

ON THE STABLE TRACE FORMULA FOR Sp_4

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ABSTRACT. For many applications of the trace formula to problems in the theory of automorphic forms one needs to stabilize the trace formula. Such a stabilization has been achieved by Arthur, building on work of Langlands and Kottwitz, subject to conjectures for orbital integrals known as fundamental lemmas. For the group Sp_4 the standard invariant fundamental lemma has been established by Hales. However, for the stabilization of the full trace formula one needs to prove a fundamental lemma for weighted orbital integrals on Sp_4 . We prove this weighted fundamental lemma and thereby make Arthur's stabilization of the Sp_4 trace formula unconditional.

1. INTRODUCTION

The stabilization of the global trace formula for a connected reductive group has been achieved by Arthur in a series of papers [4], [3] and [5] subject to certain local conjectures for orbital integrals; see [4, Assumption 5.2(i)].

Let G be such a group defined over a number field F . For each nonarchimedean place v of F such that G is unramified over F_v Arthur defines a family of groups $\mathcal{U}(G_v, F_v)$; see [4, pg 222]. This set consists of groups defined over F_v , or some unramified extension thereof, and contains G over any finite unramified extension of F_v together with groups of dimension strictly smaller than $\dim G$. In order to stabilize the trace formula Arthur requires [4, Assumption 5.2] that for almost all v the weighted fundamental lemma [4, Conjecture 5.1] and the fundamental lemma for the Lie algebra hold for all groups in $\mathcal{U}(G_v, F_v)$.

Suppose now that F is a local nonarchimedean field of characteristic zero. We consider the symplectic group Sp_4 over F . The groups G in $\mathcal{U}(\mathrm{Sp}_4, F)$ with $\dim G < \dim \mathrm{Sp}_4$, are closely related to GL_2 , SL_2 and GL_1 and the necessary fundamental lemmas can be established from results already in the literature.

For Sp_4/F the standard fundamental lemma has been by Hales in [7]. The Lie algebra form of the fundamental lemma for Sp_4 follows from this. Thus one is left with proving the weighted fundamental lemma for Sp_4 . This fundamental lemma applies to a pair (M, M') where M is a Levi subgroup of Sp_4 and M' is a proper unramified elliptic endoscopic group for M . Taking $M = \mathrm{Sp}_4$ we recover the fundamental lemma proven by Hales. However, Sp_4 also has a proper Levi subgroup which possesses proper elliptic endoscopic groups. This is the Klingen Levi subgroup which is isomorphic to $\mathrm{SL}_2 \times \mathrm{GL}_1$.

1.1. Statement of theorem. We now describe the statement of the weighted fundamental lemma for the Klingen Levi subgroup of Sp_4 . We continue with F a local field of characteristic zero and residual characteristic $p > 0$. We let \mathcal{O}_F denote the ring of integers in F .

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We let

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

and

$$\mathrm{Sp}_4 = \{g \in \mathrm{GL}_4 : J^t g^{-1} J^{-1} = g\}.$$

We let P denote the Klingen parabolic subgroup of Sp_4 ; that is the upper triangular parabolic in Sp_4 with Levi subgroup

$$M = \left\{ \begin{pmatrix} a & & & \\ & g & & \\ & & & \\ & & & a^{-1} \end{pmatrix} : g \in \mathrm{SL}_2, a \in \mathrm{GL}_1 \right\}.$$

We let $K = \mathrm{Sp}_4(\mathcal{O}_F)$ and denote by 1_K the characteristic function of K . For $\gamma \in M(F)$ which is strongly Sp_4 -regular (i.e. for which $\mathrm{Sp}_{4,\gamma}$ is a torus) we let

$$r_M^{\mathrm{Sp}_4}(\gamma) = |D_G(\gamma)|^{\frac{1}{2}} \int_{\mathrm{Sp}_{4,\gamma}(F) \backslash \mathrm{Sp}_4(F)} 1_K(g^{-1}\gamma g) v_M(g) dg$$

as in [4, §5].

The group M has a unique proper unramified elliptic endoscopic group

$$M' = U_1 \times \mathrm{GL}_1$$

where U_1 denotes the unitary group in one variable attached to the unramified quadratic extension of F .

The theorem we prove in this paper is the following, which is Conjecture 5.1 of [4] for the pair (Sp_4, M) .

Theorem 1.1. *Assume that F has odd residual characteristic. For each strongly regular stable conjugacy class $\ell' \in M'(F)$ we have*

$$\sum_{k \in \Gamma(M(F))} \Delta_{K \cap M}(\ell', k) r_M^{\mathrm{Sp}_4}(k) = \sum_{G' \in \mathcal{E}_{M'}(\mathrm{Sp}_4)} \iota_{M'}(\mathrm{Sp}_4, G') s_{M'}^{G'}(\ell').$$

The left hand side is a sum over conjugacy classes in $M(F)$. The term $\Delta_{K \cap M}$ is the transfer of Langlands and Shelstad [10] for the pair (M, M') deprived of the term Δ_{IV} . It is normalized as in [6] relative to the maximal compact group $K \cap M$ of $M(F)$.

The set $\mathcal{E}_{M'}(\mathrm{Sp}_4)$ consists of endoscopic groups for Sp_4 which contain M' as a Levi subgroup; see [2, Section 4]. The coefficient $\iota_{M'}(\mathrm{Sp}_4, G')$ is defined for $G' \in \mathcal{E}_{M'}(\mathrm{Sp}_4)$ and vanishes unless G' is elliptic. The terms $s_{M'}^{G'}(\ell')$ are defined in [4, §5].

This paper is organized as follows. In Section 2 we determine the various terms appearing in the statement of Theorem 1.1. We then give an explicit statement of Theorem 1.1 in Section 3. We prove this statement in Section 4.

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2. PRELIMINARIES

In this section we compute the various terms which appear in the statement of Theorem 1.1.

2.1. Notation. In addition to the notation introduced above we use the following notation throughout this paper.

We let \mathfrak{p}_F denote the maximal ideal of \mathcal{O}_F . We fix a uniformizer π of F and let $|\cdot|$ denote the multiplicative valuation on F normalized such that $|\pi| = q^{-1}$, where q denotes the cardinality of the residue field of F . We let U_F denote the group of units in \mathcal{O} and set $U_F^1 = 1 + \mathfrak{p}_F \subset U_F$. We take v to be the additive valuation on F normalized such that $v(\pi) = 1$. We fix the additive measure on F to give \mathcal{O}_F volume 1.

We let E denote the unramified quadratic extension of F . We use the same notation for E as we do for F , with the addition of a subscript E . We let σ denote the non-trivial element of $\text{Gal}(E/F)$ and for $x \in E$ we sometimes use \bar{x} to denote $\sigma(x)$. We let E^1 denote the subgroup of E of elements of norm one.

For $a_1, \dots, a_n \in F$ we let $\text{diag}(a_1, \dots, a_n)$ denote the diagonal matrix in $\text{GL}_n(F)$ with the entries a_1, \dots, a_n along the diagonal. Throughout we denote by \log the logarithm to the base q .

2.2. Endoscopic groups. The dual group of M is $\widehat{M} = \text{SO}_3(\mathbf{C}) \times \mathbf{C}^\times$ which sits inside $\widehat{\text{Sp}}_4 = \text{SO}_5(\mathbf{C})$ as the Levi subgroup

$$\left\{ \begin{pmatrix} b & & & \\ & h & & \\ & & & \\ & & & b^{-1} \end{pmatrix} : h \in \text{SO}_3(\mathbf{C}), b \in \mathbf{C}^\times \right\}.$$

The unique proper unramified elliptic endoscopic group for M is $M' = U_1 \times \text{GL}_1$ where U_1 denotes the unitary group in one variable attached to E/F . To obtain this endoscopic group we take the element

$$s = \text{diag}(1, -1, 1, -1, 1) \in \widehat{M}.$$

The connected component of the centralizer of s in \widehat{M} is the diagonal torus and the action of $\text{Gal}(E/F)$ on this torus is given by

$$\sigma : \text{diag}(a, b, 1, b^{-1}, a^{-1}) \mapsto \text{diag}(a, b^{-1}, 1, b, a^{-1}).$$

We now determine the elliptic endoscopic groups in the set $\mathcal{E}_{M'}(\text{Sp}_4)$ which appears in the statement of Theorem 1.1. We recall from [2, Section 4] that the groups in $\mathcal{E}_{M'}(\text{Sp}_4)$ are given by elements in $\text{SO}_5(\mathbf{C})$ of the form $\text{diag}(a, -1, 1, -1, a^{-1})$ together with the Galois action coming from that on \widehat{M}' . We get elliptic endoscopic groups if and only if $a^2 = 1$.

When $a = -1$, the connected component of such an element is isomorphic to

$$\text{SO}_4(\mathbf{C}) \cong (\text{GL}_2(\mathbf{C}) \times \text{GL}_2(\mathbf{C}))' / \mathbf{C}^\times$$

where the prime denotes the subgroup of pairs with equal determinant, and \mathbf{C}^\times is diagonally embedded in the product. The group $\text{Gal}(E/F)$ acts by permuting the $\text{GL}_2(\mathbf{C})$ factors. We let H_1 denote the endoscopic group for Sp_4 obtained in this way. Thus if we let $G_1 = (\text{GL}_2 \times \text{GL}_2)' / \text{GL}_1$ over E , then we have

$$H_1(F) = \{(g_1, g_2) \in G_1(E) : (g_1, g_2) = (\bar{g}_2, \bar{g}_1)\}.$$

When $a = +1$, the connected component of such an element is isomorphic to

$$\mathrm{SO}_3(\mathbf{C}) \times \mathrm{SO}_2(\mathbf{C})$$

with $\mathrm{Gal}(E/F)$ acting on the $\mathrm{SO}_2(\mathbf{C})$ factor. Hence we get

$$H_2 = \mathrm{SL}_2 \times \mathrm{U}_1.$$

For the groups H_1 and H_2 we take M' sitting inside as the diagonal torus. For $H \in \mathcal{E}_{M'}(\mathrm{Sp}_4)$ we have, by definition,

$$\iota_{M'}(\mathrm{Sp}_4, H) = \frac{|Z(\widehat{M}')^{\mathrm{Gal}(E/F)}/Z(\widehat{M})|}{|Z(\widehat{H})^{\mathrm{Gal}(E/F)}|}.$$

Thus $\iota_{M'}(\mathrm{Sp}_4, H_1) = 2$ and $\iota_{M'}(\mathrm{Sp}_4, H_2) = 1$ (and $\iota_{M'}(\mathrm{Sp}_4, H)$ vanishes otherwise).

2.3. Norm maps. We take T to be the diagonal torus inside Sp_4 and $T_i = M'$ to be the diagonal torus inside H_i . In this section we work over the quadratic extension E of F .

We have $\widehat{T} \subset \mathrm{SO}_5(\mathbf{C})$ and $\widehat{T}_i \subset \widehat{H}_i \subset \mathrm{SO}_4(\mathbf{C})$ both sitting inside $\mathrm{SO}_5(\mathbf{C})$ as the diagonal torus. This gives rise to an isomorphism $N_i : T \xrightarrow{\sim} T_i$. One can readily check that the norm maps N_i are given by

$$N_1 : \mathrm{diag}(a, b, b^{-1}, a^{-1}) \mapsto (\mathrm{diag}(ab, 1), \mathrm{diag}(a, b)) \in H_1$$

and

$$N_2 : \mathrm{diag}(a, b, b^{-1}, a^{-1}) \mapsto (\mathrm{diag}(a, a^{-1}), b) \in H_2.$$

2.4. Transfer factor. We let $D \in F$ be such that $E = F(\sqrt{D})$. We assume, as we may, that D is a unit in \mathcal{O}_F . For $a \in F^\times$ and $\beta = \beta_1 + \beta_2\sqrt{D} \in E^1$, with $\beta_i \in F$, we write

$$\gamma(a, \beta) = \begin{pmatrix} a & & & & \\ & \beta_1 & \beta_2 D & & \\ & \beta_2 & \beta_1 & & \\ & & & & a^{-1} \end{pmatrix} \in M(F).$$

The stable conjugacy class of $\gamma(a, \beta)$ is a union of two conjugacy classes, with

$$\gamma'(a, \beta) = \begin{pmatrix} a & & & & \\ & \beta_1 & \pi^{-1}\beta_2 D & & \\ & \pi\beta_2 & \beta_1 & & \\ & & & & a^{-1} \end{pmatrix} \in M(F)$$

representing the other conjugacy class.

Under the norm map, the stable conjugacy class of $\gamma(a, \beta)$ is sent to the (stable) conjugacy class of

$$\gamma_1(a, \beta) = \left(\begin{pmatrix} a\beta & \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \\ & \beta \end{pmatrix} \right) \in M'(F) \subset H_1(F)$$

and to

$$\gamma_2(a, \beta) = \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \beta \right) \in M'(F) \subset H_2(F).$$

In our case the transfer factor $\Delta_{K \cap M}$ is computed in [10, Section 1.1]. We have

$$\Delta_{K \cap M}(\gamma_i(a, \beta), \gamma(a, \beta)) = (-1)^{v(\beta_2)},$$

and

$$\Delta_{K \cap M}(\gamma_i(a, \beta), \gamma'(a, \beta)) = (-1)^{v(\beta_2)+1}.$$

2.5. Weight functions. For a reductive group G defined over a nonarchimedean local field F and a Levi subgroup L of G Arthur defines a weight function

$$v_L : G(F) \rightarrow \mathbf{R}$$

in [1, Section 1]. Since v_L is left invariant under $L(F)$ and right invariant under K , a fixed hyperspecial maximal subgroup, it suffices to know v_L on the unipotent radical of a parabolic subgroup with Levi component M . For each pair (G, L) appearing in the statement of the fundamental lemma we compute this function on such a unipotent radical.

The definition of the weight functions involve a choice of measure on the spaces

$$\mathfrak{a}_L = \text{Hom}(X(L)_F, \mathbf{R})$$

where $X(L)_F$ denotes the group of characters of L defined over F . The norm maps of Section 2.3 allow us to relate the measure on this space when $L = M$ and the measure on the corresponding space for M' . We fix measures on these spaces such that they are related in this way.

For Sp_4 we compute the weight function v_M on N , the unipotent radical of the upper triangular Klingen parabolic subgroup. We write an element of $N(F)$ as

$$n(x_1, x_2, x_3) = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

with $x_1, x_2, x_3 \in F$.

Lemma 2.1. *For $n(x_1, x_2, x_3) \in N(F)$ we have*

$$v_M(n(x_1, x_2, x_3)) = \log \max\{1, |x_1|, |x_2|, |x_3|\}.$$

Proof. This can be computed for Sp_4 as it is for GSp_4 in [11, Lemma 4.12]. \square

For H_1 and H_2 the calculation of the weight functions can be deduced from the calculation for GL_2 found in [11, Lemma 4.15].

We let N_1 denote the unipotent radical of the upper triangular Borel subgroup of H_1 . So we have

$$N_1(F) = \left\{ n_1(x) = \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \right) : x \in E \right\}.$$

We now compute the weight function $v_{M'}^1$ on N_1 .

Lemma 2.2. *For $x \in E$ we have*

$$v_{M'}^1(n_1(x)) = \frac{1}{2} \log \max\{1, |x|_E\}.$$

We let N_2 denote the unipotent radical of the upper triangular Borel subgroup of H_2 . So we have

$$N_2(F) = \left\{ n_2(y) = \left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}, 1 \right) : y \in F \right\}.$$

We now compute the weight function $v_{M'}^2$ on N_2 .

Lemma 2.3. *For $y \in F$ we have*

$$v_{M'}^2(n_2(y)) = \log \max\{1, |y|\}.$$

3. EXPLICIT STATEMENT OF THE FUNDAMENTAL LEMMA

We now give an explicit statement of Theorem 1.1. We take $\gamma(a, \beta) \in M(F)$ to be G -regular. This is equivalent to demanding that $a \neq \pm 1$ and $\beta \neq \pm 1$.

The Iwasawa decomposition allows us to write the integrals on H_1 and H_2 as integrals on the torus M' ; see [11, Lemma 4.16]. We have

$$2s_{M'}^{H_1}(\gamma_1(a, \beta)) = 2r_{M'}^{H_1}(\gamma_1(a, \beta)).$$

And by [11, Lemma 4.16] we have

$$2r_{M'}^{H_1}(\gamma_1(a, \beta)) = |a\beta - 1|_E \int_{\mathfrak{p}_E^{-v_E(a\beta-1)}} \log \max\{1, |x|_E\} dx$$

if $a \in U_F$ and $\beta \in U_E$; and vanishes otherwise.

Similarly we have

$$s_{M'}^{H_2}(\gamma_2(a, \beta)) = r_{M'}^{H_2}(\gamma_2(a, \beta))$$

which equals, by [11, Lemma 4.16],

$$|a^2 - 1| \int_{\mathfrak{p}_F^{-v(a^2-1)}} \log \max\{1, |y|\} dy$$

if $a \in U_F$ and $\beta \in U_E$; and vanishes otherwise.

Since the left hand side of Theorem 1.1 also vanishes unless $a \in U_F$ and $\beta \in U_E$ the statement we wish to prove is given by the following.

Theorem 3.1. *Assume that the residual characteristic of F is odd. Then for all $a \in U_F \setminus \{\pm 1\}$ and $\beta = \beta_1 + \beta_2\sqrt{D} \in (U_E \cap E^1) \setminus \{\pm 1\}$ we have*

$$(-1)^{v(\beta_2)} \left(r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) - r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) \right)$$

equal to

$$|a\beta - 1|_E \int_{\mathfrak{p}_E^{-v_E(a\beta-1)}} \log \max\{1, |x|_E\} dx + |a^2 - 1| \int_{\mathfrak{p}_F^{-v(a^2-1)}} \log \max\{1, |y|\} dy.$$

4. PROOF OF THE FUNDAMENTAL LEMMA

We now prove Theorem 3.1 using the topological Jordan decomposition. We take a and β as in the statement of the theorem. We let $K_M = K \cap M$. By the topological Jordan decomposition we can write $\gamma(a, \beta)$ uniquely as

$$\gamma(a, \beta) = u(a, \beta)s(a, \beta) = s(a, \beta)u(a, \beta)$$

with $s(a, \beta) \in K_M$ absolutely semisimple (of finite order prime to q) and $u(a, \beta) \in K_M$ topologically unipotent, that is $u(a, \beta)^{q^n}$ converges to the identity in K as $n \rightarrow \infty$. It's clear that if we write $a = a_u a_s$ and $\beta = \beta_u \beta_s$ in their topological Jordan decompositions then we have

$$u(a, \beta) = \gamma(a_u, \beta_u)$$

and

$$s(a, \beta) = \gamma(a_s, \beta_s).$$

Having fixed a and β we set $s = s(a, \beta)$ and $u = u(a, \beta)$. We let $G_1 = Z_{\mathrm{Sp}_4}(s)$, $M_1 = Z_M(s)$, $N_1 = Z_N(s)$, $P_1 = Z_P(s)$ and $K_{M_1} = K \cap M_1(F)$.

If $g \in \mathrm{Sp}_4(F)$ is such that

$$g^{-1}\gamma(a, \beta)g \in K$$

then we have $g \in G_1(F)K$; see [8]. Using this and the Iwasawa decomposition for $G_1(F)$ we can write $r_M^{\mathrm{Sp}_4}(\gamma(a, \beta))$ as an integral on $M_1(F)$ as in [11, Lemma 4.17].

Lemma 4.1. *For $a \in M_1(F)$ strongly G_1 -regular let $\varphi_a : N_1 \rightarrow N_1$ denote the inverse of the bijection $N_1 \rightarrow N_1 : n \mapsto a^{-1}n^{-1}an$ and define*

$$\sigma_{P_1}(a) = \int_{N_1(F) \cap K} v_M(\varphi_a(n)) \, dn.$$

Then we have

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = |D_{M_1}(u)|^{\frac{1}{2}} \int_{M_{1,u}(F) \backslash M_1(F)} \mathbf{1}_{K_{M_1}}(m^{-1}um) \sigma_{P_1}(m^{-1}um) \, dm.$$

We have the following possibilities for $s = s(a, \beta)$.

1. s lies in $Z(\mathrm{Sp}_4)$,
2. s lies in $Z(M)$ but not in $Z(\mathrm{Sp}_4)$,
3. s does not lie in $Z(M)$.

We now prove Theorem 3.1 in each of these cases.

4.1. Proof of Theorem 3.1 when s is central in Sp_4 . First we suppose that s is central in Sp_4 . So we have $\gamma(a, \beta) = \pm u(a, \beta)$ and hence, since scaling by -1 does not alter any of the terms in the statement of Theorem 3.1, we may assume that $\gamma(a, \beta)$ is topologically unipotent.

In [11, Section 7.2.1] certain twisted weighted orbital integrals on $\mathrm{GL}_4 \times \mathrm{GL}_1$ were computed by writing them as weighted orbital integrals on Sp_4 . If we set

$$G = \mathrm{GL}_4 \times \mathrm{GL}_1$$

and let α denote the automorphism of G given by

$$\alpha : (g, e) \mapsto (J^t g^{-1} J^{-1}, e \det g).$$

Then in the notation of [11, Section 7.2.1] we have

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = r_{(1,2,1)}^G((\gamma(a, \beta), 1)\alpha)$$

and

$$r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = r_{(1,2,1)}^G((\gamma'(a, \beta), 1)\alpha).$$

Here $r_{(1,2,1)}^G(\gamma\alpha)$ denotes the α -twisted weighted orbital integral on G with respect to the Levi subgroup of G whose projection onto the first factor is the $(1,2,1)$ Levi in GL_4 . The proof of Theorem 3.1 in this case then follows from [11, Proposition 7.1], together with the observations made at the beginning of [11, Section 7.2].

4.2. Proof of Theorem 3.1 when s is central in M . In this case we may assume that

$$s = \mathrm{diag}(a_s, 1, 1, a_s^{-1}) \in M(F)$$

with $a_s \neq 1$. Then we have the following.

Lemma 4.2. *We have $N_1 = \{I\}$ unless $a_s = -1$ in which case we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & & & x_3 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

First we suppose that $a_s \neq -1$. Then we have

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = 0$$

and

$$r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = 0.$$

On the other hand since $a_s \neq \pm 1$ we must have $v(a^2 - 1) = 0$ and since $\beta \in U_E^1$ so we have $v_E(a\beta - 1) = 0$. Thus Theorem 3.1 is proven in this case.

Now we assume that $a_s = -1$. For ease of notation we set

$$I(a) = |a^2 - 1| \int_{\mathfrak{p}_F^{-v(a^2-1)}} \log \max\{1, |x|\} dx.$$

Then we have $r_M^{\mathrm{Sp}_4}(\gamma(a, \beta))$ equal to

$$|y| \operatorname{vol}\{g \in M_{\gamma(a, \beta)}(F) \setminus M(F) : g^{-1}\gamma(a, \beta)g \in K\} I(a)$$

and $r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta))$ equal to

$$|y| \operatorname{vol}\{g \in M_{\gamma'(a, \beta)}(F) \setminus M(F) : g^{-1}\gamma'(a, \beta)g \in K\} I(a).$$

Using the double coset decompositions for SL_2 from [9, Section 2] we get

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = q^{-v(\beta_2)} \left(1 + (q+1) \sum_{m=1}^{\lfloor \frac{v(\beta_2)}{2} \rfloor} q^{2m-1} \right) I(a)$$

and

$$r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = q^{-v(\beta_2)} (q+1) \left(\sum_{m=0}^{\lfloor \frac{v(\beta_2)-1}{2} \rfloor} q^{2m} \right) I(a).$$

Hence we get

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) - r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = (-1)^{v(\beta_2)} I(a).$$

On the other hand since $\beta \in U_E^1$ we have $v_E(a\beta - 1) = 0$ and hence Theorem 3.1 is proven in this case.

4.3. Proof of Theorem 3.1 when s is elliptic. In this case we have $s = \gamma(a_s, \beta_s)$ with $\beta_s \notin F$. One can readily check that we have the following.

Lemma 4.3. *We have $N_1 = \{I\}$ unless $a_s = \pm 1$ in which case we have*

$$N_1 = \left\{ \left(\begin{array}{ccc} 1 & & x_3 \\ & 1 & \\ & & 1 \end{array} \right) \right\}.$$

First we assume that $a_s \neq \pm 1$. Then in this case we have

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = 0$$

and

$$r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = 0.$$

On the other hand, since $\beta_s \notin F$ we have $v_E(a\beta - 1) = 0$ and since $a_s \neq \pm 1$ we have $v(a^2 - 1) = 0$. Hence Theorem 3.1 is proven in this case.

Now we assume that $a_s = \pm 1$ then we have

$$r_M^{\mathrm{Sp}_4}(\gamma(a, \beta)) = |a^2 - 1| \int_{\mathfrak{p}_F^{-v(a^2-1)}} \log \max\{1, |x|\} dx.$$

Since we must have $v(\beta_2) = 0$ it follows that

$$r_M^{\mathrm{Sp}_4}(\gamma'(a, \beta)) = 0.$$

Again since $\beta_s \notin F$ so we have $v_E(a\beta - 1) = 0$ and the proof of Theorem 3.1 is complete.

REFERENCES

- [1] James Arthur. The local behaviour of weighted orbital integrals. *Duke Math. J.*, 56(2):223–293, 1988.
- [2] James Arthur. Canonical normalization of weighted characters and a transfer conjecture. *C. R. Math. Acad. Sci. Soc. R. Can.*, 20(2):33–52, 1998.
- [3] James Arthur. A stable trace formula. II. Global descent. *Invent. Math.*, 143(1):157–220, 2001.
- [4] James Arthur. A stable trace formula. I. General expansions. *J. Inst. Math. Jussieu*, 1(2):175–277, 2002.
- [5] James Arthur. A stable trace formula. III. Proof of the main theorems. *Ann. of Math. (2)*, 158(3):769–873, 2003.
- [6] Thomas C. Hales. A simple definition of transfer factors for unramified groups. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 109–134. Amer. Math. Soc., Providence, RI, 1993.
- [7] Thomas C. Hales. The fundamental lemma for $\mathrm{Sp}(4)$. *Proc. Amer. Math. Soc.*, 125(1):301–308, 1997.
- [8] David Kazhdan. On lifting. In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 209–249. Springer, Berlin, 1984.
- [9] Jean-Pierre Labesse and Robert P. Langlands. L -indistinguishability for $\mathrm{SL}(2)$. *Canad. J. Math.*, 31(4):726–785, 1979.
- [10] Robert P. Langlands and Diana Shelstad. On the definition of transfer factors. *Math. Ann.*, 278(1-4):219–271, 1987.
- [11] David Whitehouse. The twisted weighted fundamental lemma for the transfer of automorphic forms from $\mathrm{GSp}(4)$ to $\mathrm{GL}(4)$. *Astérisque*, (302):291–436, 2005. Formes automorphes. II. Le cas du groupe $\mathrm{GSp}(4)$.

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