homomorphism

$$H_M : M(F) \to \mathfrak{a}_M$$

defined by

$$\langle H_M(m), \chi \rangle = \log |\chi(m)|$$

for $m \in M(F)$ and $\chi \in X(M)_F$. Let $P \in \mathcal{P}(M)$ then using the Iwasawa decomposition

$$G(F) = N_P(F)M(F)K,$$

where N_P denotes the unipotent radical in P, we can extend H_M to a map

$$H_P : G(F) \to \mathfrak{a}_M$$

by taking H_P to be zero on $N_P(F)$ and K.

We then define for $g \in G(F)$, $v_M(g)$ to be the volume of the convex hull of the projection of

$$\{H_P(g) : P \in \mathcal{P}(M)\} \subset \mathfrak{a}_M$$

onto $\mathfrak{a}_M^G$.

Equivalently one can define v_M in terms of a (G, M)-family, which we now describe. Let $P \in \mathcal{P}(M)$ and let $\Delta_P \subset \mathfrak{a}_P^*$ denote the simple roots of (P, A_M). We have the set of "coroots"

$$\Delta_P^\vee = \left\{ \beta^\vee \in \mathfrak{a}_M^G : \beta \in \Delta_P \right\}.$$

We recall the definition of these coroots from [Art78, Section 1]. Let P_0 be a minimal parabolic subgroup of G contained in P. Each $\beta \in \Delta_P$ is the restriction to $\mathfrak{a}_P^G$ of a unique root $\beta_1 \in \Delta_{P_0}$. We then define $\beta^\vee$ to be the projection onto $\mathfrak{a}_P^G$ of $\beta_1^\vee \in \mathfrak{a}_{P_0}^G$. Let $\mathfrak{a}_{M,\mathbf{C}} = \mathfrak{a}_M \otimes_{\mathbf{R}} \mathbf{C}$, then for $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$ we define

$$\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_M^G/\mathbf{Z}[\Delta_P^\vee])^{-1} \prod_{\beta \in \Delta_P} \lambda(\beta^\vee).$$

We define, for $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$ and $g \in G(F)$,

$$v_P(\lambda, g) = \exp(-\lambda(H_P(g))).$$

We now set

$$v_M(\lambda, g) = \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x)\theta_P(\lambda)^{-1}.$$

By [Art81, Lemma 6.2] for each $g \in G(F)$ this function extends to a smooth function on $i\mathfrak{a}_M^*$. We define $v_M(g)$ to be the value of the function $v_M(\lambda, g)$ at $\lambda = 0$.

Remark 1.1. We note that $v_M(mgk) = v_M(g)$ for $m \in M(F)$ and $k \in K$.

Example 1.2. (Weight functions for GL(2).) We take M to be the diagonal torus in GL(2).

In order to compute $v_M(g)$ it is sufficient to consider the case that g is upper triangular unipotent. We take

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Of course in this case $\mathcal{P}(M) = \{P, Q\}$ where Q denotes the lower triangular Borel subgroup. We need to write

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k$$

with $k \in \mathrm{GL}(2, \mathcal{O}_F)$.

First of all we clearly have $|b| = |a|^{-1}$. Next we apply the row vector $(1, 0)$ to this identity. Applying it to the right hand side gives

$$a(1, 0)k.$$

On the other hand applying it to the left hand allows us to deduce that

$$|a| = \max\{1, |x|\}.$$

Hence we obtain

$$v_M(g) = \mathrm{vol}(\mathfrak{a}_M^G/\mathbf{Z}[\Delta_B^\vee]) \log \max\{1, |x|\}.$$

Example 1.3. As an example of a weight function for a non-maximal Levi subgroup we take the following from [Whi05, Corollary 4.8]. Let B denote the upper triangular Borel subgroup in Sp(4) and let M denote the diagonal torus. Then we have for

$$n = \begin{pmatrix} 1 & x_1 & x_2 + x_1 x_4 & x_3 \\ & 1 & x_4 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \in N_B(F),$$

$$v_M(n) = \frac{\mathrm{vol}(\mathfrak{a}_B^G/\mathbf{Z}(\Delta_B^\vee))}{2} \left(-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)\right),$$

where

$$
\begin{aligned}
A &= \log \max\{1, |x_2|, |x_4|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\} \\
B &= \log \max\{1, |x_1|, |x_2 + x_1x_4|, |x_3|\} \\
C &= \log \max\{1, |x_1|\} \\
D &= \log \max\{1, |x_1|^2, |x_3 + x_1x_2 + x_1^2x_4|\} \\
E &= \log \max\{1, |x_2|, |x_4|\} \\
F &= \log \max\{1, |x_4|\}.
\end{aligned}
$$

1.2. **Weighted orbital integrals.** Suppose now we take $f \in C_c^\infty(G(F))$ and an element $\gamma \in M(F)$. We assume that γ is strongly G-regular, i.e. γ is a semisimple element of M such that G_γ is a torus. Then we define the weighted orbital integral of f at the element γ by

$$J_M(\gamma, f) = |D_G(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\ v_M(g)\ dg.$$

We note that this is well defined since $G_\gamma = M_\gamma \subset M$ by the assumption that γ is strongly G-regular and v_M is left $M(F)$-invariant.

In the special case that $f = \mathbf{1}_K$ we define

$$r_M^G(\gamma) = J_M(\gamma, \mathbf{1}_K).$$

1.3. **Statement of the weighted fundamental lemma.** Suppose further now that G is unramified. Let M be a Levi subgroup of G. Suppose now we take an unramified elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M. Here,

(1) M' is an unramified group defined over F,
(2) $\mathcal{M}' = {}^L M' \subset {}^L M$,
(3) s'_M is a semisimple element in $\widehat{M}$ such that $\widehat{M}' = Z_{\widehat{M}}(s'_M)^0$,
(4) ξ'_M is the inclusion of $\mathcal{M}'$ in ${}^L M$.

The datum $(M', \mathcal{M}', s'_M, \xi'_M)$ satisfies the conditions of [LS87, (1.2)].

We note that $\widehat{M} \subset \widehat{G}$ as a Levi subgroup in a canonical way.

We define $\mathcal{E}_{M'}(G)$ to be the set of endoscopic data for G of the form

$$(G', \mathcal{G}', s', \xi'),$$

where

(1) s' lies in $s'_M Z(\widehat{M})^\Gamma$,
(2) $\mathcal{G}' = \mathcal{M}'\widehat{G}'$, where $\widehat{G}' = Z_{\widehat{G}}(s')^0$,
(3) ξ' is the identity embedding of $\mathcal{G}'$ into ${}^L G$,
(4) ${}^L G' = \mathcal{G}'$.

The elements in $\mathcal{E}_{M'}(G)$ are taken up to translation of s' by $Z(\widehat{G})^\Gamma$.

We note that for each $G' \in \mathcal{E}_{M'}(G)$ we have a well defined (up to $G'(F)$-conjugacy) embedding $M' \hookrightarrow G'$.

For $G' \in \mathcal{E}_{M'}(G)$ we set

$$\iota_{M'}(G, G') = |Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma||Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1},$$

where $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

Conjecture 1.4. *(The weighted fundamental lemma.) For each G and M there is a function $s_M^G(\ell)$ defined on G-regular stable conjugacy classes in $M(F)$ satisfying the following property: For any G, M and M' as above and for any element $\ell \in \mathcal{L}_{G-reg}(M')$, the strongly G-regular conjugacy classes in $M'(F)$, we have*

$$\sum_{k \in \Gamma_{G\text{-}reg}(M(F))} \Delta_{M \cap K}(\ell', k) r_M^G(k) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{M'}^{G'}(\ell').$$

1.3.1. *Remarks on the statement of the weighted fundamental lemma.*

- The sum on the left is over $\Gamma_{\text{G-reg}}(M(F))$ the set of strongly G-regular conjugacy classes in $M(F)$.
- The factor $\Delta_{M \cap K}(\ell', k)$ is the Langlands-Shelstad transfer factor normalized relative to the maximal compact subgroup $M \cap K$ of M and deprived of term Δ_{IV}.
- The left hand side is independent of the choice of the maximal compact subgroup K of $G(F)$.
- The term $\iota_{M'}(G, G')$ vanishes unless the endoscopic datum G' is elliptic for G. Moreover it is non-zero for only finitely many $G' \in \mathcal{E}_{M'}(G)$.
- The functions $s_M^G(\ell)$ are uniquely defined by taking $M' = M$ above. In this case we clearly have $G \in \mathcal{E}_M(G)$ and we obtain an inductive definition for $s_M^G(\ell)$ by the formula

$$s_M^G(\ell) = r_M^G(\ell) - \sum_{G' \in \mathcal{E}_{M'}(G) G' \neq G} \iota_{M'}(G, G') s_M^{G'}(\ell)$$

  where

$$r_M^G(\ell) = \sum_{k \in \Gamma_{\text{G-reg}}(M(F)), k \in \ell} r_M^G(k).$$

- Thus the point of the weighted fundamental lemma is to prove the identities when $M' \neq M$.
- If we take $M = G$ then from the definition above we see that $s_G^G(\ell)$ is equal to stable orbital integral of $\mathbf{1}_K$ along the stable conjugacy class ℓ. Moreover if we take G' to be an endoscopic group for G then clearly $\mathcal{E}_{G'}(G) = \{G'\}$ and so the weighted fundamental lemma reduces to the standard invariant fundamental lemma conjectured by Langlands.
- If we take M to be a minimal Levi subgroup then the weighted fundamental lemma is trivial.

1.3.2. *What's known about the weighted fundamental lemma.* It's clear from the statement that the weighted fundamental lemma asserts something non-trivial about the pair (G, M) precisely when the Levi subgroup M has proper elliptic endoscopic groups. Thus for certain groups the extension of the fundamental lemma to weighted orbital integrals is trivial. This is the case for example when $G = \mathrm{GSp}(4)$, $\mathrm{SO}(5)$ or $\mathrm{SL}(p)$ with p prime. In each of these cases the fundamental lemma for invariant orbital integrals is known. One of the first groups where something extra is needed in the case of weighted orbital integrals is for $\mathrm{Sp}(4)$.

1.3.3. *Statement of the weighted fundamental lemma for* $\mathrm{SO}(2n+1)$. The dual group of $\mathrm{SO}(2n+1)$ is $\mathrm{Sp}(2n, \mathbf{C})$. The elliptic endoscopic groups are obtained by taking elements of the form

$$s = \begin{pmatrix} I_{n_1} & & \\ & -I_{2n_2} & \\ & & I_{n_1} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbf{C})$$

with $n_1+n_2 = n$. Then the centralizer of s in $\mathrm{Sp}(2n, \mathbf{C})$ is isomorphic to $\mathrm{Sp}(2n_1, \mathbf{C})\times \mathrm{Sp}(2n_2, \mathbf{C})$. Hence we get

$$\mathcal{E}_{ell}(\mathrm{SO}(2n+1)) = \{\mathrm{SO}(2n_1+1) \times \mathrm{SO}(2n_2+1) : n_1 + n_2 = n\}\,.$$

Suppose now we take a Levi subgroup M of $\mathrm{SO}(2n+1)$. Then we have

$$M = \mathrm{SO}(2m_0+1) \times \prod_{i=1}^{k} \mathrm{GL}(m_i)$$

for integers $m_i \geq 0$ with

$$\sum_{i=0}^{k} m_i = n.$$

We assume that $m_i > 0$ for $1 \leq i \leq k$. We view

$$M = \left\{\mathrm{diag}(A_k, \dots, A_1, A_0, {}^{\tau}A_1^{-1}, \dots, {}^{\tau}A_k^{-1})\right\} \subset G.$$

We have

$$\widehat{M} = \mathrm{Sp}(2m_0, \mathbf{C}) \times \prod_{i=1}^{k} \mathrm{GL}(m_i, \mathbf{C}) \hookrightarrow \mathrm{Sp}(2n, \mathbf{C})$$

via the analogous embedding.

Let M' be an elliptic endoscopic group for M. Then, by the arguments above, we can take

$$M' = \mathrm{SO}(2m_0^+ + 1) \times \mathrm{SO}(2m_0^- + 1) \times \prod_{i=1}^{k} \mathrm{GL}(m_i)$$

for integers $m_0^+, m_0^- \geq 0$ with $m_0^+ + m_0^- = m_0$. This elliptic endoscopic group is given by the element

$$s'_M = \mathrm{diag}(I_{m_k}, \dots, I_{m_1}, I_{m_0^+}, -I_{2m_0^-}, I_{m_0^+}, I_{m_1}, \dots, I_{m_k}) \in \widehat{M}.$$

In order to determine the set $\mathcal{E}_{M'}(G)$ we need to look at endoscopic data coming from elements of the form

$$s'_M = \mathrm{diag}(\lambda_k I_{m_k}, \dots, \lambda_1 I_{m_1}, I_{m_0^+}, -I_{2m_0^-}, I_{m_0^+}, \lambda_1^{-1} I_{m_1}, \dots, \lambda_k^{-1} I_{m_k}) \in \widehat{G}.$$

We get elliptic endoscopic data if and only if $\lambda_i \in \{\pm 1\}$ for all i with $1 \leq i \leq k$. Write $[1,k] = I^+ \amalg I^-$ then we get the group

$$\mathrm{SO}(2n_{I^+} + 1) \times \mathrm{SO}(2n_{I^-} + 1),$$

where

$$n_{I^\pm} = m_0^\pm + \sum_{i \in I^\pm} m_i$$

and with M' embedded as

$$M'_{I^+} \times M'_{I^-} = \left(\mathrm{SO}(2m_0^+ + 1) \times \prod_{i\in I^+} \mathrm{GL}(m_i)\right) \times \left(\mathrm{SO}(2m_0^- + 1) \times \prod_{i\in I^-} \mathrm{GL}(m_i)\right).$$

If we assume first that $m_0^+ = m_0$ and $m_0^- = 0$ then we get the definition

$$s_M^{\mathrm{SO}(2n+1)}(\ell) = r_M^{\mathrm{SO}(2n+1)}(\ell) - \sum_{[1,k]=I^+\amalg I^-, I^-\neq\emptyset} s_{M_{I^+}}^{\mathrm{SO}(2n_{I^+}+1)}(\ell_{I^+}) s_{M_{I^-}}^{\mathrm{SO}(2n_{I^-}+1)}(\ell_{I^-}),$$

if $m_0 > 0$ and

$$s_M^{\mathrm{SO}(2n+1)}(\ell) = r_M^{\mathrm{SO}(2n+1)}(\ell) - \frac{1}{2} \sum_{[1,k]=I^+\amalg I^-, I^+, I^-\neq\emptyset} s_{M_{I^+}}^{\mathrm{SO}(2n_{I^+}+1)}(\ell_{I^+}) s_{M_{I^-}}^{\mathrm{SO}(2n_{I^-}+1)}(\ell_{I^-}),$$

here $\ell_{I^\pm}$ denotes the projection of ℓ to $M_{I^\pm}$.

If on the other hand we have $m_0^+ > 0$ and $m_0^- > 0$ then the weighted fundamental lemma asserts that for ℓ' a stable conjugacy class in $M'(F)$ we have, for a strongly $\mathrm{SO}(2n+1)$-regular stable conjugacy class in $M'(F)$,

$$\sum_{k\in\Gamma_{\mathrm{SO}(2n\,+\,1)\text{-reg}}(M(F))} \Delta_{M\cap K}(\ell, k) r_M^{\mathrm{SO}(2n+1)}(k)$$

equal to

$$\sum_{[1,k]=I^+\amalg I^-} s_{M_{I^+}}^{\mathrm{SO}(2n_{I^+}+1)}(\ell_{I^+}^+) s_{M_{I^-}}^{\mathrm{SO}(2n_{I^-}+1)}(\ell_{I^-}^-),$$

2. The weighted fundamental lemma for Sp(4)

We now take $G = \mathrm{Sp}(4)$ the symplectic group of rank 2, explicitly given as

$$\mathrm{Sp}(4) = \left\{ g \in \mathrm{GL}(4) : J^t g^{-1} J^{-1} = g \right\}$$

where

$$J = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}.$$

Up to conjugacy there are four parabolic subgroups of $\mathrm{Sp}(4)$. Taking those which contain the upper triangular matrices they are of the form

$$G, \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ && * & * \\ && * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ &&& * \end{pmatrix}, B.$$

In each of these cases the Levi components are isomorphic to

$$G, \quad \mathrm{GL}(2), \quad \mathrm{SL}(2) \times \mathrm{GL}(1), \quad \mathrm{GL}(1)^2.$$

Thus only one proper Levi subgroup possesses proper elliptic endoscopic groups. This is

$$M = \left\{ \begin{pmatrix} a && \\ & g & \\ && a^{-1} \end{pmatrix} : a \in \mathrm{GL}(1), g \in \mathrm{SL}(2) \right\}.$$

The dual group of M is $\widehat{M} = \mathrm{SO}(3, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C})$ which sits inside $\widehat{G} = \mathrm{SO}(5, \mathbf{C})$ as

$$\widehat{M} = \left\{ \begin{pmatrix} a && \\ & g & \\ && a^{-1} \end{pmatrix} : a \in \mathbf{C}^\times, g \in \mathrm{SO}(3, \mathbf{C}) \right\}.$$

Let E/F denote the unramified quadratic extension of F. The group M has a unique proper unramified elliptic endoscopic group, namely M' with $M'(F) = E^1 \times F^\times$ where E^1 denotes the group of elements in $E^\times$ with norm 1.

To obtain the endoscopic group M' we take the element

$$s = \mathrm{diag}(1, -1, 1, -1, 1) \in \widehat{M}.$$

The connected component of the centralizer of s in $\widehat{M}$ is the diagonal torus and the action of $\mathrm{Gal}(E/F)$ on this torus is given by

$$\sigma : \mathrm{diag}(a, b, 1, b^{-1}, a^{-1}) \mapsto \mathrm{diag}(a, b^{-1}, 1, b, a^{-1})$$

which is given by conjugation by the element

$$\begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix}$$

which commutes with s.

In order to compute the set $\mathcal{E}_{M'}(G)$ we need to consider elements in $sZ(\widehat{M}) \subset \widehat{G}$. Thus we need to look at elements of the form

$$\mathrm{diag}(a, -1, 1, -1, a^{-1}) \in \widehat{G}.$$

We get elliptic endoscopic data if and only if we have $a = \pm 1$.

When $a = -1$ we have the connected component of the centralizer of such an element equal to $\mathrm{SO}(4, \mathbf{C})$. We have

$$\mathrm{SO}(4, \mathbf{C}) = (\mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C}))'/\mathbf{C}^\times$$

where the prime denotes the subgroup of pairs with equal determinant, and the $\mathbf{C}^\times$ is embedded diagonally. The group $\mathrm{Gal}(E/F)$ acts by permuting the factors. Consider the group $G_1 = (\mathrm{GL}(2) \times \mathrm{GL}(2))'/\mathrm{GL}(1)$ over E, then we have

$$H_1(F) = \{(g_1, g_2) \in G_1(E) : (g_1, g_2) = (\bar{g}_2, \bar{g}_1)\}.$$

When $a = 1$ we have the connected component of the centralizer of such an element equal to

$$\mathrm{SO}(3, \mathbf{C}) \times \mathrm{SO}(2, \mathbf{C})$$

with a Galois action on $\mathrm{SO}(2, \mathbf{C})$. Hence we get the group H_2 with $H_2(F) = \mathrm{SL}(2, F) \times E^1$.

For $H \in \mathcal{E}_{M'}(\mathrm{Sp}_4)$ we have, by definition,

$$\iota_{M'}(\mathrm{Sp}_4, H) = \frac{|\, Z(\widehat{M'})^{\mathrm{Gal}(E/F)}/Z(\widehat{M}) \,|}{|\, Z(\widehat{H})^{\mathrm{Gal}(E/F)} \,|}.$$

Thus $\iota_{M'}(\mathrm{Sp}_4, H_1) = 2$ and $\iota_{M'}(\mathrm{Sp}_4, H_2) = 1$.

We let $D \in F$ be such that $E = F(\sqrt{D})$. We assume, as we may, that $D \in \mathcal{O}_F^\times$. For $a \in F^\times$ and $\beta = \beta_1 + \beta_2\sqrt{D} \in E^1$, with $\beta_i \in F$, we write

$$\gamma(a, \beta) = \begin{pmatrix} a & & & \\ & \beta_1 & \beta_2 D & \\ & \beta_2 & \beta_1 & \\ & & & a^{-1} \end{pmatrix} \in M(F).$$

The stable conjugacy class of $\gamma(a,\beta)$ in $M(F)$ is a union of two conjugacy classes, with

$$\gamma'(a,\beta) = \begin{pmatrix} a & & & \\ & \beta_1 & \varpi^{-1}\beta_2 D & \\ & \varpi\beta_2 & \beta_1 & \\ & & & a^{-1} \end{pmatrix} \in M(F)$$

representing the other conjugacy class.

Under the norm map, the stable conjugacy class of $\gamma(a,\beta)$ is sent to the conjugacy class of

$$\gamma_1(a,\beta) = \left(\begin{pmatrix} a\beta & \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \\ & \beta \end{pmatrix} \right) \in M'(F) \subset H_1(F)$$

and to

$$\gamma_2(a,\beta) = \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \beta \right) \in M'(F) \subset H_2(F).$$

In our case the transfer factor $\Delta_{K\cap M}$ is computed in [LS87, Section 1.1]. And we have

$$\Delta_{K\cap M}(\gamma_i(a,\beta), \gamma(a,\beta)) = (-1)^{v(\beta_2)}$$

and

$$\Delta_{K\cap M}(\gamma_i(a,\beta), \gamma'(a,\beta)) = (-1)^{v(\beta_2)+1}.$$

2.1. **How to compute weighted orbital integrals.** Suppose we take M as above and fix a parabolic subgroup P with Levi component M. Let N_P denote the unipotent radical of P. For a strongly G-regular element $a \in M(F)$ we define

$$\sigma_P(a) = \int_{N_P(F)\cap K} v_M(\varphi_a(n))\ dn$$

where $\varphi_a : N_P(F) \to N_P(F)$ is the inverse of the map

$$N_P(F) \to N_P(F) : n \mapsto a^{-1}n^{-1}an.$$

Then we have the following Lemma.

Lemma 2.1. *With notation as above we have*

$$r_M^G(\gamma) = |D_M(\gamma)|^{\frac{1}{2}} \int_{M_\gamma(F)\backslash M(F)} \mathbf{1}_{K_M}(m^{-1}\gamma m)\ \sigma_P(m^{-1}\gamma m)\ dm.$$

Proof. Use the Iwasawa decomposition to write the Haar measure on $G(F)$ as $dg = dm\ dn\ dk$. □

2.2. **Statement of the weighted fundamental lemma for** Sp(4)**.** We can use the previous Lemma, together with the calculations of weighted functions on GL(2), to readily compute the weighted orbital integrals on $H_1(F)$ and $H_2(F)$. Thus the statement of the weighted fundamental lemma is given by the following.

Theorem 2.2. *For all $a \in U_F \setminus \{\pm 1\}$ and $\beta = \beta_1 + \beta_2\sqrt{D} \in (U_E \cap E^1) \setminus \{\pm 1\}$ we have*

$$(-1)^{v(\beta_2)} \left(r_M^{\mathrm{Sp}_4}(\gamma(a,\beta)) - r_M^{\mathrm{Sp}_4}(\gamma'(a,\beta)) \right)$$

equal to

$$|a\beta - 1|_E \int_{|x|_E \le |a\beta-1|_E^{-1}} \log\max\{1, |x|_E\}\ dx + |a^2-1| \int_{|y|\le |a^2-1|^{-1}} \log\max\{1,|y|\}\ dy.$$

2.3. **Proof of the Fundamental Lemma.** It remains to compute the weighted orbital integrals on Sp(4).

We take P to be the upper triangular parabolic in Sp(4) with Levi component N_P. Then we have

$$N_P(F) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & 0 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} : x_i \in F \right\}.$$

One can compute, as in the case of GL(2), that we have

$$v_M(g) = \log \max\{1, |x_1|, |x_2|, |x_3|\}$$

for $g \in N_P(F)$ as above.

Suppose now that we take

$$\gamma = \begin{pmatrix} a & & \\ & B & \\ & & a^{-1} \end{pmatrix} \in M(F)$$

and write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then the map $n \mapsto \gamma^{-1} n^{-1} \gamma n$ is given in the coordinates (x_1, x_2, x_3) by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (I - a^{-1t}B) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$x_3 \mapsto (1 - a^{-2})x_3 + a^{-1}\left(b_{12}x_1^2 + (b_{22} - b_{11})x_1x_2 - b_{21}x_2^2\right).$$

Using the fact that σ_P is invariant under conjugation by an element of K_M, we have, as in [LL79, (2.1)],

$$r_M^{\mathrm{Sp}_4}(\gamma(a,\beta)) - r_M^{\mathrm{Sp}_4}(\gamma'(a,\beta)) = \sigma_P(\gamma(a,\beta)) + (q+1)\sum_{k=1}^{v(\beta_2)} (-1)^k q^{k-1} \sigma_P(z_k^{-1}\gamma(a,\beta)z_k)$$

where

$$z_k = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^k & \\ & & & 1 \end{pmatrix}.$$

One can compute this expression and prove the identity above.

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School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA
E-mail address: dw@ias.edu