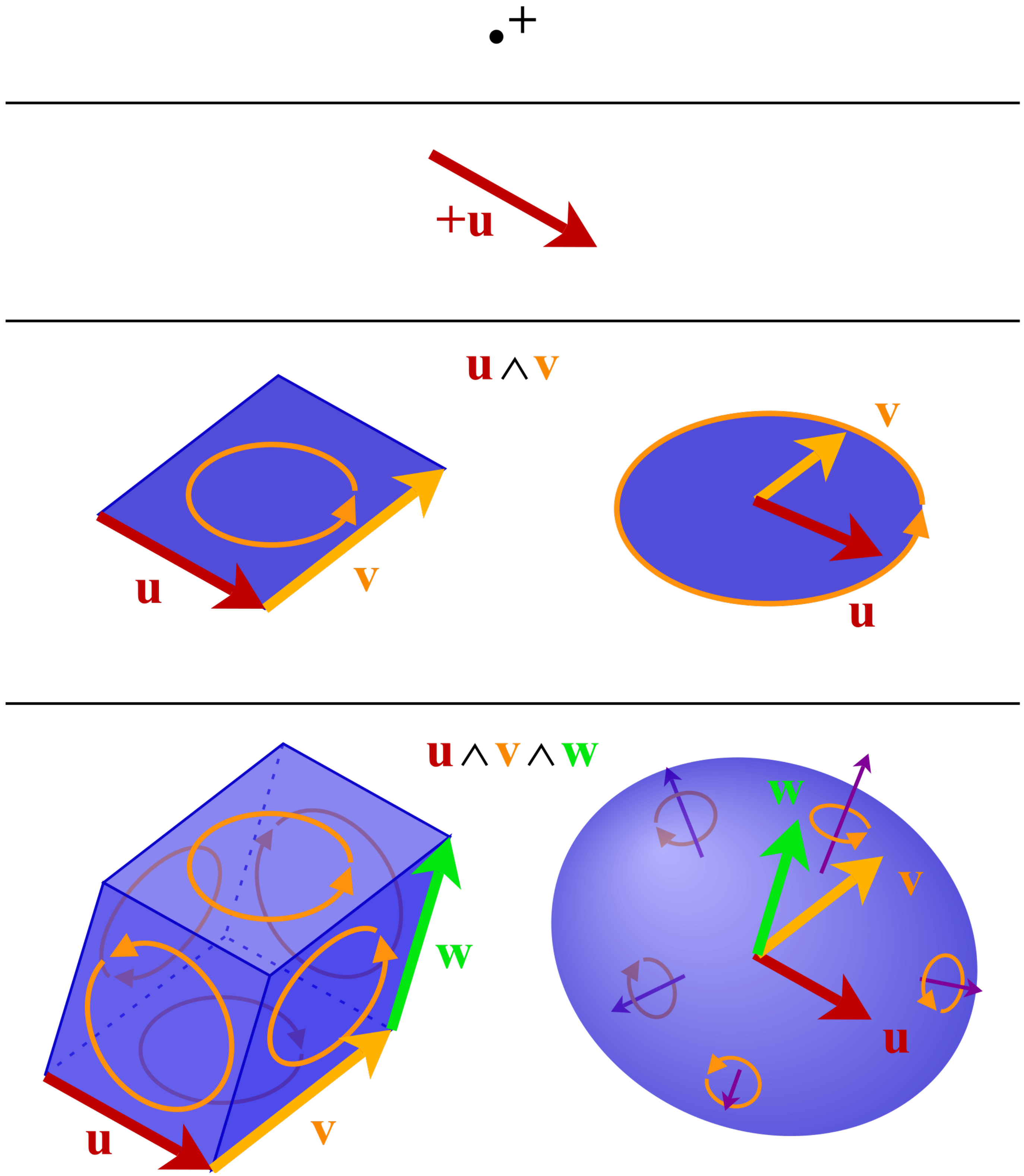
18.S996
Introduction to
Geometric Algebra with Applications in Physical Mathematics

Jörn Dunkel

L01: History & background



+
+u
u ∧ v
u
v
v
u
u ∧ v ∧ w
w
u
v
w
v
u

Online resources

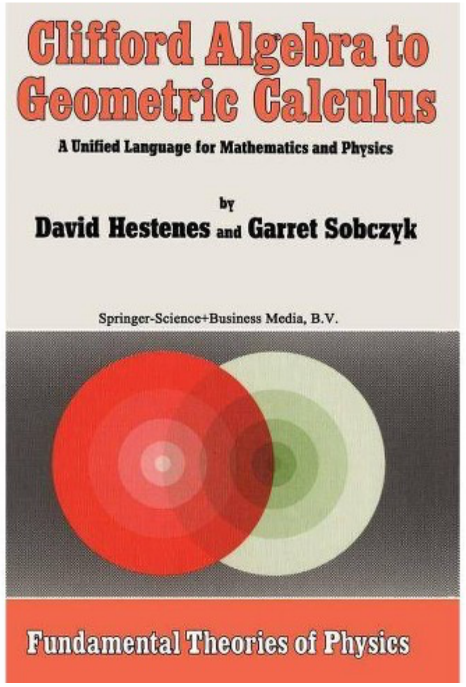
- 18.S996 CANVAS
  https://canvas.mit.edu/courses/13047

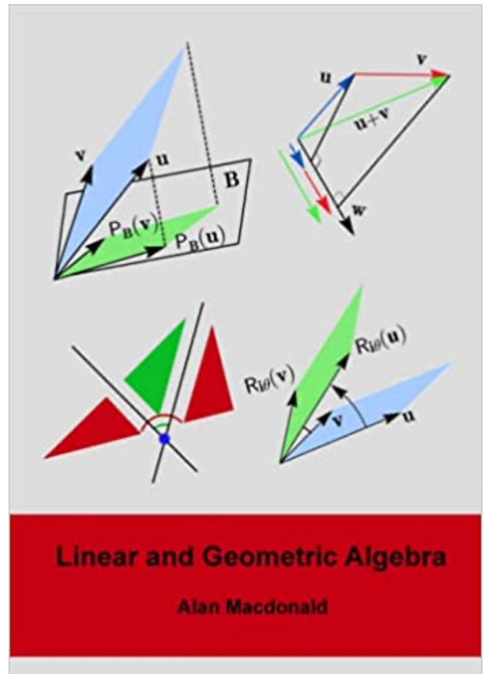
- 18.S996 archive on my homepage
  https://math.mit.edu/~dunkel/Teach/18.S996_2022S/

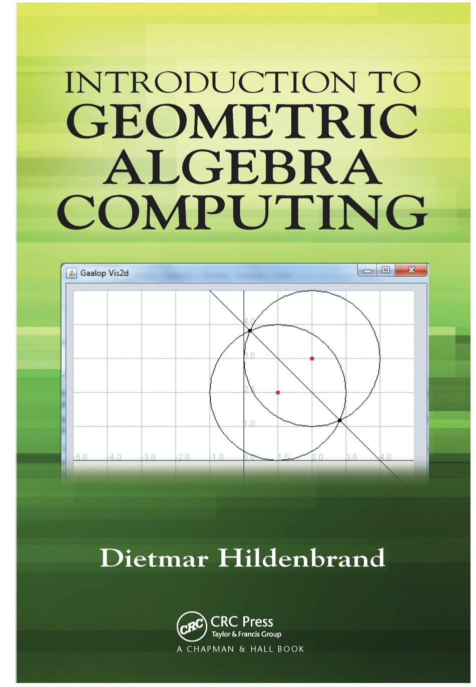
Topics

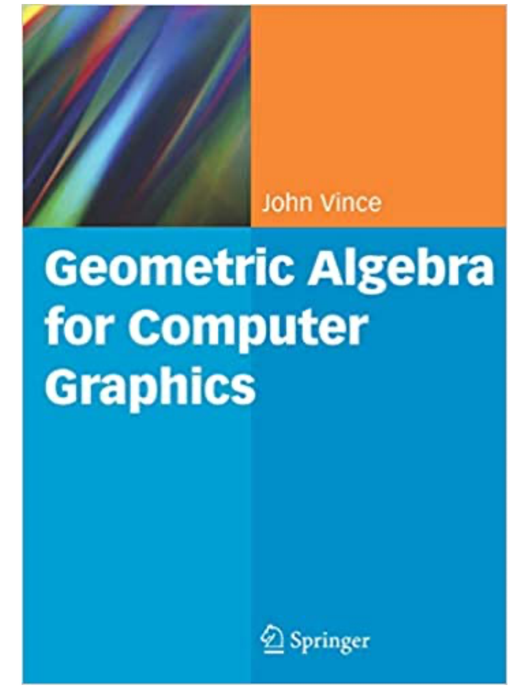
- Historical overview
- Review of complex numbers, functions, differentiation & integration
- Geometric algebra basics (dim=1, 2, 3)
- Matrix representations
- Geometric operations & functions (Cayley-Hamilton)
- Numerical frameworks
- Geometric differentiation & integration
- Fundamental Theorem of Geometric Calculus
- Applications to generalized elasticity problems (mostly 2-dim)

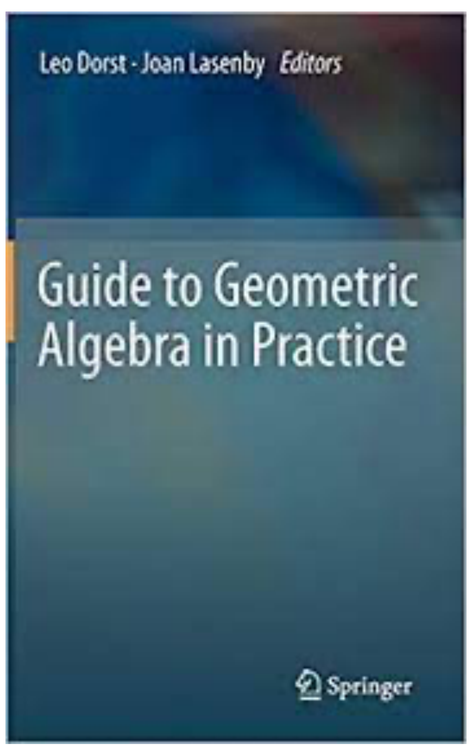
Books

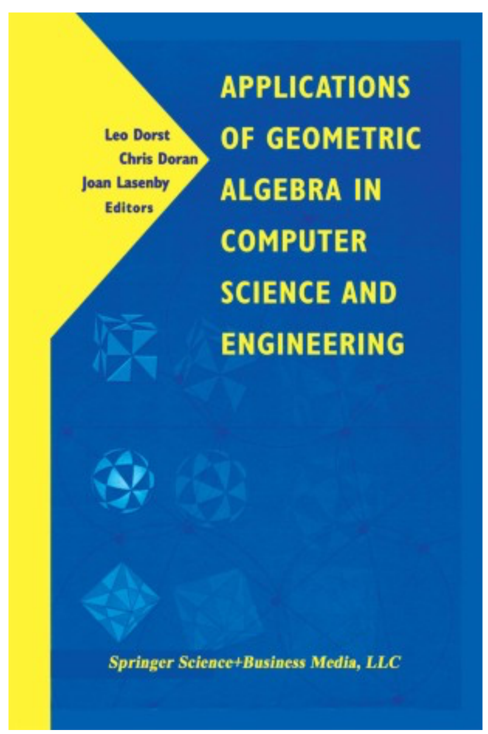

Clifford Algebra to Geometric Calculus
A Unified Language for Mathematics and Physics
by
David Hestenes and Garret Sobczyk
Springer-Science+Business Media, B.V.
Fundamental Theories of Physics


Linear and Geometric Algebra
Alan Macdonald

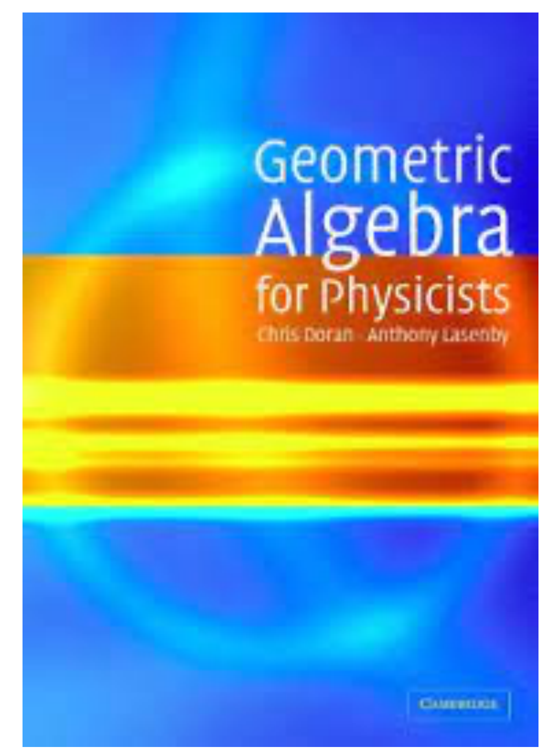

INTRODUCTION TO GEOMETRIC ALGEBRA COMPUTING
Dietmar Hildenbrand
CRC Press
A CHAPMAN & HALL BOOK


John Vince
Geometric Algebra for Computer Graphics
Springer


Leo Dorst · Joan Lasenby Editors
Guide to Geometric Algebra in Practice
Springer


Leo Dorst
Chris Doran
Joan Lasenby
Editors
APPLICATIONS OF GEOMETRIC ALGEBRA IN COMPUTER SCIENCE AND ENGINEERING
Springer Science+Business Media, LLC


GEOMETRIC ALGEBRA for Computer Science


Geometric Algebra for Physicists
Chris Doran · Anthony Lasenby

Web resources

https://en.wikipedia.org/wiki/Geometric_algebra

M

http://geocalc.clas.asu.edu

http://www.faculty.luther.edu/~macdonal/

CS

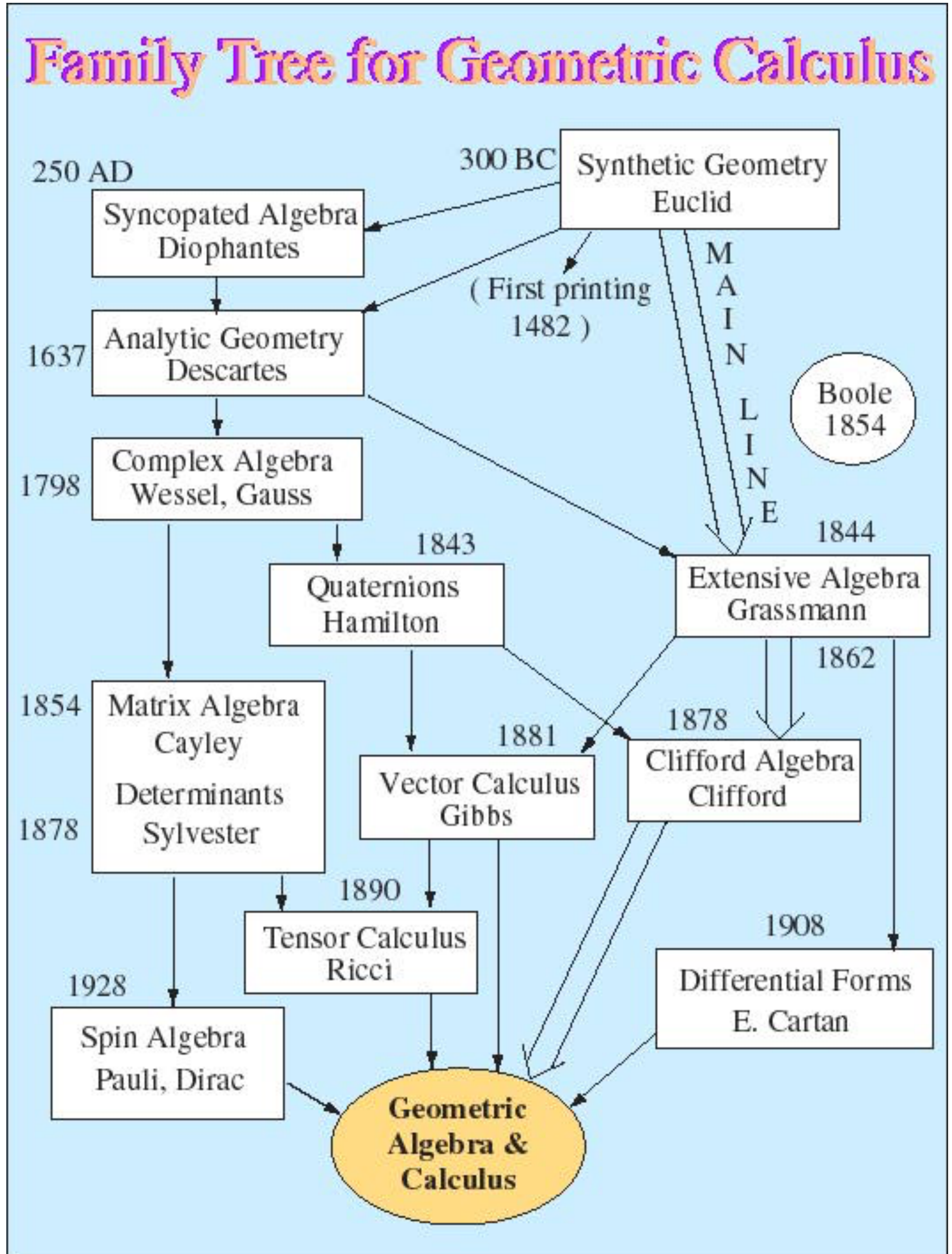
https://geometricalgebra.org

https://staff.science.uva.nl/l.dorst/clifford/index.html

P

https://www.mrao.cam.ac.uk/~anthony/recent_ga.php

David Hestenes
(ASU)



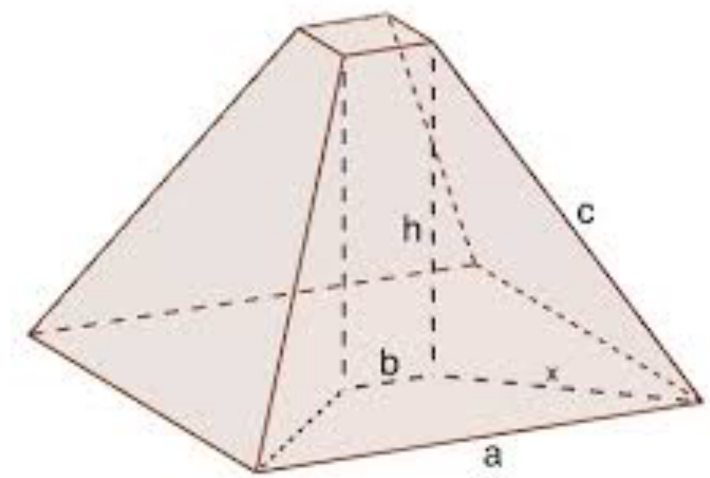
http://geocalc.clas.asu.edu/pdf/GrassmannsLegacy4.pdf

Heron of Alexandria

Wiki

c. 10 AD – c. 70 AD

frustum



$$V = \frac{1}{3}h(a^2 + ab + b^2)$$

$$h = \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2}$$

$$h = \sqrt{(15)^2 - 2\left(\frac{28-4}{2}\right)^2} = \sqrt{225 - 2(12)^2} = \sqrt{225 - 144 - 144} = \sqrt{81 - 144} \quad = \sqrt{-63}.$$

but the *Stereometria* records it as $h = \sqrt{63}$

http://umhistory.dc.umich.edu/history/Faculty_History/B/Beman,_Wooster_Woodruff.html

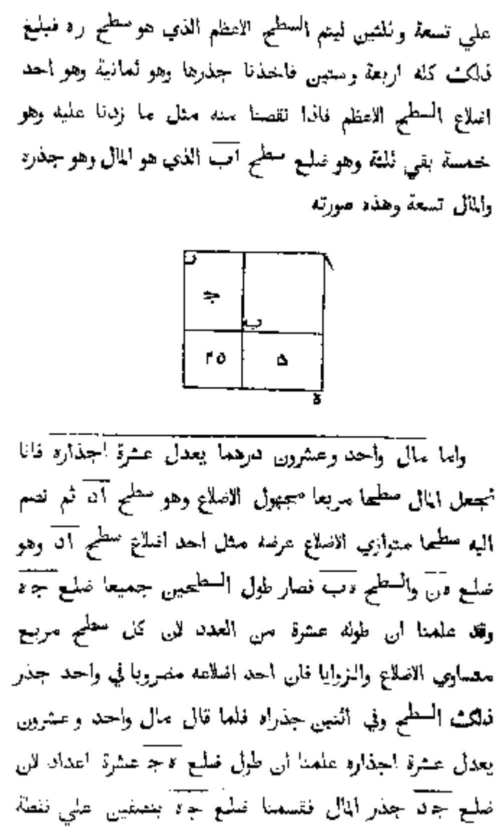
From Nahin's book:

Now, let's skip ahead in time to 1897, to a talk given that year at a meeting of the American Association for the Advancement of Science by Wooster Woodruff Beman, a professor of mathematics at the University of Michigan, and a well-known historian of the subject. I quote from that address:

We find the square root of a negative quantity appearing for the first time in the *Stereometria* of Heron of Alexandria . . . After having given a correct formula for the determination of the volume of a frustum of a pyramid with square base and applied it successfully to the case where the side of the lower base is 10, of the upper 2, and the edge 9, the author endeavors to solve the problem where the side of the lower base is 28, of the upper 4, and the edge 15. Instead of the square root of 81 − 144 required by the formula, he takes the square root of 144 − 81 . . . , i.e., he replaces $\sqrt{-1}$ by 1, and fails to observe that the problem as stated is impossible. Whether this mistake was due to Heron or to the ignorance of some copyist cannot be determined.[4]

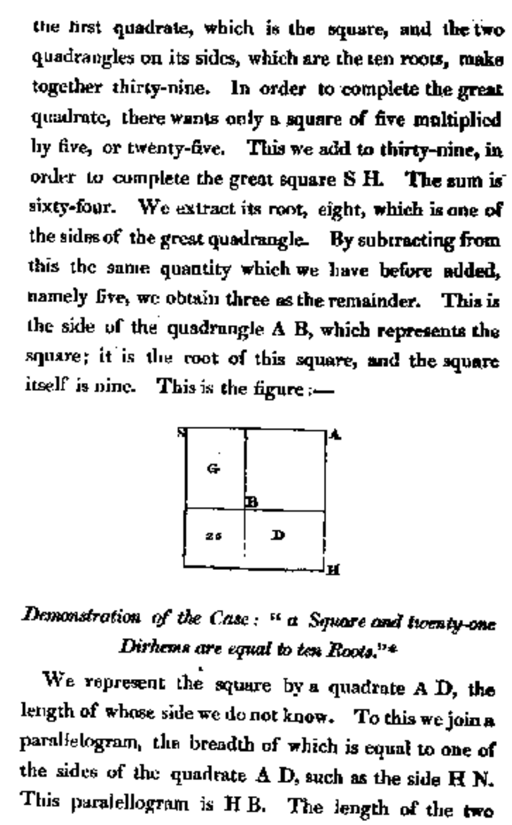
Al-Khwarizmi (780-850) in his Algebra has solution to quadratic equations of various types. Solutions agree with is learned today at school, restricted to positive solutions [9] Proofs are geometric based. Sources seem to be greek and hindu mathematics. According to G. J. Toomer, quoted by Van der Waerden,

> Under the caliph al-Ma'mun (reigned 813-833) al-Khwarizmi became a member of the "House of Wisdom" (Dar al-Hikma), a kind of academy of scientists set up at Baghdad, probably by Caliph Harun al-Rashid, but owing its preeminence to the interest of al-Ma'mun, a great patron of learning and scientific investigation. It was for al-Ma'mun that Al-Khwarizmi composed his astronomical treatise, and his Algebra also is dedicated to that ruler



على تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

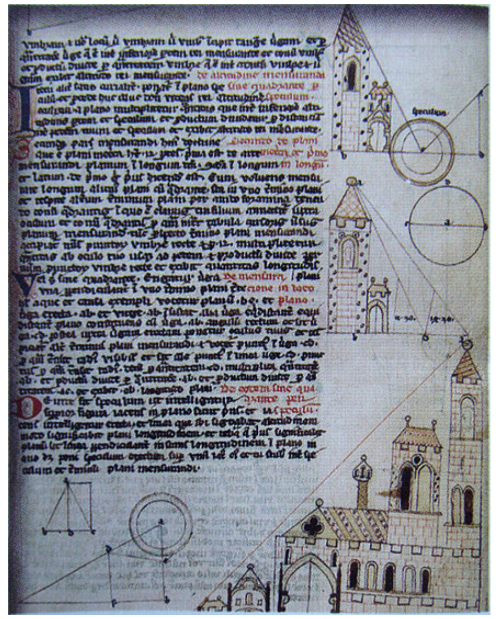
واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع هن والسطح هب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة



the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine, in order to complete the great square S H. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle A B, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—

Demonstration of the Case: "a Square and twenty-one Dirhems are equal to ten Roots."

We represent the square by a quadrate A D, the length of whose side we do not know. To this we join a parallelogram, the breadth of which is equal to one of the sides of the quadrate A D, such as the side H N. This parallelogram is H B. The length of the two



Fibonacci

The methods of algebra known to the arabs were introduced in Italy by the Latin translation of the algebra of al-Khwarizmi by Gerard of Cremona (1114-1187), and by the work of Leonardo da Pisa (Fibonacci)(1170-1250).

About 1225, when Frederick II held court in Sicily, Leonardo da Pisa was presented to the emperor. A local mathematician posed several problems, all of which were solved by Leonardo. One of the problems was the solution of the equation

$$x^3 + 2x^2 + 10x = 20$$

The general cubic equation

$$x^3 + ax^2 + bx + c = 0$$

can be reduced to the simpler form

$$x^3 + px + q = 0$$

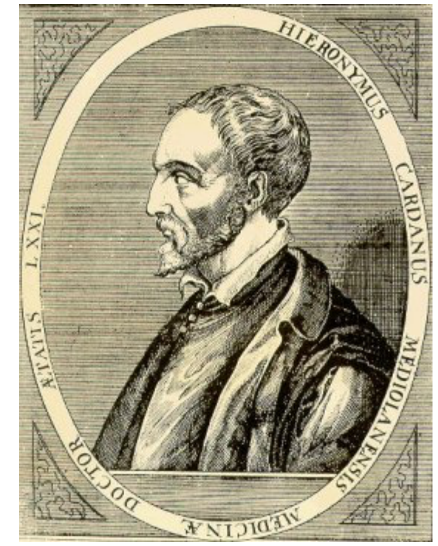
through the change of variable $x' = x + \frac{1}{3}a$. This change of variable appears for the first time in two anonymous florentine manuscripts near the end of the 14th century.

If only positive coefficients and positive values of x are admitted, there are three cases, all collectively known as *depressed cubic*:

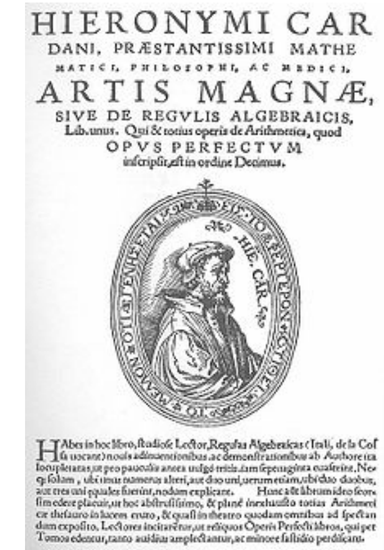
$$\begin{aligned} x^3 + px &= q \\ x^3 &= px + q \\ x^3 + q &= px \end{aligned}$$

Scipione del Ferro
(6 Feb 1465 – 5 Nov1526)

Niccolò Fontana Tartaglia
(1499/1500 – 13 Dec 1557)



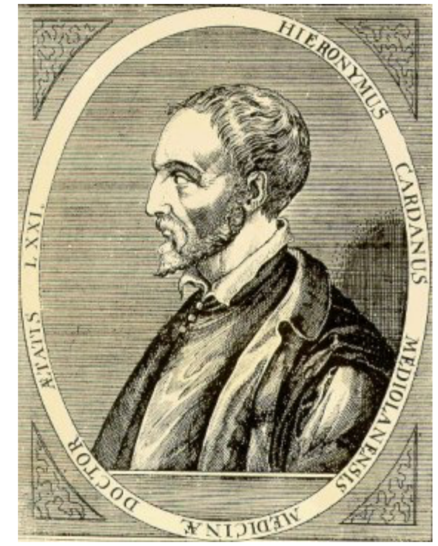
Gerolamo Cardano
(24 Sep 1501 – 21 Sep 1576)


HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

The first to solve equation (1) (and maybe (2) and (3)) was Scipione del Ferro, professor of U. of Bologna until 1526, when he died. In his deathbed, del Ferro confided the formula to his pupil Antonio Maria Fiore. Fiore challenged Tartaglia to a mathematical contest. The night before the contest, Tartaglia rediscovered the formula and won the contest. Tartaglia in turn told the formula (but not the proof) to Gerolamo Cardano, who signed an oath to secrecy. From knowledge of the formula, Cardano was able to reconstruct the proof. Later, Cardano learned that del Ferro had the formula and verified this by interviewing relatives who gave him access to del Ferro's papers. Cardano then proceeded to publish the formula for all three cases in his Ars Magna (1545). It is noteworthy that Cardano mentioned del Ferro as first author, and Tartaglia as obtaining the formula later in independent manner.

Scipione del Ferro
(6 Feb 1465 – 5 Nov1526)

Niccolò Fontana Tartaglia
(1499/1500 – 13 Dec 1557)



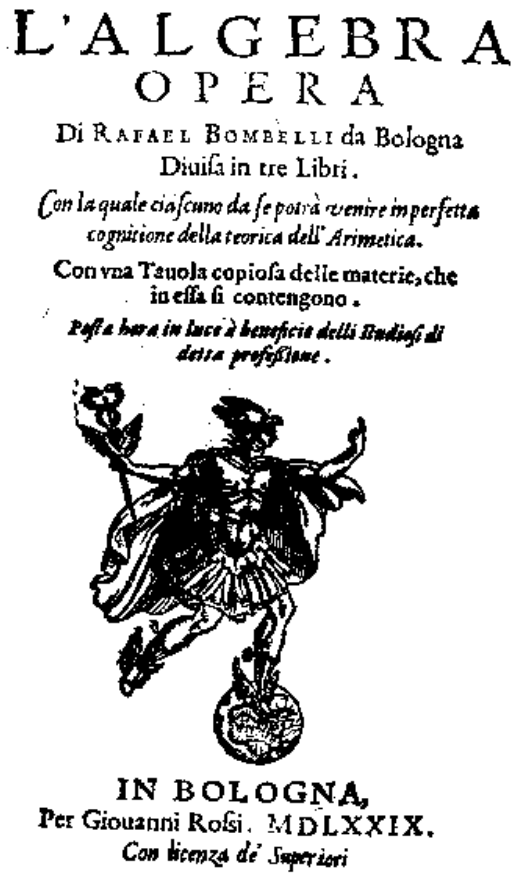
Gerolamo Cardano
(24 Sep 1501 – 21 Sep 1576)

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

Habes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa vocant) novis adinventionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro paucis antea vulgo tritis iam septuaginta evaserint. Neque solum, ubi unus numerus alteri, aut duo uni, verum etiam, ubi duo duobus, aut tres uni aequales fuerint, nodum explicant. Hunc aut librum ideo seorsim edere placuit, ut hoc abstrusissimo, & plane inexhausto totius Arithmeticae thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto avidius amplectantur, ac minore fastidio perdiscant.

According to [9], "Cardano was the first to introduce complex numbers $a + \sqrt{-b}$ into algebra, but had misgivings about it." In Chapter 37 of Ars Magna the following problem is posed: "To divide 10 in two parts, the product of which is 40".

> It is clear that this case is impossible. Nevertheless, we shall work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$.
>
> Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ making $25 - (-15)$ which is $+15$. Hence this product is 40.



L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copiosa delle materie, che in essa si contengono.
Posta hora in luce à beneficio delli Studiosi di detta professione.

IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori

Rafael Bombelli authored l'Algebra (1572, and 1579), a set of three books. Bombelli introduces a notation for $\sqrt{-1}$, and calls it "*piú di meno*".

The discussion of cubics in l'Algebra follows Cardano, but now the casus irreducibilis is fully discussed. Bombelli considered the equation

$$x^3 = 15x + 4$$

for which the Cardan formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli observes that the cubic has $x = 4$ as a solution, and then proceeds to explain the expression given by the Cardan formula as another expression for $x = 4$ as follows. He sets

$$\sqrt[3]{2 + \sqrt{-121}} = a + bi$$

from which he deduces

$$\sqrt[3]{2 - \sqrt{-121}} = a - bi$$

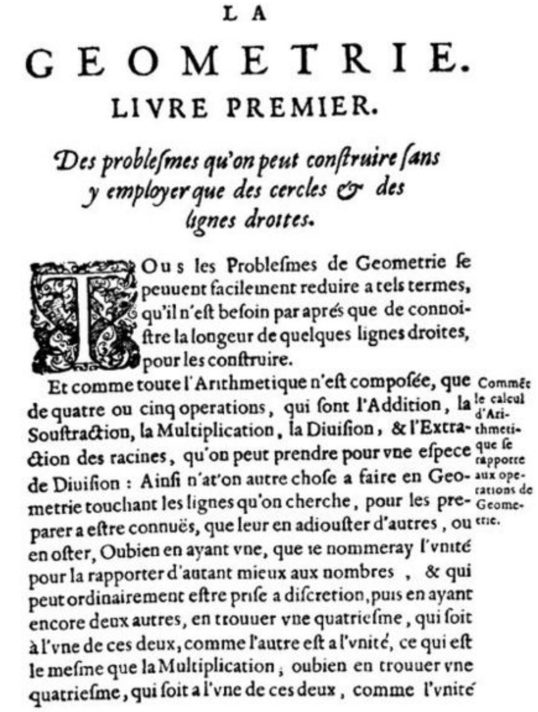
and obtains, after algebraic manipulations, $a = 2$ and $b = 1$. Thus

$$x = a + bi + a - bi = 2a = 4$$

After doing this, Bombelli commented:

> " At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof."

https://math.mit.edu/~dunkel/Teach/18.S996_2022S/history/Merino_ComplexNumbers_History.pdf



LA

GEOMETRIE.

LIVRE PREMIER.

Des problesmes qu'on peut construire sans y employer que des cercles & des lignes droites.

TOus les Problesmes de Geometrie se peuuent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision : Ainsi n'at'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connuës, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit à l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication ; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité

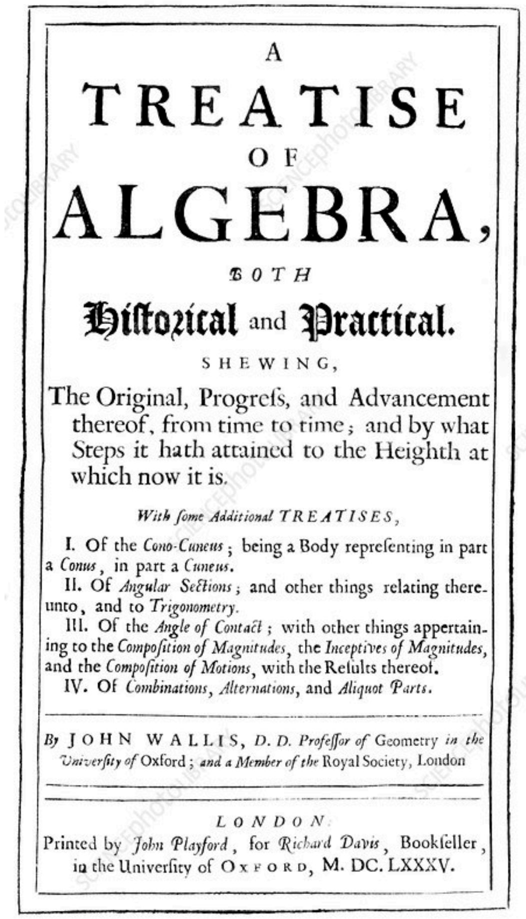
Commẽt le calcul d'Arithmetique se rapporte aux operations de Geometrie.

René Descartes (1596-1650) was a philosopher whose work, La Géométrie, includes his application of algebra to geometry from which we now have Cartesian geometry. Descartes was pressed by his friends to publish his ideas, and he wrote a treatise on science under the title "Discours de la méthod pour bien conduire sa raison et chercher la vérité dans les sciences". Three appendices to this work were La Dioptrique, Les Météores, and La Géométrie. The treatise was published at Leiden in 1637. Descartes associated imaginary numbers with geometric impossibility. This can be seen from the geometric construction he used to solve the equation $z^2 = az - b^2$, with a and b^2 both positive. According to [1], Descartes coined the term *imaginary*:

> "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines."

https://math.mit.edu/~dunkel/Teach/18.S996_2022S/history/Merino_ComplexNumbers_History.pdf

John Wallis
(3 Dec 1616 – 8 Nov 1703)



A
TREATISE
OF
ALGEBRA,
BOTH
Historical and Practical.
SHEWING,
The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With some Additional TREATISES,

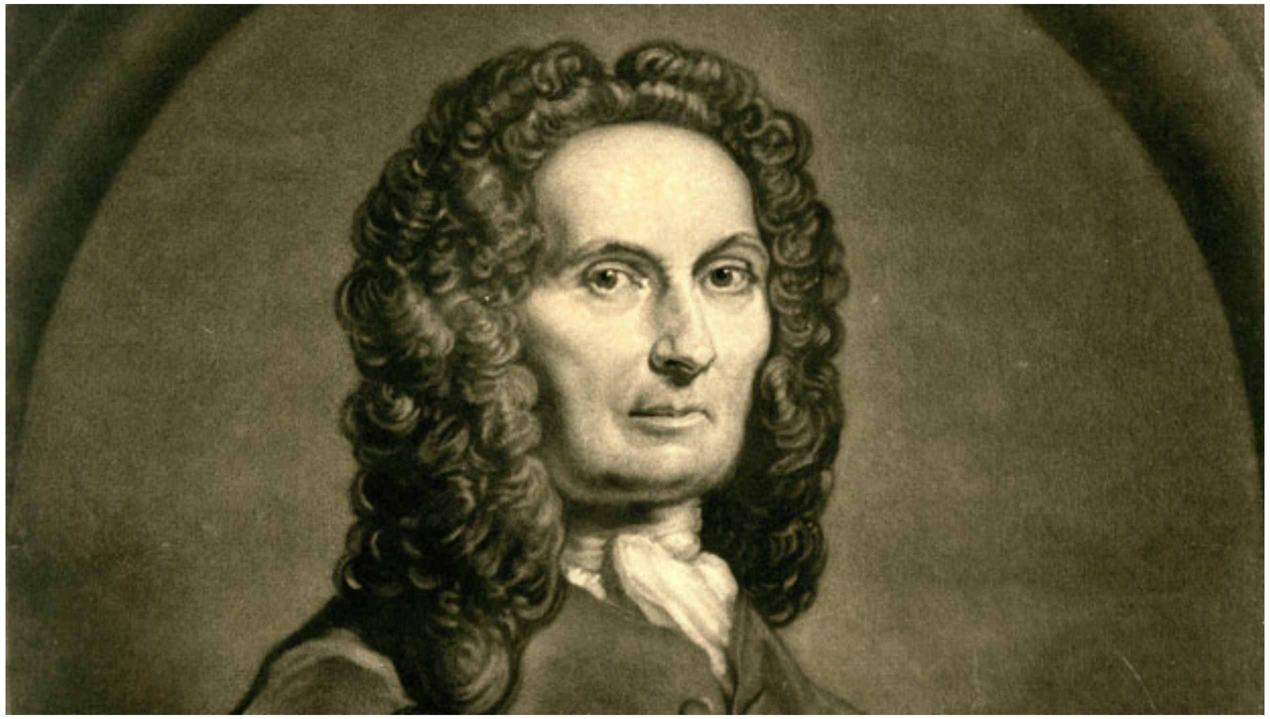
I. Of the *Cono-Cuneus*; being a Body representing in part a *Conus*, in part a *Cuneus*.
II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.
III. Of the *Angle of Contact*; with other things appertaining to the *Composition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Composition of Motions*, with the Results thereof.
IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

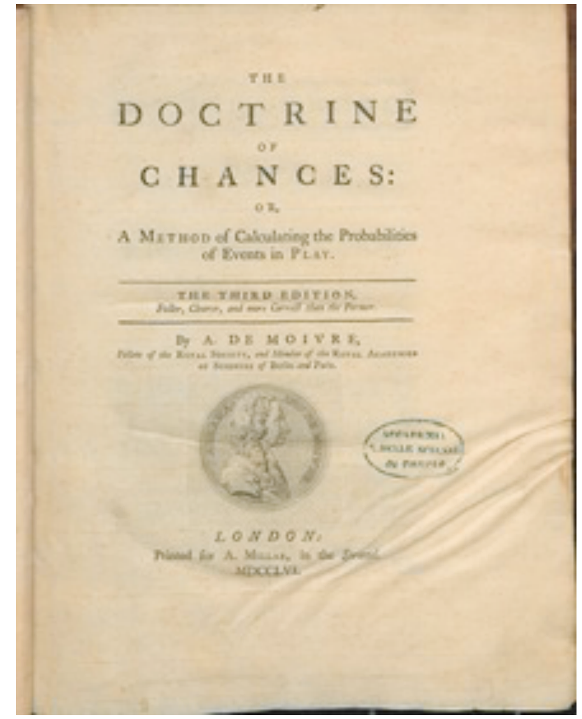
By JOHN WALLIS, *D. D. Professor of* Geometry *in the University of* Oxford; *and a Member of the* Royal Society, London.

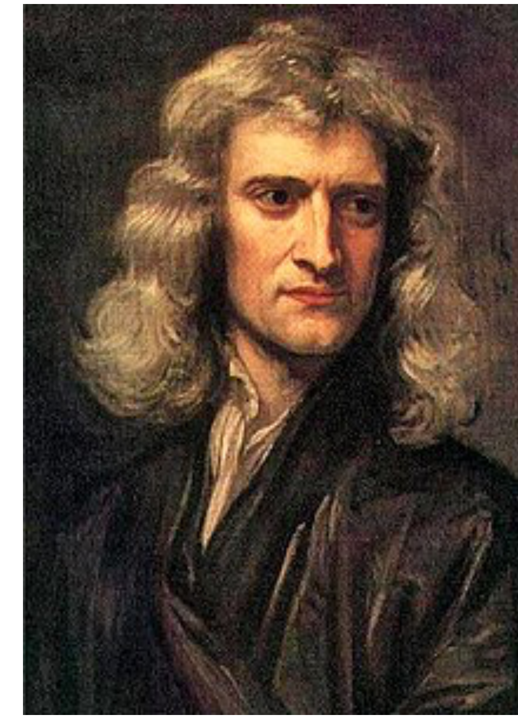
LONDON:
Printed by *John Playford*, for *Richard Davis*, Bookseller, in the University of OXFORD, M. DC. LXXXV.

John Wallis (1616-1703) notes in his *Algebra* that negative numbers, so long viewed with suspicion by mathematicians, had a perfectly good physical explanation, based on a line with a zero mark, and positive numbers being numbers at a distance from the zero point to the right, where negative numbers are a distance to the left of zero. Also, he made some progress at giving a geometric interpretation to $\sqrt{-1}$.

https://math.mit.edu/~dunkel/Teach/18.S996_2022S/history/Merino_ComplexNumbers_History.pdf




THE
DOCTRINE
OF
CHANCES:
OR,
A METHOD of Calculating the Probabilities
of Events in PLAY.
THE THIRD EDITION.
By A. DE MOIVRE,
LONDON:
MDCCLVI.

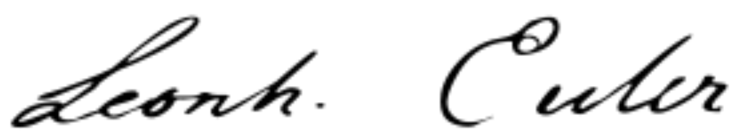


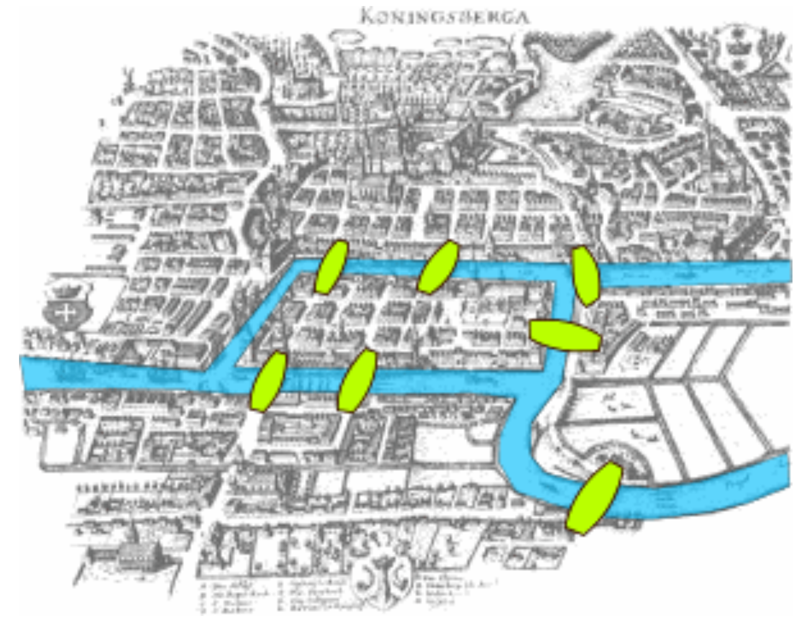
Abraham de Moivre (1667-1754) left France to seek religious refuge in London at eighteen years of age. There he befriended Newton. In 1698 he mentions that Newton knew, as early as 1676 of an equivalent expression to what is today known as de Moivre's theorem:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

where n is an integer. Apparently Newton used this formula to compute the cubic roots that appear in Cardan formulas, in the irreducible case. de Moivre knew and used the formula that bears his name, as it is clear from his writings -although he did not write it out explicitly.

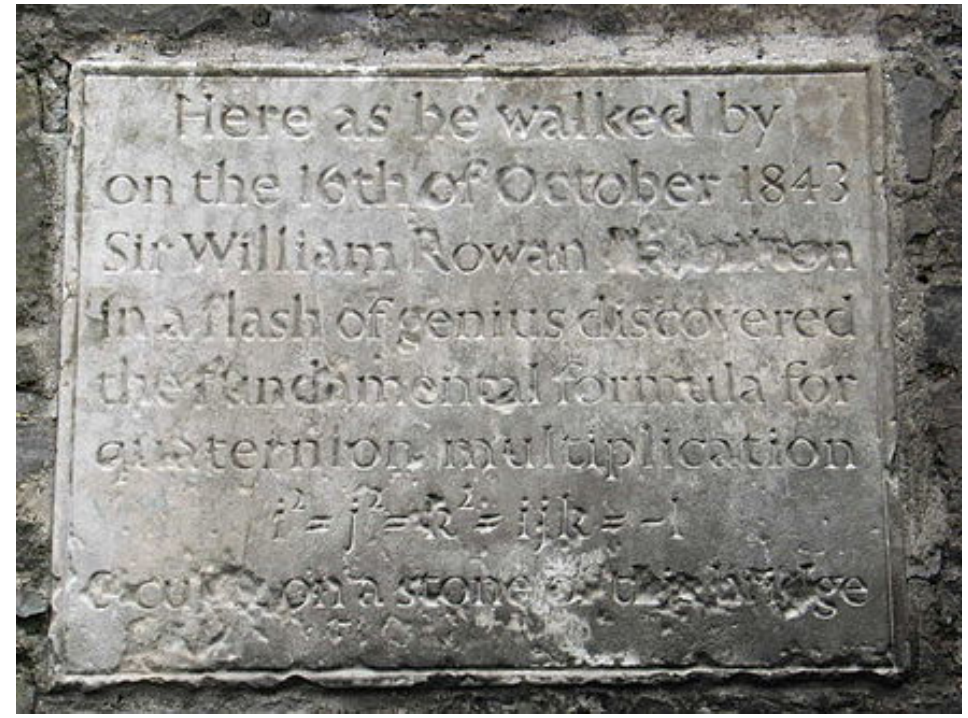
https://math.mit.edu/~dunkel/Teach/18.S996_2022S/history/Merino_ComplexNumbers_History.pdf





L. Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ [3], and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. Euler used the formula $x + iy = r(\cos\theta + i\,\sin\theta)$, and visualized the roots of $z^n = 1$ as vertices of a regular polygon. He defined the complex exponential, and proved the identity $e^{i\theta} = \cos\theta + i\,\sin\theta$.

https://math.mit.edu/~dunkel/Teach/18.S996_2022S/history/Merino_ComplexNumbers_History.pdf



Quaternion Plaque on Broom Bridge

William Rowan Hamilton (1805-65) in an 1831 memoir defined *ordered pairs* of real numbers (a, b) to be a *couple*. He defined addition and multiplication of couples: $(a, b)+(c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, bc + ad)$. This is in fact an algebraic definition of complex numbers.

1843: quaternions $\qquad a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$

Carl Friedrich Gauss (1777-1855). There are indications that Gauss had been in possession of the geometric representation of complex numbers since 1796, but it went unpublished until 1831, when he submitted his ideas to the Royal Society of Gottingen. Gauss introduced the term *complex number*

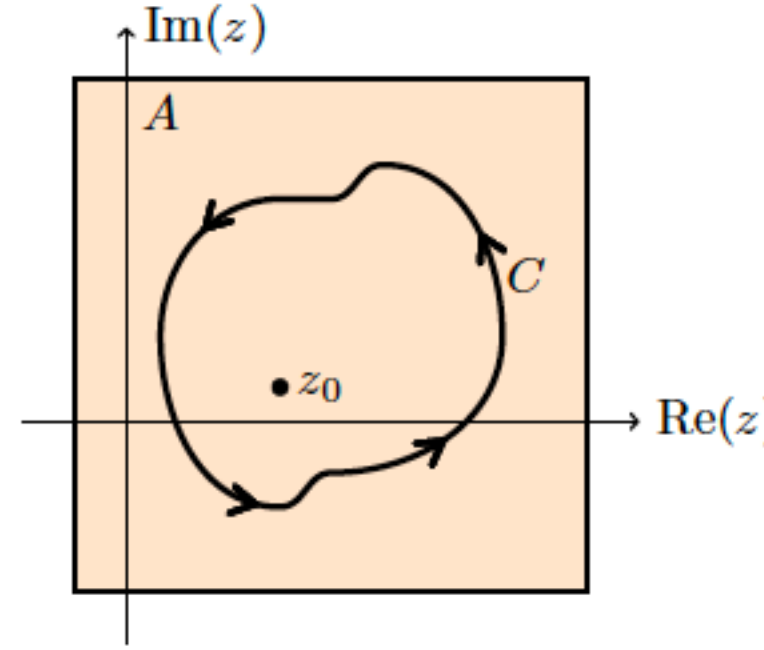
> "If this subjet has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had +1, -1 and $\sqrt{-1}$, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say,of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity."

In a 1811 letter to Bessel, Gauss mentions the theorem that was to be known later as Cauchy's theorem. This went unpublished, and was later rediscovered by Cauchy and by Weierstrass.

https://math.mit.edu/~dunkel/Teach/18.04_2019S/historic/Merino_2006.pdf

Cauchy

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$



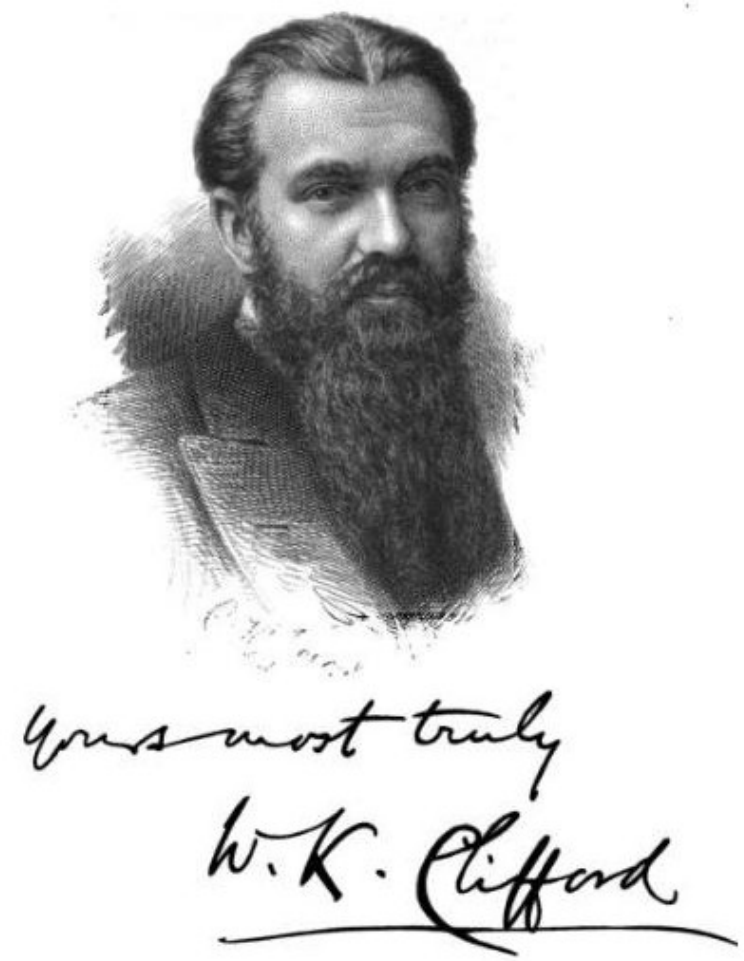
"We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it."

Augustin-Louis Cauchy (1789-1857) initiated complex function theory in an 1814 memoir submitted to the French Académie des Sciences. The term analytic function was not mentioned in his memoir, but the concept is there. The memoir was published in 1825. Contour integrals appear in the memoir, but this is not a first, apparently Poisson had a 1820 paper with a path not on the real line.

Poisson

https://math.mit.edu/~dunkel/Teach/18.04_2019S/historic/Merino_2006.pdf

Hermann Günther Grassmann
1809-1877



Exterior products

"Geometric algebra" (1878)
Special case of
Clifford algebras

$\langle e_i, e_j \rangle = 0$ for $i \neq j$, and $\langle e_i, e_i \rangle = Q(e_i)$.

The fundamental Clifford identity implies that for an orthogonal basis

$e_i e_j = -e_j e_i$ for $i \neq j$, and $e_i^2 = Q(e_i)$.

MARCEL RIESZ

Marcel Riesz
1886 - 1969

David Hestenes
(ASU)

Anthony Lasenby
(Cambridge)

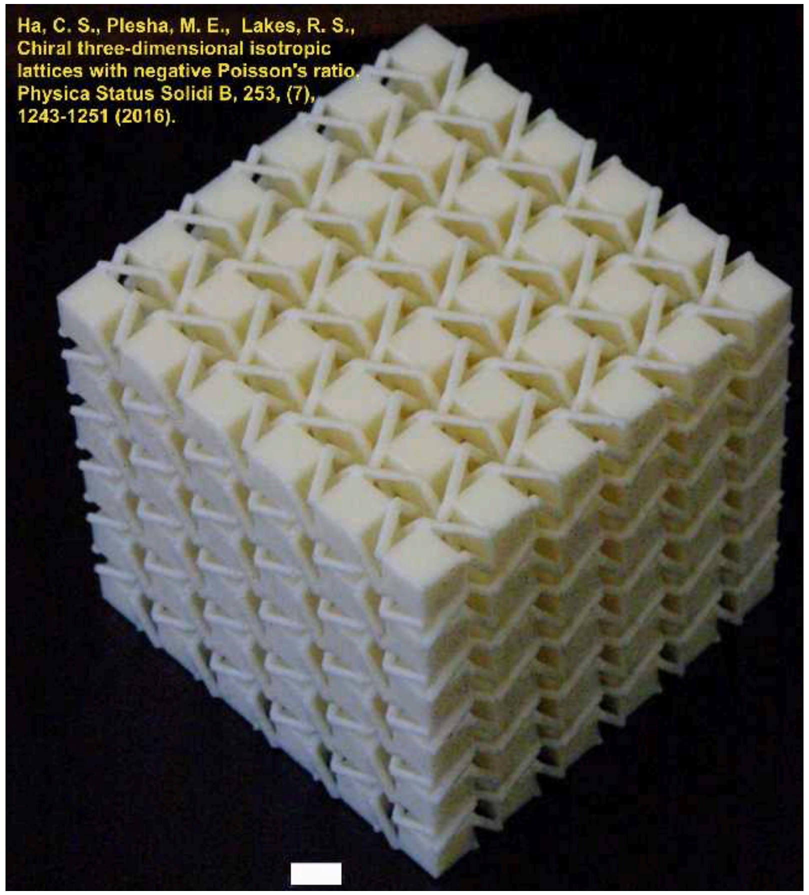
Clifford numbers & spinors

GA & physics

Goal: GA & Cosserat Elasticity

Francois Cosserat
1852 - 1915

Eugene Cosserat
1866 - 1931



Micropolar Metamaterials

Théorie des corps déformables (Theory of deformable bodies) (1909)