

Applications of the relative trace formula to central L-values

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Based on separate joint work with Kimball
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L -values

- F number field
- E/F quadratic extension
- π cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbf{A}_F)$
- π_E base change of π to E
- $\Omega : \mathbf{A}_F^\times E^\times \backslash \mathbf{A}_E^\times \rightarrow \mathbf{C}^\times$

Associated to this data is the L -function

$$L(s, \pi_E \otimes \Omega) = \varepsilon(s, \pi_E \otimes \Omega) L(1 - s, \pi_E \otimes \Omega).$$

Eg. $\Omega = 1_E$: $L(s, \pi_E) = L(s, \pi) L(s, \pi \otimes \delta_{E/F})$.

We are interested in $L(1/2, \pi_E \otimes \Omega)$.

Applications

1. Identities between $L(1/2, \pi_E \otimes \Omega)$ and period integrals over tori.
2. Average of $L(1/2, \pi_E \otimes \Omega)$ over π of fixed level and weight.
3. Subconvexity for $L(1/2, \pi_E \otimes \Omega)$ as π and Ω vary in certain ranges.

Period Integrals

Let D be a quaternion algebra over F such that

1. π transfers to π^D on D^\times/F , and
2. $E \hookrightarrow D$.

For $\varphi \in \pi^D$ define

$$P_D(\varphi) = \int_{\mathbf{A}_F^\times E^\times \backslash \mathbf{A}_E^\times} \varphi(t) \Omega^{-1}(t) dt.$$

Theorem. (Waldspurger, Jacquet)

$L(1/2, \pi_E \otimes \Omega) = 0$ if and only if $P_D(\varphi) = 0$ for all D and $\varphi \in \pi^D$.

Vanishing of Period Integrals

Note that

$$P_D \in \text{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega).$$

Since π^D and Ω both factor,

$$\text{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega) = \bigotimes_v \text{Hom}_{E_v^\times}(\pi_v^D, \Omega_v).$$

Hence if $\text{Hom}_{E_v^\times}(\pi_v^D, \Omega_v) = 0$ for some v then $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$.

Theorem. (Tunnell)

- If $\varepsilon(1/2, \pi_E \otimes \Omega) = -1$, this happens for all D .
- If $\varepsilon(1/2, \pi_E \otimes \Omega) = +1$, this happens for all but one D (characterized by local ε -factors).

We now assume that $\varepsilon(1/2, \pi_E \otimes \Omega) = +1$ and fix D as in the Theorem.

Test Vectors

For such D , $\text{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega)$ is 1-dimensional.

Assumption. π and Ω have disjoint ramification.

Gross-Prasad: Provide line V_π in π^D (fixed by a compact subgroup) such that

$$\ell(\varphi) \neq 0$$

for all $0 \neq \varphi \in V_\pi$ and $0 \neq \ell \in \text{Hom}_{\mathbf{A}_E^\times}(\pi^D, \Omega)$.

Hence,

Theorem. $L(1/2, \pi_E \otimes \Omega) = 0$ if and only if $P_D(\varphi) = 0$ for any $0 \neq \varphi \in V_\pi$.

Suffices to check just one period integral.

Refinement of Waldspurger's result

Wish to make the connection between

$$L(1/2, \pi_E \otimes \Omega)$$

and $P_D(\varphi_\pi)$ for any $0 \neq \varphi_\pi \in V_\pi$ more explicit.

That is, obtain an identity of the form

$$\frac{|P_D(\varphi_\pi)|^2}{(\varphi_\pi, \varphi_\pi)} = C(E, \pi, \Omega) L(1/2, \pi_E \otimes \Omega)$$

with $C(E, \pi, \Omega)$ explicitly determined.

Summary of partial results:

	F	E	π	Ω
Gross	\mathbb{Q}	imag	wt 2	
Zhang	tot. real	imag	wt 0, 2	
Xue	tot. real	imag		
Popa	\mathbb{Q}	real		unram

NB. Not all use Gross-Prasad test vector.

Theorem. (Martin - W.) We establish such an identity without any further restrictions.

The constant $C(E, \pi, \Omega)$ takes the form

$$\frac{1}{L(1, \pi, Ad)} \frac{1}{2\sqrt{d_{E/F^c}(\Omega)}} \prod_{\text{bad } v} C_v(E, \pi, \Omega).$$

For example, if

- π_v is Steinberg
- E_v is unramified
- Ω_v is unramified.

Then,

$$C_v(E, \pi, \Omega) = 1 - q_v^{-1}.$$

About the proof

Wish to compare,

1. Periods of base change representations on $GL(2)/E$ over a split torus, with
2. Periods of representations of D^\times/F over the non-split torus E^\times .

The relative trace formula is a tool to study periods of automorphic forms.

Theorem is established via a comparison of relative trace formulas.

Average L -values

Now make some further assumptions/notations.

- $F = \mathbf{Q}$ (or totally real)
- $E = \mathbf{Q}(\sqrt{-d})$ imaginary quadratic
- $\Omega : \mathbf{A}_{\mathbf{Q}}^{\times} E^{\times} \backslash \mathbf{A}_E^{\times} \rightarrow \mathbf{C}^{\times}$
- N squarefree integer, product of an odd number of primes all inert and unramified in E

For $k \geq 2$ denote by $\mathcal{F}(N, 2k)$

{normalized eigenforms of level N , wt $2k$ }.

Interested in $L(1/2, \pi_E \otimes \Omega)$ as we average over $\pi \in \mathcal{F}(N, 2k)$.

Theorem. (Feigon - W.)

Assume $N > dc(\Omega)$ and $2k > 2m = wt(\Omega)$,
then

$$\frac{1}{N} \binom{2k-2}{k+m-1} \sum_{\pi \in \mathcal{F}(N, 2k)} \frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \pi, Ad)}$$

is equal to

$$2L^{S(\Omega)}(1, \chi_{-d}),$$

where $S(\Omega)$ denotes the places where Ω ramifies.

Remarks.

- $\Omega = 1_E$: asymptotics known by Duke and Iwaniec-Sarnak.
- Generalizes work of Michel-Ramakrishnan
- More complicated formula for smaller level

About the proof

Let D be the quaternion algebra over \mathbb{Q} such that

$$\text{Ram}(D) = \{p \mid N\} \cup \{\infty\}.$$

We fix $E \hookrightarrow D$.

Result above gives, for $\pi \in \mathcal{F}(N, 2k)$,

$$\frac{|P_D(\varphi_\pi)|^2}{(\varphi_\pi, \varphi_\pi)} = C(E, \Omega, N, k) \frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \pi, Ad)}.$$

Instead we prove the similar result about

$$\sum_{\pi \in \mathcal{F}(N, 2k)} \frac{|P_D(\varphi_\pi)|^2}{(\varphi_\pi, \varphi_\pi)}.$$

A relative trace formula

We set $G = PD^\times / \mathbf{Q}$.

Let $f \in C_c^\infty(G(\mathbf{A}_\mathbf{Q}))$. Have a map,

$$R(f) : L^2(G(\mathbf{Q}) \backslash G(\mathbf{A}_\mathbf{Q})) \rightarrow L^2(G(\mathbf{Q}) \backslash G(\mathbf{A}_\mathbf{Q})).$$

$R(f)$ is an integral operator with kernel

$$K_f(x, y) = \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma y).$$

On the other hand using the spectral decomposition of $L^2(G(\mathbf{Q}) \backslash G(\mathbf{A}_\mathbf{Q}))$ we have,

$$K_f(x, y) = \sum_{\pi^D} \sum_{\varphi \in \mathcal{B}(\pi^D)} (\pi^D(f)\varphi)(x) \overline{\varphi(y)},$$

where

- π^D : automorphic reps of $G(\mathbf{A}_\mathbf{Q})$
- $\mathcal{B}(\pi^D)$: orthonormal basis of π^D .

Trace formula: integrate $K_f(x, y)$ over the diagonal.

Relative trace formula: integrate $K_f(x, y)$ over other interesting subgroups.

We consider

$$I^D(f) = \int \int_{(\mathbf{A}_{\mathbf{Q}}^{\times} E^{\times} \backslash \mathbf{A}_E^{\times})^2} K_f(t_1, t_2) \Omega(t_1^{-1} t_2) dt_1 dt_2.$$

We pick f to isolate

$$\{\pi^D : \pi \in \mathcal{F}(N, 2k)\},$$

so that from the spectral expansion for $K_f(x, y)$ we have,

$$I^D(f) = \sum_{\pi \in \mathcal{F}(N, 2k)} \frac{|P_D(\varphi_{\pi})|^2}{(\varphi_{\pi}, \varphi_{\pi})}.$$

Geometrically, can realize D as

$$D = \left\{ \begin{pmatrix} a & \varepsilon b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in E \right\},$$

for some $\varepsilon \in \mathbf{Q}^\times$.

Representatives for $E^\times \backslash D^\times / E^\times$ are

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & \varepsilon b \\ \bar{b} & 1 \end{pmatrix} : b \in E^\times / E^1 \right\}.$$

Hence, $I^D(f)$ is equal to the sum of,

$$\text{vol}(\mathbf{A}_\mathbf{Q}^\times E^\times \backslash \mathbf{A}_E^\times) \int_{\mathbf{A}_\mathbf{Q}^\times \backslash \mathbf{A}_E^\times} f(t) \Omega(t) dt,$$

$$\text{vol}(\mathbf{A}_\mathbf{Q}^\times E^\times \backslash \mathbf{A}_E^\times) \int_{\mathbf{A}_\mathbf{Q}^\times \backslash \mathbf{A}_E^\times} f \left(t \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \right) \Omega(t) dt,$$

and

$$\sum_{b \in E^\times / E^1} \int_{(\mathbf{A}_\mathbf{Q}^\times \backslash \mathbf{A}_E^\times)^2} f \left(t_1 \begin{pmatrix} 1 & \varepsilon b \\ \bar{b} & 1 \end{pmatrix} t_2 \right) \Omega(t_1 t_2) dt_1 dt_2.$$

Point: For N large all orbital integrals vanish except for the identity double coset.

Subconvexity

Continuing with the assumptions above. The L -function

$$L(s, \pi_E \otimes \Omega)$$

satisfies the functional equation

$$L(1-s, \pi_E \otimes \Omega) = ((Ndc(\Omega))^2)^{1/2-s} L(s, \pi_E \otimes \Omega).$$

Hence, the convexity bound is,

$$L(1/2, \pi_E \otimes \Omega) \ll ((Ndc(\Omega))^2)^{1/2}.$$

We now fix E and consider characters Ω which ramify above one prime $\ell > d$.

Theorem. (Feigon - W.)

We have, for all $\varepsilon > 0$, and $\pi \in F(N, 2k)$,

$$L(1/2, \pi_E \otimes \Omega) \ll_{E,k,\varepsilon} (N + c(\Omega)^{\frac{1}{2}+\varepsilon}) (\log N)^2.$$

Hence,

Corollary. (Feigon - W.)

For every $t \in (0, 0.16)$ and N in the range $c(\Omega)^{\frac{2.02t}{1-2.02t}} < N < c(\Omega)^{\frac{1-2t}{1+2.02t}}$,

$$L(1/2, \pi_E \otimes \Omega) \ll_{E,k,t} (Ndc(\Omega))^{\frac{1}{2}-t}.$$

Proof of the Theorem. We use the fact that

$$L(1/2, \pi_E \otimes \Omega) \ll (\log N)^2 \sum_{\pi \in \mathcal{F}(N, 2k)} \frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \pi, Ad)}.$$

Again use the relative trace formula to compute this sum.

Bound N comes from the identity coset.

Bound $c(\Omega)^{\frac{1}{2}+\varepsilon}$ comes from other cosets.