ON THE TRANSFER OF AUTOMORPHIC REPRESENTATIONS FROM GSp(4) TO GL(4)

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Abstract. A recent paper of Arthur's describes a parameterization of representations of GSp(4) in terms of those of GL(4). These results should follow from a comparison of the stable trace formula for GSp(4) with a stabilized twisted trace formula for GL(4) × GL(1). We describe the progress made towards the comparison of these trace formulas.

1. Introduction

Let

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

We consider

$$\mathrm{GSp}(4) = \left\{ (g, \lambda) \in \mathrm{GL}(4) \times \mathrm{GL}(1) : J^t g^{-1} J^{-1} = \lambda g \right\}$$

as an algebraic group over a number field F.

As we have seen from the lectures of Ito [Ito] and Hiraga [Hir], Arthur's conjectural theory of A-packets suggests that the discrete spectrum

$$L^2_{\mathrm{disc}}(\mathrm{GSp}(4, F) \backslash \mathrm{GSp}(4, \mathbf{A}_F), \chi)$$

can be broken into packets of representations parameterized by maps

$$\psi : L_F \times \mathrm{SL}(2, \mathbf{C}) \to \mathrm{GSp}(4, \mathbf{C}),$$

where L_F denotes the hypothetical Langlands group of F. Of course, L_F is not known to exist, but nonetheless we can consider the composition

$$\psi : L_F \times \mathrm{SL}(2, \mathbf{C}) \to \mathrm{GSp}(4, \mathbf{C}) \hookrightarrow \mathrm{GL}(4, \mathbf{C}).$$

Although we don't know the existence of L_F, we do know what it's four dimensional representations are, namely they correspond to automorphic representations of $\mathrm{GL}(4)/F$. Hence, one can replace the notion of an A-parameter for GSp(4) with that of a representation of GL(4). The reformulation of Arthur's conjectures in terms of representations of GL(4) can be found in [Art04].

One expects to be able to prove these conjectures by a comparison of the trace formula for GSp(4) with a suitable twisted trace formula for GL(4) × GL(1). In this paper we outline the strategy for obtaining such a comparison, and describe some of the progress made towards this comparison.

2. The trace formula

In order to accommodate the twisted trace formula we work in the setting of disconnected groups. Let G be a connected component of a reductive algebraic group G^+ defined over a number field F. The trace formula [Art88a] computes a certain linear form $I^G(f)$ on test functions $f \in C_c^\infty(G(\mathbf{A}_F))$ in two different ways. First we have the geometric expansion

$$I^G(f) = \sum_M |W_M||W_G|^{-1} \sum_{\gamma \in \Gamma(M(F))} a^M(\gamma) I_M(\gamma, f)$$

given in terms of weighted orbital integrals over conjugacy classes and we have the spectral expansion

$$I^G(f) = \sum_M |W_M||W_G|^{-1} \int_{\Pi(M)} a^M(\pi) I_M(\pi, f)\ d\pi$$

given in terms of the representations appearing in the spectral decomposition of $L^2(G^+(F)\backslash G^+(\mathbf{A}_F))$.

We can readily describe some of these terms. Suppose for simplicity that G is a connected semisimple group. Then on the geometric side we have, for a semisimple elliptic element $\gamma \in G(F)$,

$$a^G(\gamma) = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}_F))$$

and

$$I_G(\gamma, f) = \int_{G_\gamma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(g^{-1}\gamma g)\ dg.$$

On the spectral side we have, for a representation π which appears in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$,

$$a^G(\pi) = m_\pi,$$

the multiplicity with which π appears in the discrete spectrum, and

$$I_G(\pi, f) = \mathrm{tr}\,\pi(f).$$

The terms we are interested in are those appearing on the spectral side of the trace formula. Thus, by comparing the geometric expansion of the trace formula for different groups one will obtain, via the trace formula, similar comparisons on the spectral sides. Out of these one may hope to prove results predicted by functoriality. Clearly, given the form the geometric side takes, such a strategy requires one to be able to compare conjugacy classes in different groups and transfer functions via the matching of orbital integrals.

3. GSp(4) as a twisted endoscopic group

We consider now the group $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$, over either a local or global field F of characteristic zero, together with the automorphism

$$\theta : G^0 \to G^0 : (g, a) \mapsto (J^t g^{-1} J^{-1}, a \det g).$$

We set

$$G = G^0 \rtimes \theta \subset G^0 \rtimes \langle\theta\rangle = G^+.$$

The automorphism θ fixes the standard pair in G^0 and hence induces an automorphism on the dual group $\widehat{G}^0 = \mathrm{GL}(4, \mathbf{C}) \times \mathbf{C}^\times$,

$$\hat{\theta} : \widehat{G}^0 \to \widehat{G}^0 : (h, b) \mapsto (bJ^t h^{-1} J^{-1}, b).$$

We have

$$\mathrm{GSp}(4,\mathbf{C}) = \left(\widehat{G}^0\right)^{\hat{\theta}}$$

and hence $\mathrm{GSp}(4)/F$ is a twisted endoscopic group for the pair (G^0,θ) in the sense of [KS99].

The theory of twisted endoscopy provides a map, denoted A, from

$$\left\{\text{semisimple conjugacy classes in } \mathrm{GSp}(4,\overline{F})\right\}$$

to

$$\left\{\text{semisimple } \theta\text{-twisted conjugacy classes in } G^0(\overline{F})\right\},$$

which is defined over F. Moreover, when F is local, there is a conjectural transfer of functions from $G(F)$ to $\mathrm{GSp}(4,F)$. Given $f \in C_c^\infty(G^0(F))$ one conjectures the existence of a function $f' \in C_c^\infty(\mathrm{GSp}(4,F))$ such that

$$J^{st}(\gamma, f') = J^{st}(A(\gamma)\theta, f),$$

for any semisimple strongly regular element $\gamma \in \mathrm{GSp}(4,F)$. Here

$$J^{st}(\gamma, f') = |D_{\mathrm{GSp}(4)}(\gamma)|^{\frac{1}{2}} \int_{(\mathrm{GSp}(4)_\gamma \backslash \mathrm{GSp}(4))(F)} f(g^{-1}\gamma g)\, dg$$

denotes the integral of f' along the stable conjugacy class of γ and

$$J^{st}(A(\gamma)\theta, f) = |D_{G^0}(\gamma'\theta)|^{\frac{1}{2}} \int_{(G^0_{\gamma'\theta} \backslash G^0)(F)} f(g^{-1}\gamma\theta g)\, dg,$$

denotes the integral of f along the stable twisted conjugacy class $A(\gamma)$, here γ' is any element in $A(\gamma) \cap G^0(F)$.

We therefore seem to be in good shape to try to compare the trace formula for $\mathrm{GSp}(4)$ with the (twisted) trace formula for $(\mathrm{GL}(4) \times \mathrm{GL}(1)) \rtimes \theta$. We have a map between conjugacy classes, and, at least conjecturally, a way of transferring functions on $(\mathrm{GL}(4) \times \mathrm{GL}(1)) \rtimes \theta$ to those on $\mathrm{GSp}(4)$. However, the big problem is that this matching takes place over $\overline{F}$ and not over F itself. Thus one is led to the need to refine the expansions appearing in the trace formula. Ideally one would like to replace sums over conjugacy classes by sums over stable conjugacy classes and orbital integrals by stable orbital integrals.

4. Stabilization of the trace formula

We now return to the general setting of Section 2. The major problem is that the distribution $I^G(f)$ is not, in general, stable. Hence one can not directly obtain the refinement suggested at the end of Section 3. However, Langlands suggested that one should be able to stabilize the trace formula in the form

$$I^G(f) = \sum_H \iota(G,H) S^H(f^H),$$

where

- $\{H\}$ denotes the set of elliptic (twisted) endoscopic groups for G,
- $\iota(G,H)$ is an explicit constant,
- S^H is a stable distribution on H, and
- f^H denotes the conjectural Langlands-Shelstad/Kottwitz-Shelstad transfer of f.

Suppose now that $G = G^+$ is a connected group. Then such a decomposition of the trace formula has been achieved by Arthur, building on work of Langlands and Kottwitz, in a series of papers [Art02], [Art01] and [Art03] subject to the following local conjectures.

(1) Local transfer: $f_v \mapsto f_v^{H_v}$
(2) Fundamental lemma: $f_v = \mathbf{1}_{K_v}$, $f_v^{H_v} = \mathbf{1}_{K_v^{H_v}}$
(3) Weighted fundamental lemma: Asserts identities between weighted orbital integrals of $\mathbf{1}_{K_v}$ and $\mathbf{1}_{K_v^{H_v}}$.

We make the following remarks.

- The conjecture on local transfer, by work of Waldspurger, follows from a fundamental lemma for the Lie algebra of G.
- Conjectures 1 and 2 imply the global Langlands-Shelstad transfer.
- Conjecture 3 is required to stabilize the global geometric terms $a^M(\gamma)$.
- Conjecture 2 is a special case of Conjecture 3, we will return to this in Section 6 below.

4.1. **The stable trace formula for** GSp(4)**.** In the case of GSp(4) there is only one elliptic endoscopic group

$$H = (\mathrm{GL}(2) \times \mathrm{GL}(2)) \,/\, \mathrm{GL}(1)$$

with GL(1) is embedded in the product as $\{(x, x^{-1})\}$. In this case the stabilization of the trace formula gives

$$\begin{aligned} I^{\mathrm{GSp}(4)}(f) &= S^{\mathrm{GSp}(4)}(f) + \frac{1}{4} S^H(f^H) \\ &= S^{\mathrm{GSp}(4)}(f) + \frac{1}{4} I^H(f^H), \end{aligned}$$

since H has no proper elliptic endoscopic groups. In this case the local conjectures above are known. Local transfer follows from work of Hales [Hal89]. The fundamental lemma follows from work of Hales [Hal97] and Weissauer [Wei94]. And the extension of the fundamental lemma to weighted orbital integrals is trivial in this case since no proper Levi subgroups of GSp(4) possess proper elliptic endoscopic groups.

4.2. **The twisted trace formula for** $\mathrm{GL}(4) \times \mathrm{GL}(1)$**.** We take, as above, $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$ over a number field F with the automorphism θ. We take

$$G = G^0 \rtimes \theta \subset G^0 \rtimes \langle \theta \rangle = G^+.$$

As in the connected case one expects to be able to stabilize this twisted trace formula in the form

$$I^G(f) = \sum_H \iota(G, H) S^H(f^H),$$

as a sum over the elliptic twisted endoscopic groups H of (G^0, θ).

We recall the construction of the twisted endoscopic groups for (G^0, θ). We take a semisimple element $s\hat{\theta} \in \widehat{G}^0 \rtimes \hat{\theta}$ and consider the connected component of its centralizer in $\widehat{G}^0$,

$$Z_{\widehat{G}^0}(s\hat{\theta})^0 = \left\{ (h, b) \in \widehat{G}^0 : (s_1 J)^t h^{-1} (s_1 J)^{-1} = bh \right\}^0,$$

where s_1 denotes the projection of s to $\mathrm{GL}(4, \mathbf{C})$. This is a connected reductive group and in order to form an L-group one needs to put an action of $\mathrm{Gal}(\overline{F}/F)$ on it. This action is required to satisfy certain conditions; see [KS99]. Essentially it should come via conjugation by elements in $Z_{\widehat{G}^0}(s\hat{\theta})$ (but not necessarily in the connected component of the identity). A twisted endoscopic group is then the unique quasi-split group H over F such that

$$^L H = Z_{\widehat{G}^0}(s\hat{\theta})^0 \rtimes \mathrm{Gal}(\overline{F}/F).$$

We recall that H is called elliptic if $(Z(\widehat{H})^\Gamma)^0 \subset Z(\widehat{G}^0)$, where $\Gamma = \mathrm{Gal}(\overline{F}/F)$. We now describe the elliptic twisted endoscopic groups for (G^0, θ).

Taking $s \in \widehat{G}^0$ to be the identity we have

$$Z_{\widehat{G}^0}(s\hat{\theta}) = \left(\widehat{G}^0\right)^{\hat{\theta}} = \mathrm{GSp}(4, \mathbf{C})$$

and hence we get $H = \mathrm{GSp}(4)$.

When we take $s = \mathrm{diag}(1, 1, -1, -1)$ we get

$$Z_{\widehat{G}^0}(s\hat{\theta})^0 = \mathrm{GSO}(4, \mathbf{C}) \cong (\mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C}))/\mathbf{C}^\times.$$

This group sits inside $Z_{\widehat{G}^0}(s\hat{\theta})$ as an index two subgroup and we are allowed to put an action of the Galois group on this group by interchanging the $\mathrm{GL}(2, \mathbf{C})$ factors. When this Galois action is trivial we end up with the group

$$(\mathrm{GL}(2) \times \mathrm{GL}(2))'$$

with the prime denoting the subgroup of pairs with equal determinant. When the Galois action is non-trivial we have $\mathrm{Gal}(\overline{F}/F)$ acting through a quadratic extension E/F and in this case we obtain a group whose F-points are

$$\mathrm{GL}(2, E)'$$

where the prime denotes that the subgroup of elements with determinant in $F^\times$.

When we take $s = \mathrm{diag}(1, 1, 1, -1)$ we get

$$Z_{\widehat{G}^0}(s\hat{\theta})^0 \cong (\mathrm{GL}(2, \mathbf{C}) \times \mathbf{C}^\times \times \mathbf{C}^\times)'$$

with the prime denoting the group of triples (A, a, b) with $\det A = ab$. Again in this case this group sits inside $Z_{\widehat{G}^0}(s\hat{\theta})$ as an index two subgroup and we are allowed to put an action of the Galois group on this group by interchanging the $\mathbf{C}^\times$ factors. In this case we only get an elliptic endoscopic group when this Galois action is non-trivial and we end up with the group whose F-points are

$$(\mathrm{GL}(2, F) \times E^\times)/F^\times$$

for a quadratic extension E/F. Here $F^\times$ is embedded in the product as $\{(x, x^{-1})\}$.

The stabilization of the full twisted trace formula is not yet known, however it should follow the same strategy implemented by Arthur to deal with the case of a connected group. In particular one needs to prove twisted analogues of the local conjectures stated above. Thus we need to know

(1) Local twisted transfer: (perhaps this is now known by combining work of Hales [Hal94] with recent work of Waldspurger)
(2) Twisted fundamental lemma: Flicker [Fli99]
(3) Twisted weighted fundamental lemma: Whitehouse [Whi05].

5. Comparison of trace formulae

Finally, modulo the twisted stabilization, in order to compare trace formulae, and prove results about the lifting of automorphic representations from GSp(4) to GL(4) one needs to be able to do the following.

(1) Spectrally understand the distributions S^H for the twisted endoscopic groups H of (G^0, θ). In the case that $H = \mathrm{GSp}(4)$ this has been done by Weissauer [Wei98]. The remaining groups are closely related to GL(2) and one can obtain a spectral expansion for S^H using results in the literature.
(2) Understand spectral transfer from $H \neq \mathrm{GSp}(4)$ to G^0. One knows results in this direction. For example one has lifting results for the H coming from the element $s = \mathrm{diag}(1, 1, -1, -1)$ by work of Ramakrishnan [Ram00].
(3) Finally, use the resulting spectral identity to prove lifting from GSp(4) to GL(4).

The rest of this paper will now be devoted to explicating the proof of the relevant twisted weighted fundamental lemma found in [Whi05].

6. The fundamental lemma for weighted orbital integrals

We now recall the statement of Arthur's conjectured twisted weighted fundamental lemma from [Whi05, Appendix].

Let F be a local p-adic field of characteristic zero. Let G^0 be a connected reductive group defined over F and let θ be an automorphism of G^0 defined over F. We form the disconnected group $G^+ = G^0 \rtimes \langle\theta\rangle$ and take the connected component $G = G^0 \rtimes \theta \subset G^+$. We assume that G is unramified over F.

Let $M = M^0 \rtimes \theta$ be a Levi subset of G, in the sense of [Art88b, p. 228]. We define the weighted orbital integral

$$J_M(\gamma, f) = |D_G(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)\, dx,$$

where $f \in C_c^\infty(G(F))$, $\gamma \in M(F)$ is strongly G^0-regular, $G_\gamma = \mathrm{Cent}(G^0, \gamma)^0$, as in [Art88b, p. 233].

Suppose that M' represents an unramified elliptic, twisted endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M^0. Here, s'_M is a semisimple element in the connected component $\widehat{M} = \widehat{M}^0 \rtimes \hat{\theta} \subset \widehat{G} = \widehat{G}^0 \rtimes \hat{\theta}$. We suppose that $\mathcal{M}'$ is an L-subgroup of ${}^L M^0 = \widehat{M}^0 \rtimes W_F$, and that ξ'_M is the identity embedding.

We define $\mathcal{E}_{M'}(G)$ as follows. Let $Z(\widehat{M})^\Gamma$ denotes the group of Γ-invariants in the centralizer of $\widehat{M}$ in $\widehat{M}^0$, $\mathcal{E}_{M'}(G)$ is the set of twisted endoscopic data for G^0 of the form $(G', \mathcal{G}', s', \xi')$, where s' lies in $s'_M Z(\widehat{M})^\Gamma$, $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$ and ξ' is the identity embedding of $\mathcal{G}' = \mathcal{M}'\widehat{G}'$ into ${}^L G^0$. The elements in $\mathcal{E}_{M'}(G)$ are taken up to translation of s' by $Z(\widehat{G})^\Gamma$.

We can now proceed as in [Art02, §5]. Set

$$\iota_{M'}(G, G') = |Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma||Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1},$$

and

$$r_M^G(k) = J_M(k, u)$$

where $k \in M(F)$ is strongly G^0-regular, and $u = u_K$ is the stabilizer in $G(F)$ of the unit in a Hecke algebra of $G^0(F)$.

Conjecture 6.1. *(The twisted weighted fundamental lemma) Let ℓ' be a strongly G^0-regular, stable conjugacy class in $M'(F)$. Then*

$$\sum_{k \in \Gamma_{G\text{-}reg}(M(F))} \Delta_{M,K}(\ell', k) r_M^G(k)$$

equals

$$\sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{M'}^{G'}(\ell'),$$

where $s_{M'}^{G'}(\ell')$ is the function defined uniquely for the unramified connected pair (G', M') in [Art02, Conjecture 5.1]*, and $\Delta_{M,K}$ is the twisted transfer factor for M^0, normalized relative to the hyperspecial maximal compact $K \cap M^0(F)$.*

Here $\Gamma_{\text{G-reg}}(M(F))$ denotes the set of G^0-strongly regular conjugacy classes in $M(F)$ and $\Delta_{M,K}$ is the transfer factor of [KS99, Section 5.3] deprived of the Δ_{IV} term (which has already been included in the definition of the weighted orbital integrals).

To recall the definition of $s_{M'}^{G'}(\ell')$ we set, for a strongly G^0-regular stable conjugacy class ℓ', in $M'(F)$

$$r_{M'}^{G'}(\ell') = \sum_{\gamma} r_{M'}^{G'}(\gamma)$$

where the sum is taken over the finitely many $M'(F)$ conjugacy classes γ in ℓ'. Taking θ to be trivial and M' quasi-split gives an inductive definition of $s_{M'}^{G'}(\ell')$ by the formula

$$s_{M'}^{G'}(\ell') = r_{M'}^{G'}(\ell') - \sum_{G'' \in \mathcal{E}_{M'}^0(G')} \iota_{M'}(G', G'') s_{M'}^{G''}(\ell'),$$

where $\mathcal{E}_{M'}^0(G') = \mathcal{E}_{M'}(G') \setminus \{G'\}$.

7. The twisted weighted fundamental lemma for GL(4) × GL(1)

We now outline some elements of the proof of the twisted weighted fundamental lemma in [Whi05]. We work over a local non-archimedean field F of residual characteristic $p > 2$. We denote by $\mathcal{O}_F$ the ring of integers in F and by $|\ |$ the multiplicative valuation on F.

We take $M = M^0 \rtimes \theta$ with M^0 the Levi component of a standard proper parabolic subgroup of $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$ stable under θ. Thus, M^0 is one of the following:

$$\left\{ \left(\begin{pmatrix} A & \\ & B \end{pmatrix}, a \right) : A, B \in \mathrm{GL}(2), a \in \mathrm{GL}(1) \right\},$$

$$\left\{ \left(\begin{pmatrix} a & & \\ & A & \\ & & b \end{pmatrix}, c \right) : A \in \mathrm{GL}(2), a, b, c \in \mathrm{GL}(1) \right\},$$

$$\left\{ \left(\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}, e \right) : a, b, c, d, e \in \mathrm{GL}(1) \right\}.$$

We refer to these as the (2,2) Levi, the (1,2,1) Levi and the diagonal Levi respectively. For each of these groups we list their unramified elliptic θ-twisted endoscopic groups.

For the (2,2) Levi the only elliptic endoscopic group is $M' = \mathrm{GL}(2)\times\mathrm{GL}(1)$ and we have

$$\mathcal{E}_{M'}(G) = \{\mathrm{GSp}(4), (\mathrm{GL}(2)\times\mathrm{GL}(2))'\},$$

where M' sits inside GSp(4) as the Klingen Levi.

For the (1,2,1) Levi we have $M' = \mathrm{GL}(2)\times\mathrm{GL}(1)$ as an elliptic endoscopic group, and

$$\mathcal{E}_{M'}(G) = \{\mathrm{GSp}(4)\},$$

where M' sits inside GSp(4) as the Siegel Levi. We also have the group M' with $M'(F) = E^\times \times F^\times$, where E/F is the unramified quadratic extension of F, as an elliptic endoscopic group, and

$$\mathcal{E}_{M'}(G) = \{\mathrm{GL}(2,E)', (\mathrm{GL}(2,F)\times E^\times)/F^\times\}.$$

Finally for the diagonal Levi, the only elliptic endoscopic group is $\mathrm{GL}(1)^3$ and we have

$$\mathcal{E}_{M'}(G) = \{\mathrm{GSp}(4), (\mathrm{GL}(2)\times\mathrm{GL}(2))'\}.$$

We now concentrate on the twisted weighted fundamental lemma for M^0 equal to the (2,2) Levi and $M' = \mathrm{GL}(2)\times\mathrm{GL}(1)$.

We take an element $\gamma\theta \in M(F)$ which we assume to be strongly G^0-regular. After conjugation by an element in $M^0(F)$ we can assume that $\gamma\theta$ has the form

$$\gamma\theta = (A, I, c)\theta.$$

Then the statement of the twisted weighted fundamental lemma takes the form

$$r_M^G\left(\left(\begin{pmatrix} A & \\ & I\end{pmatrix}, c\right)\theta\right) = r_{M'}^{\mathrm{GSp}(4)}\begin{pmatrix} c\det A & & \\ & cA & \\ & & c\end{pmatrix} + r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}\left(\begin{pmatrix} c\det A & \\ & c\end{pmatrix}, cA\right).$$

We assume, as we may, that $(A, I, c)\theta \in M(\mathcal{O}_F)$, and since the integrals are invariant under scaling by an element in $Z(G^0(\mathcal{O}_F))$ we can assume that $c = 1$. Hence the identity to be proven is

$$r_M^G\left(\left(\begin{pmatrix} A & \\ & I\end{pmatrix}, 1\right)\theta\right) = r_{M'}^{\mathrm{GSp}(4)}\begin{pmatrix} \det A & & \\ & A & \\ & & 1\end{pmatrix} + r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}\left(\begin{pmatrix} \det A & \\ & 1\end{pmatrix}, A\right),$$

for all semisimple $A \in \mathrm{GL}(2,\mathcal{O}_F)$ such that $1 \notin \{\lambda_1, \lambda_2, \lambda_1\lambda_2^{-1}, \lambda_1^{-1}\lambda_2\}$ where λ_1, λ_2 denote the eigenvalues of A.

Our next step is to reduce these integrals, which are defined as integrals on the full group, to integrals on the Levi subgroup. We treat the twisted integrals here. By the Iwasawa decomposition we have

$$G^0(F) = M^0(F)N_P(F)K,$$

where N_P denotes the unipotent radical of the upper triangular parabolic subgroup P with Levi component M^0 and $K = G^0(\mathcal{O}_F)$. We can then write the Haar measure on $G^0(F)$ as $dg = dm\ dn\ dk$. Using this, together with the right invariance of v_M under K, we get

$$r_M^G(\gamma\theta) = |D_M(\gamma\theta)|^{\frac{1}{2}} \int_{M_{\gamma\theta}(F)\backslash M^0(F)} 1_{K_M\theta}(m^{-1}\gamma\theta m)\ \sigma_P(m^{-1}\gamma\theta m)\ dm$$

where $K_M = M^0(F) \cap K$ and, for a strongly G^0-regular element $a\theta \in M(F)$, we define

$$\sigma_P(a\theta) = \int_{N_P(F)\cap K} v_M(\varphi_{a\theta}(n))\ dn,$$

where $\varphi_{a\theta} : N_P(F) \to N_P(F)$ is the inverse of the bijection

$$N_P(F) \to N_P(F) : n \mapsto a^{-1}na\theta(n).$$

We can make a similar reduction to the weighted integrals on GSp(4) and $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$. We let T denote the centralizer of A in GL(2). Then we have

$$r_{M'}^{\mathrm{GSp}(4)} = |D_{\mathrm{GL}(2)}(A)|^{\frac{1}{2}} \int_{T(F)\backslash \mathrm{GL}(2,F)} 1_{\mathrm{GL}(2,\mathcal{O}_F)}(g^{-1}Ag)\sigma'_{P_1}(g^{-1}Ag)\ dg,$$

where

$$\sigma'_{P_1}(B) = \sigma_{P_1}\begin{pmatrix} \det B & & \\ & B & \\ & & 1 \end{pmatrix},$$

with P_1 equal to the upper triangular Klingen parabolic in GSp(4). And we have

$$r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'} = |D_{\mathrm{GL}(2)}(A)|^{\frac{1}{2}} \int_{T(F)\backslash \mathrm{GL}(2,F)} 1_{\mathrm{GL}(2,\mathcal{O}_F)}(g^{-1}Ag)\sigma'_{P_2}(g^{-1}Ag)\ dg,$$

where

$$\sigma'_{P_2}(B) = \sigma_{P_2}\left(\begin{pmatrix} \det B & \\ & 1 \end{pmatrix}, B\right),$$

with P_2 the upper triangular parabolic in $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ with Levi component M'.

The twisted integrals have been reduced to twisted integrals on $\mathrm{GL}(2)\times\mathrm{GL}(2)\times\mathrm{GL}(1)$. However, we can make a further reduction to obtain untwisted integrals on GL(2). Let

$$m = (A_1, B_1, c_1) \in M^0(F).$$

Then we have

$$m^{-1}\gamma\theta m = (A_1^{-1}Aw^tB_1^{-1}w, B_1^{-1}w^tA_1^{-1}w, \det A_1B_1)\theta,$$

where

$$w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Thus in order for $m^{-1}\gamma\theta m \in K_M\theta$ we need

$$B_1^{-1}w^tA_1^{-1}w \in \mathrm{GL}(2,\mathcal{O}_F)$$

and hence $B_1 = w^tA_1^{-1}wB'_1$ with $B'_1 \in \mathrm{GL}(2,\mathcal{O}_F)$. But then we have

$$A_1^{-1}Aw^tB_1^{-1}w = A_1^{-1}AA_1w^tB'_1w,$$

which lies in $\mathrm{GL}(2,\mathcal{O}_F)$ if and only if $A_1^{-1}AA_1 \in \mathrm{GL}(2,\mathcal{O}_F)$. Hence we deduce that

$$r_M^G(\gamma\theta) = |D_{\mathrm{GL}(2)}(A)|^{\frac{1}{2}} \int_{T(F)\backslash \mathrm{GL}(2,F)} 1_{\mathrm{GL}(2,\mathcal{O}_F)}(g^{-1}Ag)\sigma'_P(g^{-1}Ag)\ dg,$$

where

$$\sigma'_P(B) = \sigma_P\left(\begin{pmatrix} B & \\ & I \end{pmatrix}, 1\right).$$

Thus all the integrals in the statement of the twisted weighted fundamental lemma have been reduced to invariant orbital integrals on GL(2). Since the integrands are invariant under conjugation by $\mathrm{GL}(2, \mathcal{O}_F)$ they can be computed using double coset decompositions of the form

$$\mathrm{GL}(2, F) = \coprod_{i \in I} T(F) z_i \, \mathrm{GL}(2, \mathcal{O}_F)$$

for a suitable set of representatives $\{z_i : i \in I\}$. This is carried out in [Whi05, Section 5]. The main task is to compute the terms σ'_P (and σ'_{P_1}, σ'_{P_2}) which are given as the integral of Arthur's weight function v_M over certain compact open subsets of $N_P(F)$. We don't give the calculations here but, to give a flavor of the computations, we give the evaluation of these weight functions on the unipotent radicals as found in [Whi05, Section 4]. For the twisted integrals we have

$$v_M \left(\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ & 1 & x_3 & x_4 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, 1 \right) = \log \max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1 x_4 - x_2 x_3|\}.$$

On GSp(4) we have

$$v_{M'} \begin{pmatrix} 1 & x & r & s \\ & 1 & 0 & r \\ & & 1 & -x \\ & & & 1 \end{pmatrix} = \log \max\{1, |x|, |r|, |s|\},$$

and on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ we have

$$v_{M'} \left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) = \log \max\{1, |y|\}.$$

For the proof of the twisted weighted fundamental lemma for the (1,2,1) Levi and both its twisted endoscopic groups we use the twisted topological Jordan decomposition; see [Whi05, Section 4.5]. This allows us to reduce the twisted weighted integrals to untwisted weighted integrals on twisted centralizers of semisimple elements. These in turn we relate to the weighted (with respect to the Klingen Levi) orbital integrals on GSp(4) computed in the proof of the fundamental lemma for the (2,2) Levi. We then use these previous calculations in the proof of the twisted weighted fundamental lemma for the (1,2,1) Levi.

For the case of the diagonal Levi we again use the twisted topological Jordan decomposition. The main part of the proof is in establishing an identity [Whi05, Lemma 8.2] between weighted (with respect to the minimal Levi) orbital integrals on Sp(4). We again reduce these integrals to orbital integrals on the Levi itself and so we need to compute the integral of the weight function over certain open subsets of the unipotent radical of a Borel subgroup. This we do, although the weight function is far more complicated in this case; see [Whi05, Lemma 4.7].

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