APPLICATIONS OF THE RELATIVE TRACE FORMULA TO L-VALUES

DAVID WHITEHOUSE

Abstract. These notes are from a series of lectures given between 1-20 July 2008 at the Workshop on p-adic Representations at the Morningside Center of Mathematics of the Chinese Academy of Sciences. The goal of these lectures is to explain Jacquet's approach [Jac86], via the relative trace formula, to a result of Waldspurger [Wal85] on the vanishing of quadratic base change L-functions associated to cusp forms on GL(2). We will then describe some further applications of the relative trace formula ([Guo96], [RR05], [MW], [FW]) to central L-values and some progress in extending Jacquet's approach to higher rank.

Contents

Date: January 1, 2009.

1. Introduction and background

The relative trace formula is a tool, introduced by Jacquet, to study periods of automorphic forms.

1.1. **Periods.** Let F be a number field and G a reductive algebraic group defined over F. Let φ be an automorphic form on G. So $\varphi : G(F)\backslash G(\mathbf{A}_F) \to \mathbf{C}$ satisfying certain growth conditions. Let H be a subgroup of F also defined over F. Thus we have an inclusion $H(F)\backslash H(\mathbf{A}_F) \subset G(F)\backslash G(\mathbf{A}_F)$ and we can form a period integral

$$\int_{H(F)\backslash H(\mathbf{A}_F)} \varphi(h)\chi(h)\ dh \tag{1}$$

where we may allow a character $\chi : H(F)\backslash H(\mathbf{A}_F) \to \mathbf{C}^\times$. (Note that if the quotient $H(F)\backslash H(\mathbf{A}_F)$ has infinite volume then this integral may need to be regularized in an appropriate way.)

Definition 1.1. *Let π be an automorphic representation of $G(\mathbf{A}_F)$. We say π is (H,χ)-distinguised if for some automorphic form φ in the space of π,*

$$\int_{H(F)\backslash H(\mathbf{A}_F)} \varphi(h)\chi(h)\ dh \neq 0. \tag{2}$$

Familiar examples of period integrals include the following.

Example 1.2. Let $G = \mathrm{GL}(2)$ and

$$H = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\} \tag{3}$$

the upper triangular unipotent subgroup. Then the period integrals

$$\int_{F\backslash \mathbf{A}_F} \varphi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\ dx \tag{4}$$

are precisely those which give the cuspidality criterion for an automorphic representation π of GL(2). Namely π is not cuspidal if and only if π is H-distinguished.

Example 1.3. Let $G = \mathrm{GL}(2)$ and

$$H = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right\}. \tag{5}$$

Then the period integrals

$$\int_{F^\times\backslash \mathbf{A}_F^\times} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-\frac{1}{2}}\ da, \tag{6}$$

are related to the L-function $L(s,\pi)$ of the automorphic representation π associated to φ. Thus $L(\frac{1}{2},\pi) \neq 0$ if and only if π is H-distinguised.

Example 1.4. Let E/F be a quadratic extension. Consider $H = U(2, E/F) \subset G = \mathrm{GL}(2,E)$. Then a cuspidal automorphic representation of G is H-distinguished if and only if it arises via base change from F.

Frequently period integrals appear in problems related to automorphic forms and number theory.

They can be used to detect functorial images: The first and third examples above are of this form.

As in the second example they can be related to L-values, the topic of these lectures will be to discuss an example of this sort.

1.2. **The Relative Trace Formula.** We continue with the notations from the previous subsection. Consider the space $L^2(G(F)\backslash G(\mathbf{A}_F))$. The group $G(\mathbf{A}_F)$ acts on this space by right translation. Namely if $g \in G(\mathbf{A}_F)$ and $\varphi \in L^2(G(F)\backslash G(\mathbf{A}_F))$ then

$$(R(g)\varphi)(x) = \varphi(xg). \tag{7}$$

Given a smooth compactly supported function $f \in C_c^\infty(G(\mathbf{A}_F))$ we obtain a linear map

$$R(f) : L^2(G(F)\backslash G(\mathbf{A}_F)) \to L^2(G(F)\backslash G(\mathbf{A}_F)) \tag{8}$$

given by integrating f against the action of the group $G(\mathbf{A}_F)$,

$$(R(f)\varphi)(x) = \int_{G(\mathbf{A}_F)} f(y)\varphi(xy)\ dy. \tag{9}$$

We can rearrange this expression

$$(R(f)\varphi)(x) = \int_{G(\mathbf{A}_F)} f(y)\varphi(xy)\ dy \tag{10}$$

$$= \int_{G(\mathbf{A}_F)} f(x^{-1}y)\varphi(y)\ dy \tag{11}$$

$$= \int_{G(F)\backslash G(\mathbf{A}_F)} \left(\sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \right) \varphi(y)\ dy. \tag{12}$$

Thus we see that $R(f)$ is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y). \tag{13}$$

Let us assume for simplicity that the quotient $G(F)\backslash G(\mathbf{A}_F)$ is compact. Then we have

$$L^2(G(F)\backslash G(\mathbf{A}_F)) = \bigoplus_{\pi \in \mathcal{A}(G)} V_\pi \tag{14}$$

where the sum is over the automorphic representations $\mathcal{A}(G)$ of $G(\mathbf{A}_F)$ and V_π is the π-isotypic component. From this we see that we have another expression for the kernel as

$$K_f(x,y) = \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}(\pi)} (R(f)\varphi)(x)\overline{\varphi(y)}, \tag{15}$$

where $\mathcal{B}(\pi)$ is an orthonormal basis for the π-isotypic space V_π. (To check this just note that integration against $K_f(x,y)$ agrees with $R(f)$ for each basis element.)

The trace formula attempts to compute the trace of $R(f)$. This can be computed as

$$\operatorname{tr} R(f) = \int_{G(F)\backslash G(\mathbf{A}_F)} K_f(x,x)\ dx \tag{16}$$

$$= \int_{G(F)\backslash G(\mathbf{A}_F)} \sum_{\gamma \in G(F)} f(x^{-1}\gamma x)\ dx. \tag{17}$$

Interchanging summation and integration yields

$$\operatorname{tr} R(f) = \sum_{\gamma \in \Gamma(G(F))} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}_F)) \int_{G_\gamma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(x^{-1}\gamma x)\ dx, \tag{18}$$

where $\Gamma(G(F))$ denotes the set of conjugacy classes in $G(F)$ and G_γ is the centralizer of γ in G. This a powerful expression for the trace of $R(f)$. On the one hand one can try and evaluate this expression for special test functions, this leads to formulas for traces of Hecke operators acting on spaces of modular forms, or one may use this expression to try to compare traces for different groups which leads to cases of functoriality (e.g. the Jacquet-Langlands correspondence).

For the relative trace formula we consider the same kernel but we now, instead of integrating over $G(F)\backslash G(\mathbf{A}_F)$ embedded in the product diagonally we attempt to integrate $K_f(x,y)$ over

$$H(F)\backslash H(\mathbf{A}_F) \times H(F)\backslash H(\mathbf{A}_F) \hookrightarrow G(F)\backslash G(\mathbf{A}_F) \times G(F)\backslash G(\mathbf{A}_F). \tag{19}$$

In fact more generally one may take two subgroups H_1 and H_2 and integrate over their products, perhaps also with the addition of some characters. One can play the same game as with the trace formula. On the one hand from the spectral expansion for $K_f(x,y)$ we see that

$$\begin{aligned} I(f) &= \int_{H(F)\backslash H(\mathbf{A}_F)} \int_{H(F)\backslash H(\mathbf{A}_F)} K_f(h_1,h_2)\ dh_1\ dh_2 \\ &= \int_{H(F)\backslash H(\mathbf{A}_F)} \int_{H(F)\backslash H(\mathbf{A}_F)} \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}(\pi)} (R(f)\varphi)(h_1)\overline{\varphi(h_2)}\ dh_1\ dh_2 \\ &= \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}(\pi)} \int_{H(F)\backslash H(\mathbf{A}_F)} (R(f)\varphi)(h_1)\ dh_1 \int_{H(F)\backslash H(\mathbf{A}_F)} \overline{\varphi(h_2)}\ dh_2. \end{aligned}$$

On the other hand

$$\begin{aligned} I(f) &= \int_{H(F)\backslash H(\mathbf{A}_F)} \int_{H(F)\backslash H(\mathbf{A}_F)} \sum_{\gamma \in G(F)} f(h_1^{-1}\gamma h_2)\ dh_1\ dh_2 \\ &= \sum_{\gamma \in H(F)\backslash G(F)/H(F)} \operatorname{vol}(H_\gamma(F)\backslash H_\gamma(\mathbf{A}_F)) \int_{H_\gamma(\mathbf{A}_F)\backslash (H(\mathbf{A}_F)\times H(\mathbf{A}_F))} f(h_1^{-1}\gamma h_2)\ dh_1\ dh_2. \end{aligned}$$

Thus we obtain an explicit expression for sums of period integrals. As in the case of the trace formula one can use this expression to attempt to compute $I(f)$ for certain interesting f, in the examples we consider we will see this leads to the study of averages of families of L-functions, or one can use this expression to compare $I(f)$ with other relative trace formulas, in the example we consider we will see that this leads to interesting relationships between L-values and period integrals.

2. Waldspurger's Theorem

2.1. **The Jacquet-Langlands correspondence.** Let D be a quaternion algebra over F. Thus D is a simple algebra of dimension 4 over F with center F. Let v be a place of F. Then we can form the algebra $D_v = D \otimes_F F_v$ which is a quaternion algebra over the local field F_v. Over F_v there are, up to isomorphism, two such algebras (assuming $F_v \not\cong \mathbf{C}$), either $D_v \cong M(2, F_v)$ or else D_v is the unique quaternion division algebra over F_v. We recall there is a bijection

(20)
$$\{\text{quaternion algebras over } F\} \leftrightarrow \{\text{finite subsets of } \Sigma_F \setminus \Sigma_{F,\mathbf{C}} \text{ of even cardinality}\},$$

where Σ_F denotes the set of places of F and $\Sigma_{F,\mathbf{C}}$ the (possibly empty) set of complex places of F.. This map is given by assigning to D the set

$$\mathrm{Ram}(D) = \{v \in \Sigma_F : D_v \text{ is a division algebra }\}. \tag{21}$$

The group $D^\times$ is an algebraic group defined over F. The Jacquet-Langlands correspondence yields a map

$$\mathrm{JL} : \mathcal{A}(D^\times) \hookrightarrow \mathcal{A}(\mathrm{GL}(2)) \tag{22}$$

from the automorphic representations of $D^\times$ to those of $\mathrm{GL}(2)$. The map is given by taking the tensor product of local maps. Namely if $\pi = \otimes_v' \pi_v$ then $\mathrm{JL}(\pi) = \otimes_v' \mathrm{JL}_v(\pi_v)$. The local maps JL_v are somewhat involved for $v \in \mathrm{Ram}(D)$ but at the places $v \notin \mathrm{Ram}(D)$ we have $D_v^\times \cong \mathrm{GL}(2, F_v)$ and in this case JL_v is the identity map. Due to the strong multiplicity one theorem for $\mathrm{GL}(2)$ it is often just enough to know the Jacquet-Langlands correspondence at the places $v \notin \mathrm{Ram}(D)$.

Fact 2.1. The image of JL in $\mathcal{A}(\mathrm{GL}(2))$ is equal to the set of automorphic representations $\pi = \otimes_v' \pi_v$ such that π_v is a discrete series representation of $\mathrm{GL}(2, F_v)$ at the places $v \notin \mathrm{Ram}(D)$.

2.2. **Waldspurger's Theorem.** We fix the following notation:

- F a number field,
- $\mathbf{A}_F$ the adeles of F,
- π a cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbf{A}_F)$,
- E/F a quadratic extension,
- η the quadratic character of $F^\times \backslash \mathbf{A}_F^\times$ associated to the extension E/F by class field theory.

Let $X(E)$ denote the set of (isomorphism classes of) quaternion algebras D/F which contain the quadratic extension E of F. We recall that a quaternion algebra D contains the extension E if and only if all the places $v \in \mathrm{Ram}(D)$ do not split in E.

Given a cuspidal automorphic representation π of $\mathrm{PGL}(2, \mathbf{A}_F)$ we let $X(E, \pi)$ denote the (finite) set of $D \in X(E)$ such that π lies in the image of the Jacquet-Langlands correspondence from $\mathcal{A}(D^\times)$.

Let $L(s, \pi)$ denote the L-function of π. We recall this is given by a product

$$L(s, \pi) = \prod_{v \in \Sigma_F} L(s, \pi_v) \tag{23}$$

of local L-factors which converges provided $\Re s \gg 0$. In general the factors $L(s, \pi_v)$ are hard to define, but in the case that π_v is unramified it is easy. In this case associated to π_v is a pair of unramified characters $\chi_1, \chi_2 : F_v^\times \to \mathbf{C}^\times$ such that π_v

is equal to the representation of $\mathrm{GL}(2, F_v)$ induced from the character of the Borel subgroup

$$\chi : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d). \tag{24}$$

Then we have

$$L(s, \pi_v) = L(s, \chi_1)L(s, \chi_2) = \frac{1}{(1 - \chi_1(\varpi_v)q_v^{-s})(1 - \chi_2(\varpi_v)q_v^{-s})}, \tag{25}$$

where ϖ_v is a uniformizer at v and q_v denotes the cardinality of the residue class field at v. We recall that $L(s, \pi)$ has an analytic continuation to all $s \in \mathbf{C}$ and satisfies a functional equation

$$L(s, \pi) = \epsilon(s, \pi)L(1 - s, \pi), \tag{26}$$

since π has trivial central character.

We can twist our automorphic representation π by the character η. We denote by $\pi \otimes \eta$ the representation of $\mathrm{PGL}(2, \mathbf{A}_F)$ given by

$$(\pi \otimes \eta)(g) = \pi(g)\eta(\det g). \tag{27}$$

It is not hard to check that $\pi \otimes \eta$ is also automorphic. We let

$$L(s, \pi_E) = L(s, \pi)L(s, \pi \otimes \eta) \tag{28}$$

In fact π_E is an automorphic representation itself, namely of $\mathrm{GL}(2, E)$, which arises from the base change functoriality of Langlands.

Let $D \in X(E, \pi)$ and let π^D denote the automorphic representation of $PD^\times$ associated to π by the Jacquet-Langlands correspondence. For each automorphic form $\varphi \in \pi^D$ we define period integrals,

$$P_D(\varphi) = \int_{\mathbf{A}_F^\times E^\times \backslash \mathbf{A}_E^\times} \varphi(t)\ dt. \tag{29}$$

Here the embedding of $\mathbf{A}_E^\times$ into $D^\times(\mathbf{A}_F)$ is induced by $E \hookrightarrow D$.

Theorem 2.2. *(Waldspurger* [Wal85]*) $L(1/2, \pi_E) = 0$ if and only if $P_D(\varphi) = 0$ for all $D \in X(E, \pi)$ and $\varphi \in \pi^D$.*

Example 2.3. Suppose $F = \mathbf{Q}$, π is unramified everywhere and E is an imaginary quadratic field. So π is generated by a function φ on the set

$$Z(\mathbf{A_Q})\,\mathrm{GL}(2, \mathbf{Q})\backslash\,\mathrm{GL}(2, \mathbf{A_Q})/K \tag{30}$$

where

$$K = O(2, \mathbf{R}) \times \prod_p \mathrm{GL}(2, \mathbf{Z}_p) \tag{31}$$

is a maximal compact subgroup of $\mathrm{GL}(2, \mathbf{A_Q})$. By strong approximation,

(32)
$$Z(\mathbf{A_Q})\,\mathrm{GL}(2, \mathbf{Q})\backslash\,\mathrm{GL}(2, \mathbf{A_Q})/K \xrightarrow{\sim} Z(\mathbf{R})\backslash\,\mathrm{GL}(2, \mathbf{R})/K_\infty K_{fin} \cap \mathrm{GL}(2, \mathbf{Q}) \xrightarrow{\sim} \mathcal{H}/\,\mathrm{SL}(2, \mathbf{R}).$$

Note that we have $X(E, \pi) = \{M(2, F)\}$. On the other hand

$$P_D(\varphi) = \int_{\mathbf{A_Q}^\times E^\times \backslash \mathbf{A}_E^\times} \varphi(t)\ dt = \int_{\mathbf{A_Q}^\times E^\times \backslash \mathbf{A}_E^\times / \mathbf{A}_E^\times \cap K} \varphi(t)\ dt \tag{33}$$

Remark 2.4. In fact Waldspurger proves a more general result which allows one to twist by a character Ω of the idele class group of E. More precisely one may take a unitary character of $E^\times\backslash\mathbf{A}_E^\times$ which is trivial when restricted to $\mathbf{A}_F^\times$. Then the theorem above asserts that $L(1/2,\pi_E\otimes\Omega)=0$ if and only if

$$\int_{\mathbf{A}_F^\times E^\times\backslash\mathbf{A}_E^\times}\varphi(t)\Omega^{-1}(t)\ dt \tag{34}$$

for all $D\in X(E,\pi)$ and $\varphi\in\pi^D$. This result was also reproven by Jacquet using the relative trace formula [Jac87]. We shall not discuss this relative trace formula here.

3. Relative Trace Formula: Plan of Attack

Let $f\in C_c^\infty(\mathrm{PGL}(2,\mathbf{A}_F))$. We recall the definition of $K_f(x,y)$ above. We let A denote the diagonal torus in $\mathrm{PGL}(2)$. We define a distribution

$$I(f)=\lim_{c\to\infty}\int_{A(F)\backslash A^c(\mathbf{A}_F)}\int_{A(F)\backslash A^c(\mathbf{A}_F)}K_f(a,b)\eta(\det b)\ da\ db, \tag{35}$$

where

$$A^c(\mathbf{A}_F)=\left\{\begin{pmatrix}a&0\\0&1\end{pmatrix}\in A(\mathbf{A}_F):c^{-1}<|a|<c\right\}. \tag{36}$$

We recall that the kernel $K_f(x,y)$ has a spectral expansion in terms of the automorphic representations of $\mathrm{PGL}(2,\mathbf{A}_F)$. The cuspidal part is given by

$$K_{f,cusp}(x,y)=\sum_{\pi\in\mathcal{A}_{cusp}(\mathrm{PGL}(2))}\sum_{\varphi\in\mathcal{B}(\pi)}(R(f)\varphi)(x)\overline{\varphi(y)}. \tag{37}$$

We denote by $I_{cusp}(f)$ the contribution to $I(f)$ from the cuspidal kernel. Interchanging summation and integration gives

$$I_{cusp}(f)=\sum_{\pi\in\mathcal{A}_{cusp}(\mathrm{PGL}(2))}I_\pi(f), \tag{38}$$

where

$$I_\pi(f)=\sum_{\varphi\in\mathcal{B}(\pi)}\int_{A(F)\backslash A(\mathbf{A}_F)}(R(f)\varphi)(a)\ da\overline{\int_{A(F)\backslash A(\mathbf{A}_F)}\varphi(b)\eta(\det b)\ db}. \tag{39}$$

Integrals of cusp forms on $\mathrm{GL}(2)$ over the diagonal torus are related to the central value of their L-function. Given this it is not hard to show that we have $I_\pi(f)=0$ for all $f\in C_c^\infty(\mathrm{PGL}(2,\mathbf{A}_F))$ if and only if $L(1/2,\pi_E)=0$.

Let $D\in X(E)$ and set $G_D=PD^\times$. We let T denote the torus in G_D such that $T(F)=F^\times\backslash E^\times$. We define a distribution

$$I^D(f^D)=\int_{T(F)\backslash T(\mathbf{A}_F)}\int_{T(F)\backslash T(\mathbf{A}_F)}K_{f^D}(t_1,t_2)\ dt_1\ dt_2, \tag{40}$$

for $f^D\in C_c^\infty(G_D(\mathbf{A}_F))$. Again by interchanging summation and integration, and with similar notations as above, we have

$$I^D_{cusp}(f^D)=\sum_{\pi^D\in\mathcal{A}_{cusp}(G_D)}I_{\pi^D}(f^D), \tag{41}$$

where

$$I_{\pi^D}(f^D) = \sum_{\varphi \in \mathcal{B}(\pi^D)} P_D(R(f^D)\varphi)\overline{P_D(\varphi)}. \tag{42}$$

And we have $I_{\pi^D}(f^D) = 0$ for all $f^D \in C_c^\infty(G_D(\mathbf{A}_F))$ if and only if $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$.

Thus one way to prove Waldspurger's result is to compare the distributions $I_\pi(f)$ with $I_{\pi^D}(f^D)$ for $D \in X(E,\pi)$. The point is that the kernels $K_f(x,y)$ and $K_{f^D}(x,y)$ also have geometric expansions. We shall carry out a similar analysis of the geometric sides of the distributions. The upshot of the geometric comparison will be an identity of the form

$$I_{cusp}(f) = \sum_{D \in X(E)} I^D_{cusp}(f^D) \tag{43}$$

provided that f and $\{f^D : D \in X(E)\}$ match in a suitable sense. We can then use the "fundamental lemma for the Hecke algebra" to go from this identity involving all cusp forms to identities,

$$I_\pi(f) = \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D) \tag{44}$$

for each cuspidal automorphic representation π of $\mathrm{PGL}(2,\mathbf{A}_F)$. From here the proof of Waldspurger's theorem is almost immediate.

4. Geometric side of the relative trace formula

4.1. **Integral over elliptic torus.** We first carry out the analysis for the geometric expansion of $I^D(f^D)$. In this case the quotient $T(F)\backslash T(\mathbf{A}_F)$ is compact. Recall,

$$K_{f^D}(x,y) = \sum_{\gamma \in \mathrm{PGL}(2,F)} f^D(x^{-1}\gamma y). \tag{45}$$

We define a distribution

$$I^D(f^D) = \int_{T(F)\backslash T(\mathbf{A}_F)} \int_{T(F)\backslash T(\mathbf{A}_F)} K_{f^D}(t_1,t_2)\; dt_1\; dt_2. \tag{46}$$

We begin by reordering the summation in $K_{f^D}(x,y)$ according to $T(F)$ double cosets. We have,

$$K_{f^D}(x,y) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \sum_{\gamma \in T(F)\delta T(F)} f^D(x^{-1}\gamma y). \tag{47}$$

If we fix $\delta \in G_D(F)$ then we see that the map

$$T(F) \times T(F) \to T(F)\delta T(F) : (t_1,t_2) \to t_1^{-1}\delta t_2, \tag{48}$$

is surjective with kernel $T_\delta(F)$ where,

$$T_\delta = \left\{(t_1,t_2) \in T \times T : t_1^{-1}\delta t_2 = \delta\right\}. \tag{49}$$

Hence,

$$K_{f^D}(x,y) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \sum_{(\gamma_1,\gamma_2) \in T_\delta(F)\backslash(T(F)\times T(F))} f(x^{-1}\gamma_1^{-1}\delta\gamma_2 y). \tag{50}$$

Recall,

$$I^D(f^D) = \int_{T(F)\backslash T(\mathbf{A}_F)} \int_{T(F)\backslash T(\mathbf{A}_F)} K_{f^D}(t_1, t_2)\ dt_1\ dt_2. \tag{51}$$

We now use the expression (50) for $K_{f^D}(x, y)$. On interchanging the first summation and integration we obtain,

$$I^D(f^D) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \int_{T(F)\backslash T(\mathbf{A}_F)} \int_{T(F)\backslash T(\mathbf{A}_F)} \sum_{(\gamma_1,\gamma_2)\in T_\delta(F)\backslash(T(F)\times T(F))} f^D(t_1^{-1}\gamma_1^{-1}\delta\gamma_2 t_2)\ dt_1\ dt_2. \tag{52}$$

By interchanging the inner sum and integration we obtain,

$$I^D(f^D) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \int_{T_\delta(F)\backslash(T(\mathbf{A}_F)\times T(\mathbf{A}_F))} f(t_1^{-1}\delta t_2)\ dt_1\ dt_2, \tag{53}$$

which is clearly equal to

$$I^D(f^D) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \operatorname{vol}(T_\delta(F)\backslash T_\delta(\mathbf{A}_F)) \int_{T_\delta(\mathbf{A}_F)\backslash(T(\mathbf{A}_F)\times T(\mathbf{A}_F))} f(t_1^{-1}\delta t_2)\ dt_1\ dt_2. \tag{54}$$

We define

$$I^D_\delta(f^D) = \int_{T_\delta(\mathbf{A}_F)\backslash(T(\mathbf{A}_F)\times T(\mathbf{A}_F))} f(t_1^{-1}\delta t_2)\ dt_1\ dt_2. \tag{55}$$

In a compact form we have shown that,

$$I^D(f^D) = \sum_{\delta \in T(F)\backslash G_D(F)/T(F)} \operatorname{vol}(T_\delta(F)\backslash T_\delta(\mathbf{A}_F)) I^D_\delta(f^D). \tag{56}$$

4.2. **Integral over split torus.** We now give the geometric expansion for the distribution $I(f)$. In this case the quotient $A(F)\backslash A(\mathbf{A}_F)$ has infinite volume and we will need to be more careful when we carry out the analysis of the geometric side of the trace formula. Recall,

$$K_f(x, y) = \sum_{\gamma \in \mathrm{PGL}(2,F)} f(x^{-1}\gamma y). \tag{57}$$

We wish to define an distribution

$$I(f) = \lim_{c\to\infty} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} K_f(a, b)\eta(\det b)\ da\ db, \tag{58}$$

where, for $c > 0$,

$$A^c(\mathbf{A}_F) = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \in \mathrm{PGL}(2, \mathbf{A}_F) : c^{-1} < |a| < c \right\}. \tag{59}$$

We set

$$I^c(f) = \int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} K_f(a, b)\eta(\det b)\ da\ db. \tag{60}$$

We begin by carrying out the same process on $I^c(f)$ as we did above. Reordering the summation according to $A(F)$ double cosets,

$$K_f(x, y) = \sum_{\delta\in A(F)\backslash \mathrm{PGL}(2,F)/A(F)} \sum_{\gamma\in A(F)\delta A(F)} f(x^{-1}\gamma y). \tag{61}$$

If we fix $\delta \in \mathrm{PGL}(2, F)$ then we see that the map

$$A(F) \times A(F) \to A(F)\delta A(F) : (a, b) \to a^{-1}\delta b, \tag{62}$$

is surjective with kernel $A_\delta(F)$ where,

$$A_\delta = \left\{(a, b) \in A \times A : a^{-1}\delta b = \delta\right\}. \tag{63}$$

Hence,

$$K_f(x, y) = \sum_{\delta \in A(F)\backslash \mathrm{PGL}(2,F)/A(F)} \sum_{(\gamma_1,\gamma_2)\in A_\delta(F)\backslash(A(F)\times A(F))} f(x^{-1}\gamma_1^{-1}\delta\gamma_2 y). \tag{64}$$

We now use the expression (64) for $K_f(x, y)$. On interchanging the first summation and integration we obtain,

(65)

$$I^c(f) = \sum_{\delta\in A(F)\backslash \mathrm{PGL}(2,F)/A(F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \sum_{(\gamma_1,\gamma_2)\in A_\delta(F)\backslash(A(F)\times A(F))} f(a^{-1}\gamma_1^{-1}\delta\gamma_2 b)\eta(b)\, da\, db.$$

We set

(66)

$$I^c_\delta(f) = \int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \sum_{(\gamma_1,\gamma_2)\in A_\delta(F)\backslash(A(F)\times A(F))} f(a^{-1}\gamma_1^{-1}\delta\gamma_2 b)\eta(b)\, da\, db.$$

We wish to understand the limit of $I^c_\delta(f)$ as $c \to \infty$. We begin by describing the double coset space $A(F)\backslash \mathrm{PGL}(2, F)/A(F)$.

Lemma 4.1. ([Jac86, (1.3)]) *The following is a complete set of representatives for* $A(F)\backslash \mathrm{PGL}(2, F)/A(F)$*:*

$$\left\{\begin{pmatrix}1 & a\\ 1 & 1\end{pmatrix} : a \in F \setminus \{0, 1\}\right\} \tag{67}$$

and

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, n_+ = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, n_- = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\} \tag{68}$$

and

$$\left\{w = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, wn_+ = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, wn_- = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right\}. \tag{69}$$

The first set of coset representatives are called "regular". If we take

$$\delta = \begin{pmatrix}1 & x\\ 1 & 1\end{pmatrix} \in \mathrm{PGL}(2, F) \tag{70}$$

with $x \notin \{0, 1\}$ then A_δ is trivial and the integral

$$I_\delta(f) := \int_{A(\mathbf{A}_F)} \int_{A(\mathbf{A}_F)} f(a^{-1}\delta b)\eta(b)\, da\, db \tag{71}$$

converges absolutely. Thus we have $I^c_\delta(f) \to I_\delta(f)$ as $c \to \infty$. Furthermore for a given function f, $I_\delta(f)$ vanishes for all but finitely many regular elements δ.

We now need to understand the contribution to the relative trace formula from the remaining six "irregular" double cosets. We begin with the identity element. In this case we clearly have,

$$A_e = \{(a, a) : a \in A\}. \tag{72}$$

Thus the integral above is equal to

$$\int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \sum_{(\gamma_1,1)\in A(F)\times A(F)} f(a^{-1}\gamma_1^{-1}b)\eta(b)\ da\ db. \tag{73}$$

Interchanging integration and summation yields

$$\int_{A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} f(a^{-1}b)\eta(b)\ da\ db. \tag{74}$$

Making the change of variables $a \mapsto ab$ yields,

$$\int f(a^{-1}) \left(\int_{A(F)\backslash A^c(\mathbf{A}_F)} \eta(b)\ db \right)\ da. \tag{75}$$

Clearly the inner integral vanishes for any c! Hence we see that $I_\delta^c(f) = 0$ for all c. In a similar way one can show that the contribution from the double coset $A(F)wA(F) = A(F)w$ is also zero.

The analysis of the remaining double cosets is similar. We will illustrate the case of the element n_+. In this case A_{n_+} is trivial and we have

$$I_{n_+}(f) = \int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A(F)\backslash A^c(\mathbf{A}_F)} \sum_{(\gamma_1,\gamma_2)\in A(F)\times A(F)} f(a^{-1}\gamma_1^{-1}n_+\gamma_2 b)\eta(b)\ da\ db. \tag{76}$$

We interchange the first summation and integration to obtain

$$\int_{A(F)\backslash A^c(\mathbf{A}_F)} \int_{A^c(\mathbf{A}_F)} \sum_{\gamma\in A(F)} f(a^{-1}n_+\gamma b)\eta(b)\ da\ db. \tag{77}$$

We introduce the function $\varphi_f(x,y)$ on $(x,y) \in \mathbf{A}_F^\times \times \mathbf{A}_F$ given by

$$\varphi_f(x,y) = f\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right). \tag{78}$$

Then φ_f is compactly supported and the integral above is, after a change of variables, equal to

$$\int_{F^\times\backslash \mathbf{A}_F^c} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \varphi_f(ab^{-1},\gamma b)\eta(b)\ da\ db, \tag{79}$$

where $\mathbf{A}_F^c = \left\{ a \in \mathbf{A}_F^\times : c^{-1} < |a| < c \right\}$.

We set $\mathbf{A}_F^{c,+} = \left\{ a \in \mathbf{A}_F^\times : 1 < |a| < c \right\}$. Then we can break up the integration as the sum of

$$\int_{F^\times\backslash \mathbf{A}_F^{c,+}} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \varphi_f(ab^{-1},\gamma b)\eta(b)\ da\ db, \tag{80}$$

and

$$\int_{F^\times\backslash \mathbf{A}_F^{c,+}} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \varphi_f(ab,\gamma b^{-1})\eta(b)\ da\ db. \tag{81}$$

We let $\hat{\varphi}_f$ denote the Fourier transform on φ_f in the second variable. Applying Poisson summation to the second integral then gives,

$$\int_{F^\times\backslash \mathbf{A}_F^{c,+}} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \hat{\varphi}_f(ab,\gamma b)|b|\eta(b)\ da\ db. \tag{82}$$

Thus the we have $I^c_{n_+}(f)$ equal to the sum of

$$\int_{F^\times\backslash \mathbf{A}_F^{c,+}} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \varphi_f(ab^{-1},\gamma b)\eta(b)\ da\ db, \tag{83}$$

and

$$\int_{F^\times\backslash \mathbf{A}_F^{c,+}} \sum_{\gamma\in F^\times} \int_{\mathbf{A}_F^c} \hat{\varphi}_f(ab,\gamma b)|b|\eta(b)\ da\ db. \tag{84}$$

It is now evident that this expressions have a limit as $c \to \infty$ which we denote by $I_{n_+}(f)$. Recalling Tate's thesis we see that $I_{n_+}(f)$ is equal to the value at $s=0$ of the analytic continuation of

$$\int_{\mathbf{A}_F^\times} \int_{\mathbf{A}_F^\times} \varphi_f(a,b)|b|^s\eta(b)\ d^\times a\ d^\times b. \tag{85}$$

The contributions from the elements wn_+, n_- and wn_- can be dealt with similarly.

Thus we can write

$$I(f) = \sum_{\delta\in A(F)\backslash \operatorname{PGL}(2,F)/A(F)} I_\delta(f) \tag{86}$$

where $I_\delta(f)$ is defined in the appropriate way as above.

5. Comparison of the relative trace formulas

We are attempting to establish that if f and $\{f^D : D \in X(E)\}$ match in a suitable (and as yet to be defined) sense then

$$I(f) = \sum_{D\in X(E)} I^D(f^D).$$

From the geometric expansions above we see that these distributions have an expansion in terms of orbital integrals over double cosets. Our goal in this section is to explain how to match the double cosets which appear on either side of this desired identity and then explain how to match functions.

5.1. Matching of double cosets.

We now that F is a field of characteristic zero, either a number field or a local field.

Lemma 5.1. *([Jac86, (1.3)]) The following is a complete set of representatives for $A(F)\backslash \operatorname{PGL}(2,F)/A(F)$:*

$$\left\{\begin{pmatrix}1 & a\\ 1 & 1\end{pmatrix} : a \in F\setminus\{0,1\}\right\} \tag{87}$$

and

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, n_+ = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, n_- = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\} \tag{88}$$

and

$$\left\{w = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, wn_+ = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, wn_- = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right\}. \tag{89}$$

The first set of coset representatives are called regular.

Suppose now we take D to be a quaternion algebra containing the quadratic extension E of F. Then D can be realized as

$$D_\epsilon = \left\{ \begin{pmatrix} \alpha & \epsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in E \right\}, \tag{90}$$

here $\bar{\alpha}$ denotes the Galois conjugate of α. We note that the isomorphism class of D_ϵ depends on the image of ϵ in $F^\times / N_{E/F} E^\times$.

Lemma 5.2. *([Jac86, (1.1)]) The following is a complete set of representatives for $T(F) \backslash G_\epsilon(F) / T(F)$:*

$$\left\{ \begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix} \in G_\epsilon(F) : \beta \in E^\times / E^1 \right\} \tag{91}$$

and

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \right\}. \tag{92}$$

Again the first set of coset representatives are called regular.

Given these two lemmas we see that we have a bijection between the regular cosets in $A(F) \backslash \mathrm{PGL}(2, F) / A(F)$ and those in $T(F) \backslash G_\epsilon(F) / T(F)$ as we vary over $\epsilon \in F^\times / N_{E/F} E^\times$.

Lemma 5.3. *We have a bijection between the regular cosets in $A(F) \backslash \mathrm{PGL}(2, F) / A(F)$ and $\coprod_{\epsilon \in F^\times / N_{E/F} E^\times} T(F) \backslash G_\epsilon(F) / T(F)$ given by*

$$N : \begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \epsilon\beta\bar{\beta} \\ 1 & 1 \end{pmatrix}. \tag{93}$$

5.2. **Matching of functions.** We have seen how to match up (most of) the double coset spaces $\coprod_{\epsilon \in F^\times / N_{E/F} E^\times} T(F) \backslash G_\epsilon(F) / T(F)$ and $A(F) \backslash \mathrm{PGL}(2, F) / A(F)$. We now wish to show we can match up functions which preserve this bijection. That is we hope to establish the following:

Theorem 5.4. *(Global matching) Let $f \in C_c^\infty(\mathrm{PGL}(2, \mathbf{A}_F))$ then for each $\epsilon \in F^\times / N_{E/F} E^\times$ there exists $f_\epsilon \in C_c^\infty(G_\epsilon(\mathbf{A}_F))$ such that if $\delta \in G_\epsilon(F)$ is a regular element then*

$$I_\delta(f_\epsilon) = I_{N\delta}(f), \tag{94}$$

where $N\delta$ denotes the norm of δ as defined in Lemma 5.3.

We recall that $C_c^\infty(\mathrm{PGL}(2, \mathbf{A}_F))$ is spanned by functions of the form

$$f = \prod_v f_v \tag{95}$$

where f_v is the characteristic function of $\mathrm{PGL}(2, \mathcal{O}_{F_v})$ for all but finitely many v. Furthermore for such factorizable functions,

$$I_\delta(f) = \int_{A_\delta(\mathbf{A}_F) \backslash (A(\mathbf{A}_F) \times A(\mathbf{A}_F))} f(a^{-1} \delta b) \eta(b) \, d^\times a \, d^\times b \tag{96}$$

$$= \prod_v \int_{A_\delta(F_v) \backslash (A(F_v) \times A(F_v))} f(a_v^{-1} \delta b_v) \eta_v(b_v) \, d^\times a_v \, d^\times b_v. \tag{97}$$

Thus in order to prove the global Theorem 5.4 it suffices to prove the following local Theorems:

Theorem 5.5. *(Local matching) Let E/F be a quadratic extension of local fields. Let $f \in C_c^\infty(\mathrm{PGL}(2,F))$ then for each $\epsilon \in F^\times/N_{E/F}E^\times$ there exists $f_\epsilon \in C_c^\infty(G_\epsilon(F))$ such that if $\delta \in G_\epsilon(F)$ is a regular element then*

$$I_\delta(f_\epsilon) = I_{N\delta}(f), \tag{98}$$

where $N\delta$ denotes the norm of δ as defined in Lemma 5.3.

Theorem 5.6. *(Fundamental lemma) Let E/F be an unramified quadratic extension of non-archimedean local fields. Let $\mathbf{1}_K \in C_c^\infty(\mathrm{PGL}(2,F))$ be the characteristic function of $K = \mathrm{PGL}(2,\mathcal{O}_F)$. Then in Theorem 5.5 we can take $f_\epsilon = 0$ if $\epsilon \notin N_{E/F}E^\times$ and if $\epsilon \in N_{E/F}E^\times$ then we can take f_ϵ to be the characteristic function of a maximal compact subgroup in $G_\epsilon(F)$.*

In fact, for later use, we will need to prove a stronger version of this result where we allow any Hecke function.

Remark 5.7. Of course we can only apply these matching results at the places of F which are inert in E. However, if v is a place which is split in E then $G_\epsilon(F_v) \cong \mathrm{PGL}(2,F_v)$ and matching of functions is trivial in this case.

In the remainder of this section we will discuss the proof of Theorem 5.5.

Given $\epsilon \in F^\times/N_{E/F}E^\times$ and $f_\epsilon \in C_c^\infty(G_\epsilon(F))$ we consider the function $H(x, f_\epsilon)$ of $x \in F \setminus \{0,1\}$ defined by $H(x) = 0$ if $x \notin \epsilon NE^\times$, and

$$H(\epsilon\beta\bar{\beta}, f_\epsilon) = I_\delta(f_\epsilon) \tag{99}$$

where

$$\delta = \begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix}. \tag{100}$$

We set $H(x, f_\epsilon) = 0$ if $x \notin \epsilon N_{E/F}E^\times$. Similarly given $f \in C_c^\infty(\mathrm{PGL}(2,F))$ we can define a function $H(x,f)$ of $x \in F \setminus \{0,1\}$ by

$$H(a,f) = I_\delta(f) \tag{101}$$

where

$$\delta = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}. \tag{102}$$

The strategy for proving Theorem 5.4 is clear. We will find criteria which assure that a function $H(x)$ on $F \setminus \{0,1\}$ is of the form $H(x, f_\epsilon)$. We will then verify that the functions $H(x,f)$ satisfy the necessary criteria.

Before continuing onto the general case we examine a special case which will motivate what is to come.

Lemma 5.8. *Let $H : F \setminus \{0,1\} \to \mathbf{C}$ be a smooth compactly supported function such that*

1. *$H(x) = 0$ for $x \notin \epsilon NE^\times$,*
2. *$H(x) = 0$ for x in a neighborhood of 0 and 1.*

Then there exists $f_\epsilon \in C_c^\infty(G_\epsilon(F))$ such that $H(x) = H(x, f_\epsilon)$ for all $x \in F \setminus \{0,1\}$.

Proof. We have a continuous surjective map

(103)

$$P : T(F) \times E^{\times} \setminus \{\beta \in E^{\times} : \epsilon\beta\bar{\beta}\} \times T(F) \to G_{\epsilon}^{reg}(F)$$

$$\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \bar{\alpha}_1 \end{pmatrix}, \beta, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \bar{\alpha}_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} \alpha_1 & 0 \\ 0 & \bar{\alpha}_1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \bar{\alpha}_2 \end{pmatrix}, \tag{104}$$

with fibers isomorphic to E^1. We extend H to a continuous map

$$H : T(F) \times E^{\times} \setminus \{\beta \in E^{\times} : \epsilon\beta\bar{\beta}\} \times T(F) \to \mathbf{C} \tag{105}$$

$$\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \bar{\alpha}_1 \end{pmatrix}, \beta, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \bar{\alpha}_2 \end{pmatrix} \right) \mapsto H(\epsilon\beta\bar{\beta}). \tag{106}$$

We note that H is constant along the fibers of P and we obtain a continuous, compactly supported function f_ϵ on $G_{\epsilon}^{reg}(F)$ by defining

$$f_\epsilon(P(t_1, \beta, t_2)) = \frac{H(\epsilon\beta\bar{\beta})}{\mathrm{vol}(T(F))^2}. \tag{107}$$

Then $H = H_{f_\epsilon}$. □

Thus in order to deal with functions not supported on the regular set we need to examine the behavior of $H_{f_\epsilon}(x)$ for x near 0, 1 and ∞.

Lemma 5.9. *(*[Jac86, Proposition 2.3]*) Let H be a smooth function on $F^{\times}$. There exists $f_\epsilon \in C_c^{\infty}(G_\epsilon(F))$ such that $H(x) = H(x, f_\epsilon)$ if and only if the following conditions are satisfied:*

1. *H is zero on the complement of $\epsilon N_{E/F} E^{\times}$,*
2. *H is zero in a neighborhood of 1,*
3. *there exists a smooth function A on a neighborhood of 0 in F such that for x sufficiently close to 0, one has,*
$$H(x) = A(x)(1 + \eta(\epsilon x))$$
4. *there exists a smooth function B on a neighborhood of 0 in F such that for $|x|$ sufficiently large, one has,*
$$H(x) = B(x^{-1})(1 + \eta(\epsilon x))$$

Furthermore if f_ϵ, A and B satisfy these conditions then,

$$2A(0) = \mathrm{vol}(T(F)) \int_{T(F)} f_\epsilon(t)\ dt,$$

and

$$2B(0) = \mathrm{vol}(T(F)) \int_{T(F)} f_\epsilon \left(\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} t \right)\ dt.$$

Suppose that first that we are given $f_\epsilon \in C_c^{\infty}(G_\epsilon(F))$. We set $H(x) = H(x, f_\epsilon)$. To begin, it is clear from its definition that $H(x) = 0$ if $x \notin \epsilon N_{E/F} E^{\times}$. Suppose that H is not zero is a neighborhood of 1. So we can find a sequence of elements

We now check (2). Since f is compactly supported there exists a compact set $C \subset G_\epsilon(F)$ such that if $I_\delta(f) \neq 0$ then $\delta \in T(F)CT(F)$. We note that $C' = T(F)CT(F)$ is also compact in $G_\epsilon(F)$. We let

$$\tilde{G}_\epsilon(F) = \left\{ \begin{pmatrix} \alpha & \epsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{GL}(2, E) \right\}. \tag{108}$$

Then we can write $C' = Z(F)\tilde{C}'$ for a compact set $\tilde{C}'$ of $\tilde{G}_\epsilon(F)$. Suppose now that $H(x)$ does not vanish in a neighborhood of 1. Then we can find a sequence of elements

$$g_i = \begin{pmatrix} 1 & \epsilon\beta_i \\ \bar{\beta}_i & 1 \end{pmatrix} \in \tilde{C}' \tag{109}$$

such that $\epsilon\beta_i\bar{\beta}_i \to 1$. But we can also write $g_i = z_i c_i$ with $z_i \in Z(F)$ and $c_i \in \tilde{C}'$. Taking determinants we see that

$$1 - \epsilon\beta_i\bar{\beta}_i = z_i^2 \det c_i. \tag{110}$$

Since $1 - \epsilon\beta_i\bar{\beta}_i \to 0$ and c_i is forced to stay in the compact set $\tilde{C}'$ we get that $z_i \to 0$. Hence also $g_i \to 0$. But clearly this is not true.

Next we check (3). We will now make the further assumption that F is non-archimedean. Let $\mathfrak{p}_E$ be the prime ideal in E. Then for n sufficiently large we have,

$$f\left(g\begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix}\right) = f(g) \tag{111}$$

for all $\beta \in \mathfrak{p}_E^n$. Now,

$$H(\epsilon\beta\bar{\beta}) = \int_{T(F)}\int_{T(F)} f\left(\begin{pmatrix} \alpha_1 & \\ & \bar{\alpha}_1 \end{pmatrix}\begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix}\begin{pmatrix} \alpha_2 & \\ & \bar{\alpha}_2 \end{pmatrix}\right) \, d\alpha_1 \, d\alpha_2. \tag{112}$$

Making the change of variables $\alpha_1 \mapsto \alpha_1\alpha_2^{-1}$ gives,

$$H(\epsilon\beta\bar{\beta}) = \int_{T(F)}\int_{T(F)} f\left(\begin{pmatrix} \alpha_1 & \\ & \bar{\alpha}_1 \end{pmatrix}\begin{pmatrix} 1 & \epsilon\beta\alpha_2\bar{\alpha}_2^{-1} \\ \bar{\beta}\bar{\alpha}_2\alpha_2^{-1} & 1 \end{pmatrix}\right) \, d\alpha_1 \, d\alpha_2. \tag{113}$$

Hence if $\epsilon\beta\bar{\beta}$ is sufficiently close to 0 then,

$$H(\epsilon\beta\bar{\beta}) = \mathrm{vol}(T(F)) \int_{T(F)} f\left(\begin{pmatrix} \alpha_1 & \\ & \bar{\alpha}_1 \end{pmatrix}\right) \, d\alpha_1. \tag{114}$$

The proof of condition (4) is similar to (3).

For the converse statement we assume we are given a function $H(x)$ satisfying the assumptions of the Lemma. We define f_ϵ to be the function on $G_\epsilon(F)$ which is invariant on the left and right by $T(F)$ and such that

$$f_\epsilon\begin{pmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{pmatrix} = \frac{H(\epsilon\beta\bar{\beta})}{\mathrm{vol}(T(F))^2}, \tag{115}$$

and

$$f_\epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{2A(0)}{\mathrm{vol}(T(F))^2}, \tag{116}$$

and

$$f_\epsilon\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} = \frac{2B(0)}{\mathrm{vol}(T(F))^2}. \tag{117}$$

By Lemma 5.2 this gives a well defined function on $G_\epsilon(F)$. One can then readily check that, under the assumptions of the Lemma, f_ϵ is a smooth compactly supported function on $G_\epsilon(F)$.

We now need to show that the regular orbital integrals associated to a function $f \in C_c^\infty(\mathrm{PGL}(2,F))$ satisfy the same conditions. Recall that given $f \in C_c^\infty(\mathrm{PGL}(2,F))$ we have defined a function $H(x,f)$ of $x \in F \setminus \{0,1\}$ by

$$H(x,f) = I_\delta(f) \tag{118}$$

where

$$\delta = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}. \tag{119}$$

Lemma 5.10. *(*[Jac86, Propositions 3.3 & 3.4]*) Let $f \in C_c^\infty(\mathrm{PGL}(2,F))$. Let $H(x) = H(x,f)$. Then,*

1. *H is smooth on $F^\times$,*
2. *H is zero in a neighborhood of 1,*
3. *there exists a neighborhood U of 0 and two smooth functions A_i $(i = 1,2)$ such that for $x \in U$, one has,*
$$H(x) = A_1(x) + A_2(x)\eta(x),$$
4. *there exists a neighborhood V of 0 and two smooth functions B_i $(i = 1,2)$ such that for x sufficiently large, one has,*
$$H(x) = B_1(x^{-1}) + B_2(x^{-1})\eta(x).$$

Furthermore if f, A_i and B_i satisfy these conditions then,

$$A_1(0) = I_{n_-}(f)$$
$$A_2(0) = I_{n_+}(f)$$
$$B_1(0) = I_{wn_+}(f)$$
$$B_2(0) = I_{wn_-}(f).$$

We will outline the proof of this result.

To begin we cover $\mathrm{PGL}(2,F)$ by two open sets. We have

$$\mathrm{PGL}(2,F) = A(F)N(F)N'(F) \cup A(F)N(F)wN(F) \tag{120}$$

where N denotes the upper triangular unipotent subgroup of $\mathrm{PGL}(2,F)$ and N' denotes the lower triangular unipotent subgroup. By multiplying out one can readily check

$$A(F)N(F)N'(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,F) : d \neq 0 \right\} \tag{121}$$

and

$$A(F)N(F)wN(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,F) : c \neq 0 \right\} \tag{122}$$

from which the claim readily follows.

Thus we can write $f = f_1 + f_2$ with f_1 (resp. f_2) supported in the first (resp. second) open set. We set

$$\varphi(g) = \int_{A(F)} f(ag)\, da \tag{123}$$

and define φ_1 and φ_2 similarly so that $\varphi = \varphi_1 + \varphi_2$. For $(u,v) \in F \times F$ we define

$$\Phi_1(u,v) = \varphi_1\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\right) \tag{124}$$

and

$$\Phi_2(u,v) = \varphi_2\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} w \begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\right). \tag{125}$$

These functions are compactly supported on $F \times F$. Recall

$$H(x) = \int_{A(F)}\int_{A(F)} f\left(a\begin{pmatrix}1 & x\\ 1 & 1\end{pmatrix} b\right)\eta(b)\ da\ db. \tag{126}$$

One can check that in the notations above

$$H(x) = \int_{F^\times}\Phi_1(a^{-1}(1-x)^{-1}x, a)\eta(a)\ d^\times a + \int_{F^\times}\Phi_2(a(1-x)^{-1}, a^{-1})\eta(a)\ d^\times a. \tag{127}$$

These integrals are compactly supported in $a \in F^\times$ from which we deduce that $H(x)$ is a smooth function of $x \in F \setminus \{0,1\}$.

Next we check that $H(x)$ vanishes in a neighborhood of 1. Pick C sufficiently large so that $\Phi_i(x,y) \neq 0$ for $i \in \{1,2\}$ implies that $|x| < C$ and $|y| < C$. Thus for the second integral we need $|a(1-x)^{-1}| < C$ and $|a^{-1}| < C$. Hence we need $C^{-2} < |1-x|$ for the second integral to be non-zero. For the first integral we need $|(1-x)^{-1}x| < C^2$, which implies that $|1-x|$ must be bounded away from 0. Thus $H(x)$ vanishes in a neighborhood of 1.

Next we study the behavior of $H(x)$ near 0. We begin with the following.

Lemma 5.11. *(*[Jac86, Lemma 3.2]*) Let Φ be a Schwartz-Bruhat function in two variables. Then there exist two Schwartz-Bruhat functions A_1 and A_2 in one variable such that for x different from 0,*

$$\int_{F^\times}\Phi(a^{-1}x, a)\eta(a)\ d^\times a = A_1(x) + A_2(x)\eta(x).$$

Furthermore if F is real and Φ is compactly supported, then A_1 and A_2 can be chosen to have compact support.

Proof. The proof of this Lemma is left to the reader in [Jac86]. We give a proof here when F is non-archimedean. We choose $C > 0$ sufficiently small so that $\Phi(x,y)$ is constant when $|x| < C$ and $|y| < C$. We break up the integral

$$\int_{F^\times}\Phi(a^{-1}x, a)\eta(a)\ d^\times a \tag{128}$$

into the sum of three integrals

$$\int_{|a|>C}\Phi(a^{-1}x,a)\eta(a)\ d^\times a + \int_{C^{-1}|x|<|a|<C}\Phi(a^{-1}x,a)\eta(a)\ d^\times a + \int_{|a|<C^{-1}|x|}\Phi(a^{-1}x,a)\eta(a)\ d^\times a \tag{129}$$

The function

$$x \mapsto \int_{|a|>C}\Phi(a^{-1}x,a)\eta(a)\ d^\times a \tag{130}$$

is clearly a smooth function of $x \in F$. For the third integral we make the substitution $b = a^{-1}x$ to obtain

$$\int_{|a|<C^{-1}|x|}\Phi(a^{-1}x,a)\eta(a)\ d^\times a = \eta(x)\int_{|b|>C}\Phi(b, b^{-1}x)\eta(b)\ d^\times b, \tag{131}$$

which again has the desired form. Finally, from the choice of C we see that,

$$\int_{C^{-1}|x|<|a|<C} \Phi(a^{-1}x, a)\eta(a)\ d^\times a = \Phi(0,0)\int_{C^{-1}|x|<|a|<C} \eta(a)\ d^\times a, \tag{132}$$

and one can readily check that this has the form predicted by the statement of the Lemma. □

We now return to our expression

$$H(x) = \int_{F^\times} \Phi_1(a^{-1}(1-x)^{-1}x, a)\eta(a)\ d^\times a + \int_{F^\times} \Phi_1(a(1-x)^{-1}, a^{-1})\eta(a)\ d^\times a. \tag{133}$$

We wish to prove that there exist smooth functions $A_1(x)$ and $A_2(x)$ so that

$$H(x) = A_1(x) + A_2(x)\eta(x) \tag{134}$$

for all $x \in F \setminus \{0,1\}$. To begin with it's clear that

$$\int_{F^\times} \Phi_1(a(1-x)^{-1}, a^{-1})\eta(a)\ d^\times a \tag{135}$$

is a smooth function of x (recall that the integral vanishes for x near 1). Thus we need only consider the first integral

$$\int_{F^\times} \Phi_1(a^{-1}(1-x)^{-1}x, a)\eta(a)\ d^\times a. \tag{136}$$

We apply the Lemma above which gives smooth compactly supported functions B_1 and B_2 such that

$$\int_{F^\times} \Phi_1(a^{-1}(1-x)^{-1}x, a)\eta(a)\ d^\times a = B_1(x(1-x)^{-1}) + B_2(x(1-x)^{-1})\eta(x(1-x)^{-1}). \tag{137}$$

We need to check the behavior of these functions for x small. To begin it's clear that $A_i(x) = B_i(x(1-x)^{-1})$ is smooth in a neighborhood of 0 and if x sufficiently small then $x-1$ is a norm from E and hence $\eta(x(1-x)^{-1}) = \eta(x)$. This completes the proof of condition (3) of the Lemma.

The proof of condition (4) is similar to (3).

5.3. **The fundamental lemma for the Hecke algebra.** Suppose now that E/F is an unramified quadratic extension of non-archimedean local fields. We set $K = \mathrm{PGL}(2, \mathcal{O}_F)$. For $\epsilon \in N_{E/F}E^\times$ we fix an isomorphism $I_\epsilon : G_\epsilon(F) \xrightarrow{\sim} \mathrm{PGL}(2,F)$ such that $I(T(F)) \subset K$. Let $\mathcal{H}(\mathrm{PGL}(2,F), K)$ denote the Hecke algebra of smooth, compactly supported, bi-K-invariant functions on $\mathrm{PGL}(2,F)$.

Proposition 5.12. *(The fundamental lemma* [Jac86, Proposition 5.1]*) Let $f \in \mathcal{H}(\mathrm{PGL}(2,F), K)$. Then f matches with $(f_\epsilon : \epsilon \in F^\times/NE^\times)$ where $f_\epsilon = f \circ I_\epsilon$ if $\epsilon \in NE^\times$ and $f_\epsilon = 0$ otherwise. Furthermore,*

$$I_{n_+}(f) = I_{n_-}(f) = \frac{1}{2}\,\mathrm{vol}(T(F)) \int_{T(F)} f(t)\ dt. \tag{138}$$

We will not repeat the proof here, but simply refer to [Jac86, Section 5]. We recall that by the Cartan decomposition a basis for $\mathcal{H}(\mathrm{PGL}(2,F), K)$ is given by the characteristic functions of

$$K \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} K, \tag{139}$$

where ϖ is a uniformizer of F and $m > 0$. Using the fact that for $m > 0$ a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}_F) \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}(2, \mathcal{O}_F) \tag{140}$$

if and only if

(1) $v_F(\det g) = m$,
(2) $a, b, c, d \in \mathcal{O}_F$, and
(3) at least one of a, b, c or d is a unit in $\mathcal{O}_F$,

one can calculate explicitly the regular orbital integrals for $f \in \mathcal{H}(\mathrm{PGL}(2, F), K)$ and verify the proposition.

5.4. **Global comparison.** We now return to the global setting. We again take E/F to be a quadratic extension of number fields.

We now write the geometric expansion for the relative trace formula as

$$I(f) = I_{reg}(f) + I_{irreg}(f), \tag{141}$$

where

$$I_{reg}(f) = \sum_{\delta \in A(F)\backslash \mathrm{PGL}(2,F)^{reg}/A(F)} I_\delta(f) \tag{142}$$

and

$$I_{irreg}(f) = I_{n_+}(f) + I_{n_-}(f) + I_{wn_+}(f) + I_{wn_-}(f). \tag{143}$$

Similarly for $\epsilon \in F^\times/NE^\times$,

$$I^\epsilon(f) = I^\epsilon_{reg}(f) + I^\epsilon_{irreg}(f), \tag{144}$$

where

$$I^\epsilon_{reg}(f) = \sum_{\delta \in T(F)\backslash G^\epsilon(F)^{reg}/T(F)} I_\delta(f) \tag{145}$$

and

$$I^\epsilon_{irreg}(f) = \mathrm{vol}(T(F)\backslash T(\mathbf{A}_F)) \int_{T(\mathbf{A}_F)} f(t)\, dt + \mathrm{vol}(T(F)\backslash T(\mathbf{A}_F)) \int_{T(\mathbf{A}_F)} f\left(t \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\right) dt. \tag{146}$$

From the local propositions above we can now define a correspondence mapping $f \in C_c^\infty(\mathrm{PGL}(2, \mathbf{A}_F))$ to $(f^\epsilon) \in \oplus_{\epsilon \in F^\times/NE^\times} G^\epsilon(\mathbf{A}_F)$. It is perhaps worth remarking that, thanks to the fundamental lemma, f^ϵ is zero for all but finitely many ϵ. For functions related by this correspondence we have,

$$I_{reg}(f) = \sum_{\epsilon \in F^\times/NE^\times} I^\epsilon_{reg}(f^\epsilon). \tag{147}$$

In order to prove that

$$I(f) = \sum_{\epsilon \in F^\times/NE^\times} I^\epsilon(f^\epsilon), \tag{148}$$

it remains to prove the following Lemma.

Lemma 5.13. *We have,*

$$I_{n_+}(f) + I_{n_-}(f) = \sum_{\epsilon \in F^\times / NE^\times} \mathrm{vol}(T(F)\backslash T(\mathbf{A}_F)) \int_{T(\mathbf{A}_F)} f(t)\ dt$$

and

$$I_{wn_+}(f) + I_{wn_-}(f) = \sum_{\epsilon \in F^\times / NE^\times} \mathrm{vol}(T(F)\backslash T(\mathbf{A}_F)) \int_{T(\mathbf{A}_F)} f\left(t \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\right)\ dt.$$

This is proven in [Jac86, (10.4)]. We outline the proof here. Recall that

$$I_{n_+}(f) = \int_{\mathbf{A}_F^\times} \int_{\mathbf{A}_F^\times} f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) |b|^s \eta(b)\ d^\times a\ d^\times b \Big|_{s=0} \tag{149}$$

and

$$I_{n_-}(f) = \int_{\mathbf{A}_F^\times} \int_{\mathbf{A}_F^\times} f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right) |b|^s \eta(b)\ d^\times a\ d^\times b \Big|_{s=0} \tag{150}$$

Suppose that $\varphi = \prod_v \varphi_v \in C_c^\infty(\mathbf{A}_F)$. Then from Tate's thesis

$$\int_{\mathbf{A}_F^\times} \varphi(t)|t|^s \eta(t)\ d^\times t \Big|_{s=0} = L(0,\eta) \prod_{v \in S_1} \frac{\int_{F_v^\times} \varphi_v(t_v)\eta_v(t_v)\ dt_v}{L(0,\eta_v)} \prod_{v \in S_2} \varphi_v(0)\epsilon(1, \mathbf{1}_v, \psi_v), \tag{151}$$

where S_1 denotes the set of places of F which are inert in E and S_2 the set of places in F which split in E. We recall that from Tate's thesis that for $s > 1$,

$$\int_{\mathbf{A}_F^\times} \varphi(t)|t|^s \eta(t)\ dt = L(s,\eta) \prod_v \frac{\int_{F_v^\times} \varphi_v(t_v)|t_v|_v^s \eta_v(t)\ d^\times t_v}{L(s,\eta_v)}, \tag{152}$$

with almost all factors in the product equal to 1. If $v \in S_1$ so that the character η_v is non-trivial then

$$\frac{\int_{F_v^\times} \varphi_v(t_v)|t_v|_v^s \eta_v(t)\ d^\times t_v}{L(s,\eta_v)} \Big|_{s=0} = \frac{\int_{F_v^\times} \varphi_v(t_v)\eta_v(t)\ d^\times t_v}{L(0,\eta_v)}. \tag{153}$$

On the other hand if $v \in S_2$, so that η_v is trivial, then by the local functional equation

$$\frac{\int_{F_v^\times} \varphi_v(t_v)|t_v|_v^s\ d^\times t_v}{L(s,\eta_v)} = \epsilon(1-s, \mathbf{1}_v, \psi_v) \frac{\int_{F_v^\times} \hat{\varphi}_v(t_v)|t_v|_v^{1-s}\ d^\times t_v}{L(1-s,\eta_v)} \tag{154}$$

where $\hat{\varphi}_v$ denotes the Fourier transform of φ_v with respect to the additive character ψ_v. We can now evaluate the right hand side at $s = 0$ to yield,

$$\epsilon(1, \mathbf{1}_v, \psi_v) \frac{\int_{F_v^\times} \hat{\varphi}_v(t_v)|t_v|_v\ d^\times t_v}{L(1,\eta_v)} = \epsilon(1, \mathbf{1}_v, \psi_v) \int_{F_v^\times} \hat{\varphi}_v(t_v)\ dt_v \tag{155}$$

$$= \epsilon(1, \mathbf{1}_v, \psi_v)\varphi_v(0). \tag{156}$$

We return to our orbital integral

$$I_{n_+}(f) = \int_{\mathbf{A}_F^\times} \int_{\mathbf{A}_F^\times} f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) |b|^s \eta(b)\ d^\times a\ d^\times b \Big|_{s=0}. \tag{157}$$

We apply the above to the function

$$\varphi(b) = \int_{\mathbf{A}_F^\times} f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \, d^\times a \in C_c^\infty(\mathbf{A}_F). \tag{158}$$

To yield

$$I_{n_+}(f) = L(0,\eta) \prod_{v\in S_1} \frac{I_{n_+}(f_v)}{L(0,\eta_v)} \prod_{v \in S_2} \epsilon(1, \mathbf{1}_v, \psi_v) \int_{F_v^\times} f_v \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} \, d^\times a_v, \tag{159}$$

and

$$I_{n_-}(f) = L(0,\eta) \prod_{v\in S_1} \frac{I_{n_-}(f_v)}{L(0,\eta_v)} \prod_{v \in S_2} \epsilon(1, \mathbf{1}_v, \psi_v) \int_{F_v^\times} f_v \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} \, d^\times a_v. \tag{160}$$

On the other hand we have

$$\mathrm{vol}(T(F)\backslash T(\mathbf{A}_F)) \int_{T(\mathbf{A}_F)} f(t) \, dt = 2L(1,\eta) \prod_v \int_{T(F_v)} f_v(t_v) \, dt_v. \tag{161}$$

It now remains to use the local identities between the irregular orbital integrals established above and the functional equation relating $L(1,\eta)$ to $L(0,\eta)$.

6. Aside on unramified representations

Before continuing onto the spectral identity obtained via the relative trace formula we recall some facts about unramified representations.

Let v be a finite place of F and let π_v be an irreducible smooth representation of $G(F_v)$. Recall that π_v is unramified if $\pi_v^{K_v} \neq \{0\}$ where $K_v = G(\mathcal{O}_{F_v})$. In this case $\dim_{\mathbf{C}} \pi_v^{K_v} = 1$. Let $f_v \in \mathcal{H}(G(F_v), K_v)$. Then $\pi_v(f_v)$ preserves the space $\pi_v^{K_v}$ and hence acts on it by a scalar $\lambda_{\pi_v}(f_v) \in \mathbf{C}$. In this way we get a map

(162)

$$\{\text{irreducible representations of } G(F_v)\} \to \{\text{characters of } \mathcal{H}(G(F_v), K_v)\} : \pi_v \mapsto \lambda_{\pi_v},$$

which is in fact a bijection. We recall that $\mathcal{H}(G(F_v), K_v)$ is commutative, explicitly we have the Satake isomorphism,

$$\mathcal{H}(G(F_v), K_v) \xrightarrow{\sim} \mathbf{C}[T, T^{-1}]^{\mathbf{Z}/2\mathbf{Z}} : f \mapsto \hat{f}, \tag{163}$$

where the non-trivial element of $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{C}[T, T^{-1}]$ via $T \mapsto T^{-1}$. Hence, via the Satake isomorphism, any character of $\mathcal{H}(G(F_v), K_v)$ can be seen to be of the form

$$\lambda_z : \mathcal{H}(G(F_v), K_v) \to \mathbf{C} : f \mapsto \hat{f}(z), \tag{164}$$

for some $z \in \mathbf{C}^\times$.

7. Spectral Identity

Let $f \in C_c^\infty(\mathrm{PGL}(2, \mathbf{A}_F))$ and $\{f^D \in C_c^\infty(G_D(\mathbf{A}_F)) : D \in X(E)\}$ match in the sense above. Then we have,

$$I(f) = \sum_{D \in X(E)} I^D(f^D). \tag{165}$$

For ease of notation we set $G = \mathrm{PGL}(2)$. Also assume f is K_v-finite at the archimedean places.

We now wish to apply this identity to the spectral expansions for $I(f)$ and $I^D(f^D)$. We recall that $K_f(x,y)$ is the kernel for the map

$$R(f): L^2(G(F)\backslash G(\mathbf{A}_F)) \to L^2(G(F)\backslash G(\mathbf{A}_F)) \tag{166}$$

defined by

$$(R(f)\varphi)(x) = \int_{G(\mathbf{A}_F)} f(y)\varphi(xy)\, dy. \tag{167}$$

The space $L^2(G) = L^2(G(F)\backslash G(\mathbf{A}_F))$ decomposes as

$$L^2(G) = L^2_{cusp}(G) \oplus L^2_{Eis}(G). \tag{168}$$

Here $L^2_{Eis}(G)$ denotes the orthogonal complement of $L^2_{cusp}(G)$ in $L^2(G)$. Langlands' theory of Eisenstein series gives an explicit decomposition of $L^2_{Eis}(G)$. We recall that

$$L^2_{cusp}(G) = \bigoplus_{\pi\in\mathcal{A}_{cusp}(G)} V_\pi \tag{169}$$

where V_π denotes the π-isotypic subspace of $L^2_{cusp}(G)$. The space $L^2_{Eis}(G)$ may be further decomposed as

$$L^2_{Eis}(G) = L^2_{res}(G) \oplus L^2_{cont}(G). \tag{170}$$

Explicitly we have,

$$L^2_{cont}(G) = \bigoplus_{\delta: F^\times\backslash\mathbf{A}_F^\times \to \{\pm 1\}} V_\delta \tag{171}$$

where

$$V_\delta = \mathbf{C}.\left\{\delta\circ\det : G(F)\backslash G(\mathbf{A}_F) \to \mathbf{C}\right\}. \tag{172}$$

The space $L^2_{cont}(G)$ is purely continuous and is furnished by integrals of Eisenstein series.

For each $D \in X(E)$ a similar decomposition holds for $L^2(G_D(F)\backslash G_D(\mathbf{A}_F))$, although we note that unless $D \cong M(2,F)$ we have $L^2_{cont}(G_D) = \{0\}$.

Since $R(f)$ preserves each space the kernel $K_f(x,y)$ has a decomposition

$$K_f(x,y) = K_{f,cusp}(x,y) + K_{f,res}(x,y) + K_{f,cont}(x,y). \tag{173}$$

We recall that

$$K_{f,cusp}(x,y) = \sum_{\pi\in\mathcal{A}_{cusp}(G)}\sum_{\varphi\in\mathcal{B}(\pi)} (R(f)\varphi)(x)\overline{\varphi(y)}, \tag{174}$$

where $\mathcal{B}(\pi)$ denotes an orthonormal basis of V_π. The kernel $K_{f,res}(x,y)$ has a similar expression.

Using the decomposition of the kernel we can write

$$I(f) = I_{cusp}(f) + I_{res}(f) + I_{cont}(f). \tag{175}$$

Care needs to be taken with the convergence of $I_{cont}(f)$, we refer to [Jac86, Section 8] for details. We can write

(176)

$$I_{cusp}(f) = \sum_{\pi\in\mathcal{A}_{cusp}(G)}\sum_{\varphi\in\mathcal{B}(\pi)} \int_{A(F)\backslash A(\mathbf{A}_F)} (R(f)\varphi)(a)\, da \overline{\int_{A(F)\backslash A(\mathbf{A}_F)} \varphi(b)\eta(\det b)\, db}.$$

One can also write down a similar expression for $I_{res}(f)$.

Similarly for $D \in X(E)$ and $f \in C_c^\infty(G_D(\mathbf{A}_F))$ we have,

$$I^D(f) = I^D_{cusp}(f) + I^D_{res}(f) + I^D_{cont}(f), \tag{177}$$

with

$$I_{cusp}(f) = \sum_{\pi \in \mathcal{A}_{cusp}(G_D)} \sum_{\varphi \in \mathcal{B}(\pi)} \int_{T(F)\backslash T(\mathbf{A}_F)} (R(f)\varphi)(t_1)\ dt_1 \overline{\int_{T(F)\backslash T(\mathbf{A}_F)} \varphi(t_2)\ dt_2}, \tag{178}$$

where we recall that T is the torus in G_D equal to the image of $E^\times$ via the inclusion $E^\times \hookrightarrow D^\times$.

Proposition 7.1. *Assume that $f \in C_c^\infty(G(\mathbf{A}_F))$ and $(f^D) \in \oplus_{D \in X(E)} C_c^\infty(G_D(\mathbf{A}_F))$ match. Then,*

$$I_{cusp}(f) = \sum_{D \in X(E)} I^D_{cusp}(f^D). \tag{179}$$

Before saying some words about the proof we remark that if we restrict our test functions then we already have this identity. For example if we take $f = \prod_v f_v$ with f_{v_0} the matrix coefficient of a supercuspidal representation at some place v_0 of F which is split in E then,

$$I(f) = I_{cusp}(f). \tag{180}$$

Furthermore under this restriction we see that $f^D = \prod_v f^D_v$ with $f^D_{v_0}$ also the matrix coefficient of a supercuspidal representation and hence

$$\sum_{D \in X(E)} I^D(f^D) = \sum_{D \in X(E)} I^D_{cusp}(f^D). \tag{181}$$

Proof. Take $f = \prod_v f_v$. Then we have

$$I_{cusp}(f) + I_{res}(f) + I_{cont}(f) = \sum_{D \in X(E)} \left(I^D_{cusp}(f^D) + I^D_{res}(f^D) + I^D_{cont}(f^D) \right). \tag{182}$$

We rewrite this as

$$\left(I_{cusp}(f) - \sum_{D \in X(E)} I^D_{cusp}(f^D) \right) = - \left(I_{res}(f) - \sum_{D \in X(E)} I^D_{res}(f^D) \right) - \left(I_{cont}(f) - \sum_{D \in X(E)} I^D_{cont}(f^D) \right). \tag{183}$$

We fix a finite place v_0 of F which is split in E. We fix $f_v \in C_c^\infty(G(F_v))$ for $v \neq v_0$ and view this expression as an identity of distributions on $f_{v_0} \in \mathcal{H}(G(F_{v_0}, K_{v_0})$. Let $\hat{f}_{v_0}$ denote the Satake transform of f_{v_0}. Then the left hand side of this expression takes the form

$$\sum_t a_t \hat{f}_{v_0}(t) \tag{184}$$

where $t \in \mathbf{C}$ satisfies either $|t| = 1$ or $t \in (q_{v_0}^{-1}, q_{v_0})$. This is because only $\pi = \otimes \pi_v \in \mathcal{A}_{cusp}(G)$ with π_{v_0} unramified contribute to the sum and furthermore π_{v_0} must be generic and unitary. On the other hand the right hand side takes the form

$$\int_{-\infty}^{\infty} \varphi(t) \hat{f}_{v_0}(q_{v_0}^{-2it})\ dt + c\hat{f}_{v_0}(q_{v_0}^{-1}), \tag{185}$$

for some integrable function $\varphi(t)$ and some constant c. Now one uses the uniqueness of the decomposition of a measure in to a discrete sum of delta distributions and a continuous measure. $\square$

Recall that we have

$$I(f) = \sum_{\pi \in \mathcal{A}_{cusp}(G)} I_\pi(f). \tag{186}$$

The Jacquet-Langlands correspondence gives an injection

$$\mathrm{JL} : \mathcal{A}_{cusp}(G_D) \hookrightarrow \mathcal{A}_{cusp}(G) : \pi = \otimes_v \pi_v \mapsto \mathrm{JL}(\pi) = \otimes_v \mathrm{JL}(\pi_v), \tag{187}$$

defined in terms of maps on local representations. At the places where D splits, i.e. where $D_v \cong M(2, F_v)$, JL is the identity. Recall that for $\pi \in \mathcal{A}_{cusp}(G)$ we have defined

$$X(E, \pi) = \left\{ D \in X(E) : \pi = \mathrm{JL}(\pi^D) \text{ for some } \pi^D \in \mathcal{A}_{cusp}(G_D) \right\}. \tag{188}$$

Thus we have,

$$\sum_{D \in X(E)} I^D_{cusp}(f) = \sum_{D \in X(E)} \sum_{\pi^D \in \mathcal{A}_{cusp}(G_D)} I^D_{\pi^D}(f^D) \tag{189}$$

$$= \sum_{\pi \in \mathcal{A}_{cusp}(G)} \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D). \tag{190}$$

Proposition 7.2. *Assume that $f \in C_c^\infty(G(\mathbf{A}_F))$ and $(f^D) \in \bigoplus_{D \in X(E)} C_c^\infty(G_D(\mathbf{A}_F))$ match. Then for $\pi \in \mathcal{A}_{cusp}(G)$,*

$$I_\pi(f) = \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D). \tag{191}$$

Proof. It suffices to prove the Proposition for $f = \prod_v f_v$. We take such a function f and fix a finite set of places S containing the archimedean places of F and those of F which ramify in E. We assume that $f_v \in \mathcal{H}(G(F_v), K_v)$ for all $v \notin S$. Then we claim that,

(1) $I_\pi(f) = 0$ unless π is unramified outside S, and
(2) if π^S is unramified then $I_\pi(f) = I_\pi(f_S \mathbf{1}_{K^S}) \lambda_{\pi^S}(f^S)$.

The first part is clear since $R(f)$ maps π to π^{K^S}. For the second part we write $\pi = \otimes'_v \pi_v$. We fix inner products on the local representations π_v so that they are compatible with the global inner product on π. We can take an orthonormal basis $\mathcal{B}(\pi)$ for the space of π to be of the form $\mathcal{B}(\pi) = \otimes'_v \mathcal{B}(\pi_v)$, where $\mathcal{B}(\pi_v)$ denotes an orthonormal basis of π_v. For $v \notin S$ we take

$$\mathcal{B}(\pi_v) = \left\{ \varphi_{v,0} \in \pi_v^{K_v}, \varphi_{v,1}, \ldots \right\}. \tag{192}$$

For $v \in S$ we choose $\mathcal{B}(\pi_v) = \{\varphi_{v,0}, \varphi_{v,1}, \ldots\}$ arbitrarily. So

$$\mathcal{B}(\pi) = \{\otimes_v \varphi_{v,i_v} : i_v = 0 \text{ for almost all } v\}. \tag{193}$$

Since $f = \prod_v f_v$ we see that

$$R(f) \otimes_v \varphi_{v,i_v} = \otimes_v R(f_v) \varphi_{v,i_v}. \tag{194}$$

Clearly for $v \notin S$,

$$R(f_v)\varphi_{v,i_v} \begin{cases} 0, & \text{if } i_v \neq 0; \\ \lambda_{\pi_v}(f_v)\varphi_{v,0}, & \text{if } i_v = 0. \end{cases} \tag{195}$$

Hence we see that

$$I(f) = \sum_{\pi \in \mathcal{A}_{cusp}(G)} I_\pi(f_S \mathbf{1}_{K^S}) \lambda_{\pi^S}(f^S). \tag{196}$$

Similarly we have, thanks to the fundamental lemma,

$$\sum_{D \in X(E)} I^D(f^D) = \sum_{\pi \in \mathcal{A}_{cusp}(\pi)} \left(\sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D_S \mathbf{1}_{K^S}) \right) \lambda_{\pi^S}(f^S). \tag{197}$$

Thus we have
(198)

$$\sum_{\pi \in \mathcal{A}_{cusp}(G)} I_\pi(f_S \mathbf{1}_{K^S}) \lambda_{\pi^S}(f^S) = \sum_{\pi \in \mathcal{A}_{cusp}(\pi)} \left(\sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D_S \mathbf{1}_{K^S}) \right) \lambda_{\pi^S}(f^S).$$

We now consider f_S to be fixed and view this as an identity on the Hecke algebra $\mathcal{H}(G^S, K^S)$. Using independence of characters we see that for any unramified representation σ of G^S,

$$\sum_{\pi \in \mathcal{A}_{cusp}(G) : \pi^S = \sigma} I_\pi(f_S \mathbf{1}_{K^S}) = \sum_{\pi \in \mathcal{A}_{cusp}(\pi) : \pi^S = \sigma} \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D_S \mathbf{1}_{K^S}). \tag{199}$$

Finally we use strong multiplicity one to deduce that there is only at most one $\pi \in \mathcal{A}_{cusp}(G)$ such that $\pi^S = \sigma$. Hence we deduce that for all $\pi \in \mathcal{A}_{cusp}(G)$ we have,

$$I_\pi(f_S \mathbf{1}_{K^S}) = \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D_S \mathbf{1}_{K^S}). \tag{200}$$

Since $C_c^\infty(G(\mathbf{A}_F))$ is spanned by functions of the form $f_S \mathbf{1}_{K^S}$ the proof of the Proposition is complete. $\square$

8. Proof of Waldspurger's theorem

Finally in this section we are able to prove Waldspurger's theorem.

Theorem 8.1. *(Waldspurger* [Wal85]*) $L(1/2, \pi_E) = 0$ if and only if $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$ and $D \in X(E, \pi)$.*

From the previous section we have obtained an identity

$$I_\pi(f) = \sum_{D \in X(E,\pi)} I^D_{\pi^D}(f^D) \tag{201}$$

provided f and f^D match. The following Lemma relates the vanishing of the terms in the Theorem to these distributions.

Lemma 8.2. *Let $\pi \in \mathcal{A}_{cusp}(G)$. Then,*

1. *$L(1/2, \pi_E) = 0$ if and only if $I_\pi(f) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$, and*
2. *for $D \in X(E, \pi)$, $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$ if and only if $I^D_{\pi^D}(f^D) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$.*

Proof. We note that the only if direction is clear in both (1) and (2) since $I_\pi(f)$ and $I^D_{\pi^D}(f^D)$ are defined in terms of periods related to the objects on the left hand sides of the statements.

We begin by proving the if direction of (1). Assume that $I_\pi(f) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$. As is well known that there exists $\varphi_1 \in \pi$ such that

$$\int_{F^\times \backslash \mathbf{A}_F^\times} \varphi_1 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \, d^\times a = L(s,\pi) \tag{202}$$

and $\varphi_2 \in \pi$ such that

$$\int_{F^\times \backslash \mathbf{A}_F^\times} \varphi_2 \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \eta(b)|b|^{s-\frac{1}{2}} \, d^\times b = L(s,\pi \otimes \eta). \tag{203}$$

We can pick $f \in C_c^\infty(G(\mathbf{A}_F))$ such that $R(f)\varphi_2 = \varphi_1$ and $R(f)\varphi = 0$ for all $\varphi \in \pi$ which are orthogonal to φ_2. Hence for this choice of f we have,

$$I_\pi(f) = \frac{\int_{A(F)\backslash A(\mathbf{A}_F)} \varphi_1(a) \, da \overline{\int_{A(F)\backslash A(\mathbf{A}_F)} \varphi_2(b)\eta(\det b) \, db}}{(\varphi_2,\varphi_2)} \tag{204}$$

$$= \frac{L(1/2,\pi_E)}{(\varphi_2,\varphi_2)}. \tag{205}$$

From which we deduce that $L(1/2,\pi_E) = 0$.

For the if direction of (2), the same argument as above would show that if $I_{\pi^D}^D(f') = 0$ for all $f' \in C_c^\infty(G_D(\mathbf{A}_F))$ then $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$. The problem is that we are only considering functions $f^D \in C_c^\infty(G_D(\mathbf{A}_F))$ which match with $f \in C_c^\infty(G(\mathbf{A}_F))$. We note that there is no restriction at the split places over our choice of function, at the inert places we need to use a density argument of Jacquet [Jac86, (4.2)] to obtain that $I_{\pi^D}^D(f') = 0$ for all $f' \in C_c^\infty(G_D(\mathbf{A}_F))$ if and only if $I_{\pi^D}^D(f^D) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$. This completes the proof of the lemma. □

Thus from the lemma we see that if $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$ and $D \in X(E,\pi)$ then $I_\pi(f) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$ and hence $L(1/2,\pi_E) = 0$. Conversely if $L(1/2,\pi_E) = 0$ then,

$$\sum_{D \in X(E,\pi)} I_{\pi^D}^D(f^D) = 0 \tag{206}$$

for all $f \in C_c^\infty(G(\mathbf{A}_F))$. But it is not yet clear that this need imply that $I_{\pi^D}^D(f^D) = 0$ for all $f \in C_c^\infty(G(\mathbf{A}_F))$. If we knew, for some reason, that $I_{\pi^D}^D \equiv 0$ for all but at most one $D \in X(E,\pi)$ then we would be done. In order to obtain this we need some local inputs.

Let $D \in X(E,\pi)$. Then by a change of variables it is easy to see that

$$\varphi \mapsto \int_{T(F)\backslash T(\mathbf{A}_F)} \varphi(t) \, dt \in \mathrm{Hom}_{T(\mathbf{A}_F)}(\pi^D,\mathbf{C}), \tag{207}$$

where $\mathbf{C}$ denotes the trivial representation of $T(\mathbf{A}_F)$. Since $\pi^D = \otimes \pi_v^D$ so

$$\mathrm{Hom}_{T(\mathbf{A}_F)}(\pi^D,\mathbf{C}) = \bigotimes \mathrm{Hom}_{T(F_v)}(\pi_v^D,\mathbf{C}). \tag{208}$$

Thus if $\mathrm{Hom}_{T(F_v)}(\pi_v^D,\mathbf{C}) = 0$ for some place v of F then $\mathrm{Hom}_{T(\mathbf{A}_F)}(\pi^D,\mathbf{C}) = 0$ and hence $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$. We now see that this is often the case.

Theorem 8.3. *(Waldspurger, Tunnell) Let v be a place of F and take π_v to be an irreducible, unitary, generic representation of $G(\mathbf{A}_F)$.*

(1) *For any v we have $\dim_{\mathbf{C}} \mathrm{Hom}_{A(F_v)}(\pi_v,\mathbf{C}) = 1$.*

(2) *Suppose v is inert in E. Then we write $X(E_v) = \{D_1, D_2\}$ with $D_1 \cong M(2, F_v)$ and D_2 the unique quaternion division algebra over F_v.*
   (a) *If $\epsilon(1/2, \pi_{v,E_v}) = \eta_v(-1)$ then $\dim_{\mathbf{C}} \operatorname{Hom}_{T(F_v)}(\pi_v^{D_1}, \mathbf{C}) = 1$ and $\dim_{\mathbf{C}} \operatorname{Hom}_{T(F_v)}(\pi_v^{D_2}, \mathbf{C}) = 0$.*
   (b) *If $\epsilon(1/2, \pi_{v,E_v}) = -\eta_v(-1)$ then $\dim_{\mathbf{C}} \operatorname{Hom}_{T(F_v)}(\pi_v^{D_1}, \mathbf{C}) = 0$ and $\dim_{\mathbf{C}} \operatorname{Hom}_{T(F_v)}(\pi_v^{D_2}, \mathbf{C}) = 1$.*

Proof. We make a couple of remarks, but leave the reader to see the references for the proof.

For the first point we note that we can construct a non-zero element in $\operatorname{Hom}_{A(F_v)}(\pi_v, \mathbf{C}) = 1$ using the Whittaker model for π_v. Let $\psi : F_v \to \mathbf{C}^\times$ be a non-trivial character and let $\mathcal{W}(\pi_v, \psi)$ denote the Whittaker model for π_v. We obtain an element $\ell \in \operatorname{Hom}_{A(F_v)}(\mathcal{W}(\pi_v, \psi), \mathbf{C})$ via

$$\ell(W) = \int_{F_v^\times} W \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \, d^\times a \Big|_{s=\frac{1}{2}}. \tag{209}$$

It is known that for some $W \in \mathcal{W}(\pi_v, \psi)$, $\ell(W) = L(1/2, \pi_v)$, which is non-zero for π_v unitary and generic.

For the second point we illustrate the dichotomy in the archimedean case. So we have $F_v = \mathbf{R}$ and $E_v = \mathbf{C}$. In this case $T(F_v) = \mathbf{R}^\times \backslash \mathbf{C}^\times = \{\pm 1\} \backslash S^1$ is a maximal compact subgroup of $\mathrm{PGL}(2, \mathbf{R})$. The representations of $\{\pm 1\} \backslash S^1$ are well understood, they are of the form $\chi_n : z \mapsto z^n$ with $n \in 2\mathbf{Z}$. Suppose first that π is a principal series representation of $\mathrm{PGL}(2, \mathbf{R})$ then we have

$$\pi|_{T(\mathbf{R})} = \bigoplus_{n \in 2\mathbf{Z}} \chi_n. \tag{210}$$

From this it is not hard to see that $\dim_{\mathbf{C}} \operatorname{Hom}_{T(\mathbf{R})}(\pi, \mathbf{C}) = 1$ and a non-zero element is given by taking the inner product of an element of π with a non-zero element in the one-dimensional space $\pi^{T(\mathbf{R})}$. On the other hand if π is the discrete series representation of weight $2k$, then

$$\pi|_{T(\mathbf{R})} = \bigoplus_{n \in 2\mathbf{Z}, |n| \geq 2k} \chi_n. \tag{211}$$

Hence we see $\operatorname{Hom}_{T(\mathbf{R})}(\pi, \mathbf{C}) = \{0\}$ in this case. On the other hand if we write

$$D = \left\{ \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C} \right\}, \tag{212}$$

for the quaternions. Then we see that $D^\times$ has a natural 2-dimensional representation V given by the inclusion $D^\times \hookrightarrow \mathrm{GL}(2, \mathbf{C})$. Then $\pi^D = \operatorname{Sym}^{2k-2} V \otimes det^{-(k-1)}$. In this case it is easy to see that

$$\pi^D\big|_{T(\mathbf{R})} = \bigoplus_{n \in 2\mathbf{Z}, |n| < 2k} \chi_n, \tag{213}$$

and we see that $\dim_{\mathbf{C}} \operatorname{Hom}_{T(\mathbf{R})}(\pi^D, \mathbf{C}) = 1$. □

We note that if $\pi = \otimes \pi_v \in \mathcal{A}_{cusp}(G)$ then the conditions on π_v in the statement of the theorem are necessarily satisfied. We note also that if v splits in E then $\epsilon(1/2, \pi_{v,E_v}) = 1$. Hence given $\pi \in \mathcal{A}_{cusp}(G)$ there is at most one $D \in X(E, \pi)$ such that $\operatorname{Hom}_{T(\mathbf{A}_F)}(\pi^D, \mathbf{C}) \neq 0$, namely D should be chosen to ramify at the places of F where $\epsilon(1/2, \pi_{E_v}) = -\eta_v(-1)$.

We note that if $\epsilon(1/2,\pi_E) = \prod_v \epsilon(1/2,\pi_{v,E_v}) = -1 = -\eta(-1)$ then the D we want should be ramified at an odd number of places! Hence, $\mathrm{Hom}_{T(\mathbf{A}_F)}(\pi^D,\mathbf{C}) = \{0\}$ for all $D \in X(E,\pi)$ and so $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$ and $D \in X(E,\pi)$. Of course on the other hand $L(1/2,\pi_E) = 0$ in this case and Waldspurger's theorem follows in this case.

On the other hand if $\epsilon(1/2,\pi_E) = +1$ then we take $D^* = D^*(E,\pi)$ to be the quaternion algebra over F which is ramified at the (even number of) places v such that $\epsilon(1/2,\pi_{v,E_v}) = -\eta_v(-1)$. Then the identity obtained from the trace formula reduces to

$$I_\pi(f) = I^{D^*}_{\pi^{D^*}}(f^{D^*}). \tag{214}$$

Waldspurger's theorem now follows in this case as well and we are done.

9. Refinements

Having established the non-vanishing result it is natural to ask if we can find a direct connection between $L(1/2,\pi_E)$ and the period integrals $P_D(\varphi)$. From now on we assume that $\epsilon(1/2,\pi_E) = 1$ and that D is the quaternion algebra over F as determined by the local theorem above, so D is ramified precisely at the places v of F for which $\epsilon(1/2,\pi_{v,E_v}) = -\eta_v(-1)$. We have,

$$P_D \in \mathrm{Hom}_{T(\mathbf{A}_F)}(\pi^D,\mathbf{C}) = \otimes_v \mathrm{Hom}_{T(F_v)}(\pi^D_v,\mathbf{C}). \tag{215}$$

We fix $0 \neq \ell_v \in \mathrm{Hom}_{T(F_v)}(\pi^D_v,\mathbf{C})$, then we have

$$P_D = C \otimes_v \ell_v \tag{216}$$

for some constant $C \in \mathbf{C}$. Clearly $P_D(\varphi) = 0$ for all $\varphi \in \pi^D$ if and only if $C = 0$.

Suppose we can find nice vectors $\varphi_v \in \pi^D_v$ such that $\ell_v(\varphi_v) \neq 0$. Then we see that if we set $\varphi = \otimes_v \varphi_v$ then $P_D(\varphi) = 0$ if and only if $C = 0$. Hence to check if $L(1/2,\pi_E) = 0$ it suffices to compute $P_D(\varphi)$ for this particular choice of φ. Furthermore one could then ask for a formula which relates $L(1/2,\pi_E)$ to $P_D(\varphi)$ up to some explicitly determined constants. In fact, as we shall see, the formula we seek will take the form

$$L(1/2,\pi_E) = C(E,\pi)|P_D(\varphi)|^2. \tag{217}$$

9.1. **Test vectors.** Suppose first that v is non-archimedean and split in E. In this case $G_D(F_v) \xrightarrow{\sim} \mathrm{PGL}(2,F_v)$ and the isomorphism can be chosen so that $T(F_v)$ maps to the diagonal torus $A(F_v)$. Let $\psi : F \to \mathbf{C}^\times$ be a non-trivial additive character. We consider the Whittaker model $\mathcal{W}(\pi_v,\psi)$. Then as remarked above we can take $\ell_v : \mathcal{W}(\pi,\psi) \to \mathbf{C}$ to be defined by

$$\ell_v(W) = \int_{F_v^\times} W\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} |a|^{s-\frac{1}{2}}\, d^\times a\bigg|_{s=\frac{1}{2}}. \tag{218}$$

In this case if we take W to be a new vector in $\mathcal{W}(\pi,\psi)$, i.e. W is fixed by $K_0(\mathfrak{p}_v^{n(\pi_v)})$ where $n(\pi_v)$ denotes the conductor of π_v, then

$$\ell_v(W) = L(1/2,\pi_v) \neq 0. \tag{219}$$

A more general result can be found in [GP91] which can be used at any non-archimedean place v. We set $\mathfrak{p}_v^{n(\pi_v)}$ be the conductor of π_v. Let R_{π_v} denote an order of discriminant $\mathfrak{p}_v^{n(\pi_v)}$ in D_v such that $R_{\pi_v} \cap E_v = \mathcal{O}_{E_v}$. An order in D_v is a subring R which contains $\mathcal{O}_{F_v}$ and such that $R \otimes_{\mathcal{O}_{F_v}} F_v = D$. Let

$$R^\perp = \{\beta \in D : \operatorname{tr}(\alpha\beta) \in \mathcal{O}_{F_v} \forall \alpha \in R\}. \tag{220}$$

Then $R^\perp/R$ is a finite $\mathcal{O}_{F_v}$ module. The reduced discriminant $d(R)$ is defined by $\# R^\perp/R = q_v^{2d(R)}$. Such orders $R_{n(\pi_v)}$ exist and are unique up to conjugation by $E_v^\times$. We let $K_{\pi_v} = R_{\pi_v}^\times$, a compact open subgroup of $D_v^\times$.

Proposition 9.1. *([GP91, Proposition 2.6]) Assume first that either v is unramified in E or $n(\pi_v) \le 1$. Then the space $(\pi^D)^{K_{\pi_v}}$ is 1-dimensional and $\ell_v(\varphi_v) \neq 0$ for any $0 \neq \varphi_v \in (\pi^D)^{K_{\pi_v}}$. If v is ramified in E and $n(\pi_v) \ge 2$ then $(\pi^D)^{K_{\pi_v}}$ has dimension 2. A uniformizer ϖ_{E_v} in E_v fixes a line in this subspace and $\ell_v(\varphi_v) \neq 0$ for any non-zero φ_v on this line.*

We make a couple of remarks. When v is split in E the space defined by Gross and Prasad agrees with the newspace defined by the fixed space under $K_0(\mathfrak{p}_v^{n(\pi_v)})$. If v is unramified and inert in E then $E_v^\times \subset F_v^\times K_{\pi_v}$, hence $(\pi_v^D)^{E_v^\times} \subset (\pi_v^D)^{K_{\pi_v}}$ and ℓ_v is defined by taking the inner product with a non-zero in $(\pi_v^D)^{E_v^\times}$.

If v is archimedean then we can also choose suitable test vectors. If v is inert so that $F_v = \mathbf{R}$ and $E_v = \mathbf{C}$ then we can pick φ_v to be a non-zero vector in π_v^D which is fixed by $T(F_v)$. In the remaining cases, i.e. when v splits in E, the linear form ℓ_v can again be realized by integrating the Whittaker model over a split torus. One can then choose a non-zero vector $\varphi_v \in \pi_v^D$ which lies in an appropriate K_v-type so that $\ell_v(\varphi_v) \neq 0$.

9.2. **Exact formulas.** In each case we have found a one-dimensional subspace $V(\pi_v, E_v) \subset \pi_v^D$ such that $\ell_v(\varphi_v) \neq 0$ for any $0 \neq \varphi_v \in V(\pi_v, E_v)$. We set $V(\pi, E) = \otimes V(\pi_v, E_v)$. Recall that for $f' \in C_c^\infty(G_D(\mathbf{A}_F))$,

$$I^D_{\pi^D}(f') = \sum_{\varphi \in \mathcal{B}(\pi)} P_D(R(f)\varphi)\overline{P_D(\varphi)}. \tag{221}$$

By the way the spaces $V(\pi_v, E_v)$ are defined, it is not hard to see that we can choose $f' = \prod_v f'_v \in C_c^\infty(G_D(\mathbf{A}_F))$ such that

$$I^D_{\pi^D}(f') = \frac{|P_D(\varphi)|^2}{(\varphi, \varphi)} \tag{222}$$

for any $0 \neq \varphi \in V(\pi, E)$.

Suppose we can find $f = \prod_v f_v \in C^\infty(G(\mathbf{A}_F))$ so that $f^D = f'$ with f' as above. Then from our relative trace formula identity we would have

$$I_\pi(f) = I^D_{\pi^D}(f') = \frac{|P_D(\varphi)|^2}{(\varphi, \varphi)}. \tag{223}$$

We continue to assume that we can find such an $f \in C_c^\infty(G(\mathbf{A}_F))$. We recall that

$$I_\pi(f) = \sum_{\varphi \in \mathcal{B}(\pi)} \int_{F^\times \backslash \mathbf{A}_F^\times} (R(f)\varphi) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \, d^\times a \overline{\int_{F^\times \backslash \mathbf{A}_F^\times} \varphi \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \eta(b) \, d^\times b}. \tag{224}$$

We want to see how to compute this expression, and relate it to $L(1/2, \pi_E)$.

We fix an additive character $\psi : F\backslash \mathbf{A}_F \to \mathbf{C}^\times$ and let $\mathcal{W}(\pi_v, \psi_v)$ denote the Whittaker model of π_v with respect to the additive character ψ_v. We denote by

$\mathcal{W}(\pi,\psi) = \otimes \mathcal{W}(\pi_v,\psi_v)$ the Whittaker model of π with respect to ψ. Given a K-finite vector $\varphi \in \pi$ we define

$$W_\varphi(g) = \int_{F\backslash \mathbf{A}_F} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\overline{\psi(x)}\, dx. \tag{225}$$

This gives an isomorphism between the K-finite vectors in π and the space $\mathcal{W}(\pi,\psi)$. We recall that

$$\varphi(g) = \sum_{a\in F^\times} W_\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right). \tag{226}$$

For $W \in \mathcal{W}(\pi,\psi)$ and a unitary character $\chi : F^\times\backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ we define

$$Z(s,W,\chi) = \int_{\mathbf{A}_F^\times} W\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\chi(a)|a|^{s-\frac{1}{2}}\, d^\times a, \tag{227}$$

which converges if $\Re s \gg 0$. Clearly we have,

$$\int_{F^\times\backslash \mathbf{A}_F^\times} \varphi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\chi(a)|a|^{s-\frac{1}{2}}\, d^\times a = Z(s,W_\varphi,\chi). \tag{228}$$

Similarly for $W_v \in \mathcal{W}(\pi_v,\psi_v)$ we define

$$Z(s,W_v,\chi_v) = \int_{F_v^\times} W_v\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}\chi_v(a_v)|a_v|_v^{s-\frac{1}{2}}\, d^\times a_v, \tag{229}$$

which converges for $\Re s \gg 0$.

Suppose now $W = \prod_v W_v \in \mathcal{W}(\pi,\psi)$. Then, for $\Re s \gg 0$,

$$Z(s,W,\chi) = \prod_v \int_{F_v^\times} Z(s,W_v,\chi_v) \tag{230}$$

$$= L(s,\pi\otimes\chi)\prod_v \frac{Z(s,W_v,\chi_v)}{L(s,\pi_v\otimes\chi_v)}. \tag{231}$$

We set

$$\lambda_{\chi_v}(s,W_v) = \frac{Z(s,W_v,\chi_v)}{L(s,\pi_v\otimes\chi_v)}, \tag{232}$$

then

$$Z(s,W,\chi) = L(s,\pi\otimes\chi)\prod_v \lambda_{\chi_v}(s,W_v), \tag{233}$$

with $\lambda_{\chi_v}(s,W_v) \equiv 1$ for almost all v.

For each v a $G(F_v)$-invariant inner product is given on $\mathcal{W}(\pi_v,\psi_v)$ is given by

$$(W_{1,v},W_{2,v}) = \int_{F_v^\times} W_{1,v}\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}\overline{W_{2,v}\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}}\, d^\times a_v. \tag{234}$$

Hence for $f = \prod_v f_v$,

$$I_\pi(f) = c(\pi)L(1/2,\pi_E)\prod_v \Theta_{\pi_v,\eta_v}(f_v) \tag{235}$$

where

$$\Theta_{\pi_v,\eta_v}(f_v) = \sum_{W_v\in\mathcal{B}(\mathcal{W}(\pi_v,\psi_v))} \lambda_{\mathbf{1}_v}(R(f_v)W_v)\overline{\lambda_{\eta_v}(W_v)}, \tag{236}$$

and $c(\pi)$ a constant depending on the normalization of the inner products on π and $\mathcal{W}(\pi, \psi)$.

Thus to get an exact formula we need to do the following:

(1) Find functions f_v on $G(F_v)$ which match with the function $f'_v \in C_c^\infty(G_D(F_v))$ which picks out the test space $V(\pi_v, E_v)$, and
(2) compute $\Theta_{\pi_v,\eta_v}(f_v)$ for these functions.

This is carried out in the older version of [MW] which can be found on my website at `www-math.mit.edu/~dw`. In this way we recover results of Gross [Gro87] and [Zha01].

Example 9.2. Let $F = \mathbf{Q}$ and $E = \mathbf{Q}(\sqrt{-d})$ an imaginary quadratic field of discriminant $-d$. Let N be the product of an odd number of primes, each unramified and inert in E. And let $2k \geq 2$ be an even integer. We take $\pi \in \mathcal{F}(N, 2k)$, the set of cuspidal automorphic representations π of $\mathrm{PGL}(2, \mathbf{A_Q})$ of exact level N and weight $2k$. Thus each such π can be identified with a newform in the space $S_{2k}(N)$ which is an eigenvector for all the Hecke operators. Then

$$\frac{|P_D(\varphi)|^2}{(\varphi,\varphi)} = \frac{1}{12\sqrt{d}}\frac{(2k-1)!}{(k!)^2}\prod_{p|N}(1-p^{-1})\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)}, \tag{237}$$

here $0 \neq \varphi \in V(E, \pi)$ and D is the quaternion algebra with $\mathrm{Ram}(D) = \{p \mid N\} \cup \{\infty\}$.

We can also use the relative trace formula, and this identity, to study these L-values averaged over all $\pi \in \mathcal{F}(N, 2k)$.

For $\pi \in \mathcal{F}(N, 2k)$ let $f \in C_c^\infty(G_D(\mathbf{A_Q}))$ which picks out the test space $V(\pi, E)$. Then f' depends only on N, $2k$ and E. Hence if we use the relative trace formula then

$$I^D_{cusp}(f) + I^D_{res}(f) = I^D(f) = I^D_{reg}(f) + I^D_{irreg}(f). \tag{238}$$

Then we have

$$I^D_{cusp}(f) = \sum_{\pi\in\mathcal{F}(N,2k)} \frac{|P_D(\varphi_\pi)|^2}{(\varphi_\pi,\varphi_\pi)}, \tag{239}$$

where $0 \neq \varphi_\pi \in V(\pi, E)$, and hence

$$I^D_{cusp}(f) = \frac{1}{12\sqrt{d}}\frac{(2k-1)!}{(k!)^2}\prod_{p|N}(1-p^{-1})\sum_{\pi\in\mathcal{F}(N,2k)}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)}. \tag{240}$$

The identity (238) then gives a finite closed expression for

$$\sum_{\pi\in\mathcal{F}(N,2k)}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)}. \tag{241}$$

In particular if N is large then one can show that $I^D_{irreg}(f) = 0$.

Theorem 9.3. *([FW]) Assume that $N > d$, with the same assumptions as in the example above. Then*

$$\frac{2}{N}\binom{2k-2}{k-1}\sum_{\pi\in\mathcal{F}(N,2k)}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)} = 4L(1,\eta) \tag{242}$$

if $2k > 2$, *and*

$$\frac{2}{N}\binom{2k-2}{k-1}\sum_{\pi\in\mathcal{F}(N,2k)}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)} = 4L(1,\eta) - 48\sqrt{d}\prod_{p|N}\frac{1}{p-1}L(1,\eta)^2, \tag{243}$$

when $2k = 2$.

10. Preliminaries

This section aims to give an overview of things we'll need in these lectures.

I will assume familiarity with the adelic theory for automorphic forms on GL(2); see [Kud03] for the passage from the classical viewpoint to the adelic one. Good background reading for automorphic representations of GL(2) can be found in the books of Gelbart [Gel75] and Bump [Bum97]. For the theory of L-functions for GL(2) (and GL(n)) one can consult the notes of Cogdell [Cog03]. For an overview of the relative trace formula one can see the article of Jacquet [Jac05].

10.1. Automorphic forms on PGL(2).

In this section we recall some necessary facts about the spectral decomposition of $L^2(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$.

We recall that the group $\mathrm{PGL}(2,\mathbf{A}_F)$ acts on the space

$$L^2(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F)) \tag{244}$$

via right translation. We denote by

$$B = \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right\} \tag{245}$$

and

$$N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right\}. \tag{246}$$

Remark 10.1. Throughout these notes we will abuse notation by representing elements in PGL(2) by matrices in GL(2).

We recall that a function $\varphi \in L^2(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$ is called a cusp form if

$$\int_{N(F)\backslash N(\mathbf{A}_F)} \varphi(ng)\, dn \tag{247}$$

vanishes for almost all $g \in G(\mathbf{A}_F)$. We let

$$L^2_{cusp}(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F)) \tag{248}$$

denote the subspace of cusp forms in $L^2(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$. It is clear, from the definition of cuspidality (247), that this is space is preserved by the action of $\mathrm{PGL}(2,\mathbf{A}_F)$.

Fact 10.2. As a representation of $\mathrm{PGL}(2,\mathbf{A}_F)$,

$$L^2_{cusp}(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F)) = \widehat{\bigoplus}_{\pi\in\mathcal{A}_{cusp}(\mathrm{PGL}(2,F))} V_\pi.$$

We denote by $L^2_{cont}(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$ the orthogonal complement of $L^2_{cusp}(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$ in $L^2(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$. The theory of Eisenstein series [Lan76] gives a further decomposition of the space $L^2_{cont}(\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbf{A}_F))$.

Fact 10.3. (Strong multiplicity one) Let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{PGL}(2, \mathbf{A}_F)$. Assume that $\pi_{1,v} \cong \pi_{2,v}$ for all but finitely many places v of F, then $\pi_1 = \pi_2$.

10.2. **The Jacquet-Langlands correspondence.** Let D be a quaternion division algebra over F. We take G to be the algebraic group defined over F with $G(F) = D^\times/F^\times$. Since D is a division algebra the quotient $G(F)\backslash G(\mathbf{A}_F)$ is compact and hence, as a representation of $G(\mathbf{A}_F)$,

$$L^2(G(F)\backslash G(\mathbf{A}_F)) = \widehat{\bigoplus}_{\pi' \in \mathcal{A}(G)} V_{\pi'}. \tag{249}$$

Here the sum is taken over the irreducible automorphic representations $\mathcal{A}(G)$ of $G(\mathbf{A}_F)$ and for each $\pi' \in \mathcal{A}(G)$, $V_{\pi'}$ denotes the space of π'. The Jacquet-Langlands correspondence [JL70] yields an injection $\mathrm{JL} : \mathcal{A}(G) \hookrightarrow \mathcal{A}(\mathrm{PGL}(2))$, where $\mathcal{A}(\mathrm{PGL}(2))$ denotes the set of automorphic representations of $\mathrm{PGL}(2, \mathbf{A}_F)$. The set $\mathcal{A}(G)$ can be decomposed as,

$$\mathcal{A}(G) = \mathcal{A}_{cusp}(G) \amalg \mathcal{A}_{res}(G), \tag{250}$$

where

$$\mathcal{A}_{cusp}(G) = \{\pi' \in \mathcal{A}(G) : \mathrm{JL}(\pi') \text{ is cuspidal}\} \tag{251}$$

and

$$\mathcal{A}_{res}(G) = \left\{\delta \circ N_D : \delta : F^\times\backslash \mathbf{A}_F^\times \to \{\pm 1\}\right\}. \tag{252}$$

The compatibility between the local and global Jacquet-Langlands correspondence yields the following.

Fact 10.4. The image of $\mathcal{A}_{cusp}(G)$ under JL is equal to the set of cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $\mathrm{PGL}(2, \mathbf{A}_F)$ such that π_v is a discrete series representation of $\mathrm{PGL}(2, F_v)$ at all places v where D ramifies.

10.3. ***L*-functions of cusp forms on** $\mathrm{PGL}(2)$**.** We recall that associated to an automorphic representation $\pi = \otimes \pi_v$ of $\mathrm{PGL}(2)$ is an L-function $L(s, \pi)$. This is defined by a product,

$$L(s, \pi) = \prod_v L(s, \pi_v) \tag{253}$$

taken over the places v of F which converges if $\Re s$ is sufficiently large. Here $L(s, \pi_v)$ denotes the L-function of the local representation π_v. If π_v is unramified then

$$L(s, \pi_v) = \frac{1}{(1 - \alpha_v q_v^{-s})(1 - \alpha_v^{-1} q_v^{-s})}, \tag{254}$$

where $\{\alpha_v, \alpha_v^{-1}\}$ are the Satake parameters associated to π_v and q_v denotes the cardinality of the residue class field at v. The definition of the local L-factors at the ramified places is more subtle.

The L-function $L(s, \pi)$ satisfies an analytic continuation to the whole complex plane, and has a functional equation,

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi). \tag{255}$$

This is obtained by showing that, for a suitable element φ_0 in the space of π,

$$L(s, \pi) = \int_{F^\times \backslash \mathbf{A}_F^\times} \varphi_0 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s - \frac{1}{2}} d^\times a. \tag{256}$$

Furthermore for any element φ in the space of π the quotient

$$\frac{\int_{F^\times\backslash\mathbf{A}_F^\times}\varphi\begin{pmatrix}a&0\\0&1\end{pmatrix}|a|^{s-\frac{1}{2}}d^\times a}{L(s,\pi)} \tag{257}$$

is a holomorphic function of $s\in\mathbf{C}$.

In particular we note that $L(\frac{1}{2},\pi)=0$ if and only if

$$\int_{F^\times\backslash\mathbf{A}_F^\times}\varphi\begin{pmatrix}a&0\\0&1\end{pmatrix}d^\times a=0 \tag{258}$$

for all φ in the space of π.

10.4. **Automorphic kernel.** Let G be a semisimple algebraic group defined over a number field F; for example G could be PGL(2) or $PD^\times$ for some quaternion algebra D. Let $f\in C_c^\infty(G(\mathbf{A}_F))$. The group $G(\mathbf{A}_F)$ acts on $L^2(G(F)\backslash G(\mathbf{A}_F))$ by right translation. Integrating f against the action of $G(\mathbf{A}_F)$ gives a linear map

$$R(f):L^2(G(F)\backslash G(\mathbf{A}_F))\to L^2(G(F)\backslash G(\mathbf{A}_F)) \tag{259}$$

defined by

$$(R(f)\varphi)(x)=\int_{G(\mathbf{A}_F)}f(y)\varphi(xy)\ dy. \tag{260}$$

We begin by showing that $R(f)$ is an integral operator. To begin, by making a change of variables,

$$(R(f)\varphi)(x)=\int_{G(\mathbf{A}_F)}f(x^{-1}y)\varphi(y)\ dy. \tag{261}$$

If we average over $G(F)$ then we see that this integral is equal to

$$(R(f)\varphi)(x)=\int_{G(F)\backslash G(\mathbf{A}_F)}\sum_{\gamma\in G(F)}f(x^{-1}\gamma y)\varphi(\gamma y)\ dy, \tag{262}$$

and since φ is invariant on the left by $G(F)$,

$$(R(f)\varphi)(x)=\int_{G(F)\backslash G(\mathbf{A}_F)}\sum_{\gamma\in G(F)}f(x^{-1}\gamma y)\varphi(y)\ dy. \tag{263}$$

We set

$$K_f(x,y)=\sum_{\gamma\in G(F)}f(x^{-1}\gamma y). \tag{264}$$

Clearly $K_f(x,y)$ is a function of $(x,y)\in(G(F)\backslash G(\mathbf{A}_F))\times(G(F)\backslash G(\mathbf{A}_F))$ and

$$(R(f)\varphi)(x)=\int_{G(F)\backslash G(\mathbf{A}_F)}K_f(x,y)\varphi(y)\ dy. \tag{265}$$

Thus $R(f)$ is an integral operator with kernel $K_f(x,y)$.

We now give an expression for $K_f(x,y)$ in terms of the spectrum of $L^2(G(F)\backslash G(\mathbf{A}_F))$. Assume first that the quotient $G(F)\backslash G(\mathbf{A}_F)$ is compact, for example we could take G to be $PD^\times$ with D a quaternion division algebra over F. In this case we have, as a representation of $G(\mathbf{A}_F)$,

$$L^2(G(F)\backslash G(\mathbf{A}_F))=\widehat{\bigoplus_{\pi\in\mathcal{A}(G)}}m_\pi\pi. \tag{266}$$

For each $\pi \in \mathcal{A}(G)$ let $\mathcal{B}(\pi)$ be an orthonormal basis for the π-isotypic subspace of $L^2(G(F)\backslash G(\mathbf{A}_F))$. Then

$$K_f(x,y) = \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \mathcal{B}(\pi)} (R(f)\varphi)(x)\overline{\varphi(y)}. \tag{267}$$

If the quotient $G(F)\backslash G(\mathbf{A}_F)$ is not compact then we have

$$L^2(G(F)\backslash G(\mathbf{A}_F)) = L^2_{disc}(G(F)\backslash G(\mathbf{A}_F)) \oplus L^2_{cont}(G(F)\backslash G(\mathbf{A}_F)). \tag{268}$$

Since $R(f)$ preserves each term we have a decomposition

$$K_f(x,y) = K_{f,disc}(x,y) + K_{f,cont}(x,y). \tag{269}$$

The term $K_{f,disc}(x,y)$ takes the same form as in the compact case. The continuous spectrum is furnished by Eisenstein series and $K_{f,cont}(x,y)$ has an expansion in terms of them. For suitable of choices of $f \in C_c^\infty(G(\mathbf{A}_F))$ (for example if $f = \prod_v f_v$ such that for some place v, f_v is a matrix coefficient of a supercuspidal representation of $G(F_v)$) one can ensure that $K_{f,cont}(x,y) \equiv 0$. For such test functions $K_f(x,y)$ has a spectral expansion as in the compact case involving only a sum over the discrete spectrum.

References

[Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR MR1431508 (97k:11080)

[Cog03] J. W. Cogdell, *Analytic theory of L-functions for* GL_n, An introduction to the Langlands program (Jerusalem, 2001), Birkhäuser Boston, Boston, MA, 2003, pp. 197–228. MR MR1990380

[FW] Brooke Feigon and David Whitehouse, *Averages of central L-values of Hilbert modular forms with an application to subconvexity*, Submitted.

[Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83. MR MR0379375 (52 #280)

[GP91] Benedict H. Gross and Dipendra Prasad, *Test vectors for linear forms*, Math. Ann. **291** (1991), no. 2, 343–355. MR MR1129372 (92k:22028)

[Gro87] Benedict H. Gross, *Heights and the special values of L-series*, Number theory (Montreal, Que., 1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187. MR MR894322 (89c:11082)

[Guo96] Jiandong Guo, *On the positivity of the central critical values of automorphic L-functions for* GL(2), Duke Math. J. **83** (1996), no. 1, 157–190. MR MR1388847 (97h:11055)

[Jac86] Hervé Jacquet, *Sur un résultat de Waldspurger*, Ann. Sci. Éc. Norm. Supér., IV. **19** (1986), no. 2, 185–229.

[Jac87] ———, *Sur un résultat de Waldspurger. II*, Compositio Math. **63** (1987), no. 3, 315–389. MR MR909385 (89c:11085)

[Jac05] Hervé Jacquet, *A guide to the relative trace formula*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 257–272. MR MR2192826 (2006g:11100)

[JL70] Hervé Jacquet and Robert P. Langlands, *Automorphic forms on* GL(2), Springer-Verlag, Berlin, 1970, Lecture Notes in Mathematics, Vol. 114. MR MR0401654 (53 #5481)

[Kud03] Stephen S. Kudla, *From modular forms to automorphic representations*, An introduction to the Langlands program (Jerusalem, 2001), Birkhäuser Boston, Boston, MA, 2003, pp. 133–151. MR MR1990378

[Lan76] Robert P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, Vol. 544, Springer-Verlag, Berlin, 1976. MR MR0579181 (58 #28319)

[MW] Kimball Martin and David Whitehouse, *Central L-values and toric periods for* GL(2), Submitted.

[RR05] Dinakar Ramakrishnan and Jonathan Rogawski, *Average values of modular L-series via the relative trace formula*, Pure Appl. Math. Q. **1** (2005), no. 4, 701–735. MR MR2200997

[Wal85] Jean-Loup Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242. MR MR783511 (87g:11061b)

[Zha01] Shou-Wu Zhang, *Gross-Zagier formula for* GL_2, Asian J. Math. **5** (2001), no. 2, 183–290. MR MR1868935 (2003k:11101)

Department of Mathematics, MIT, 2-304, 77 Massachusetts Avenue, Cambridge, MA 02139-4307

E-mail address: dw@math.mit.edu