Escape Problems & Stochastic Resonance

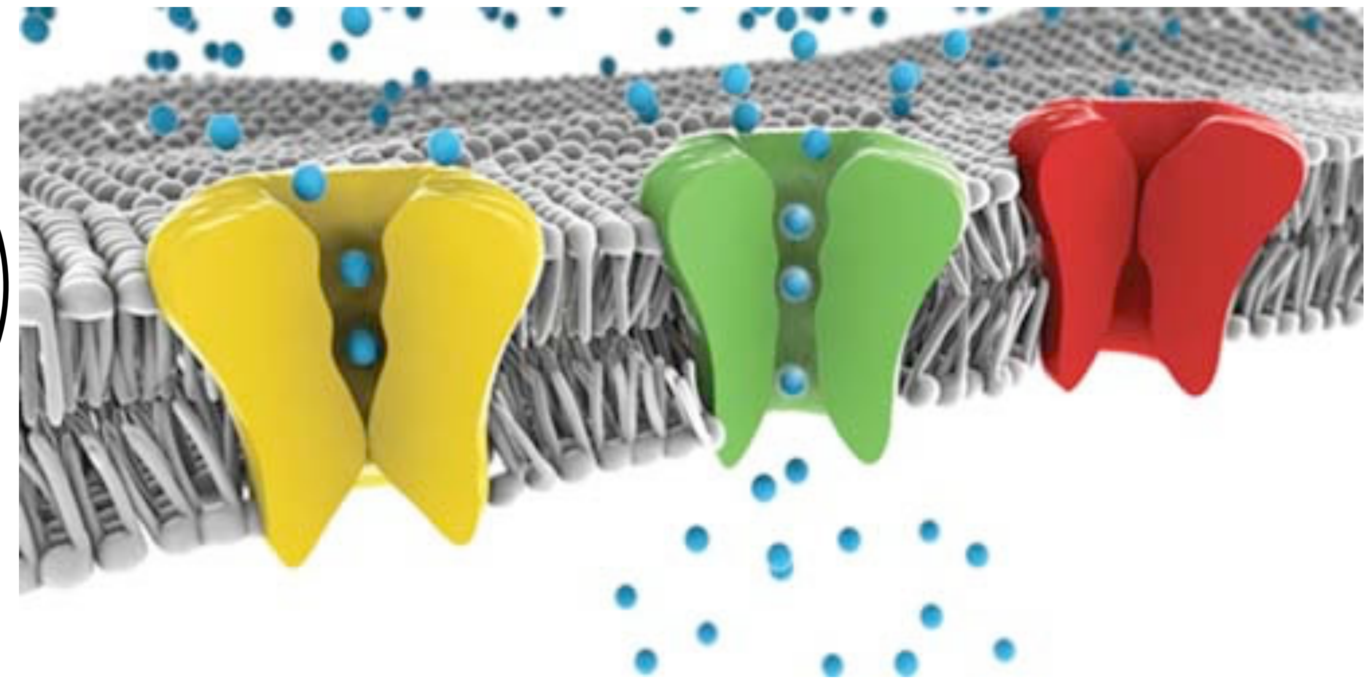
18.S995 - L08 & 09

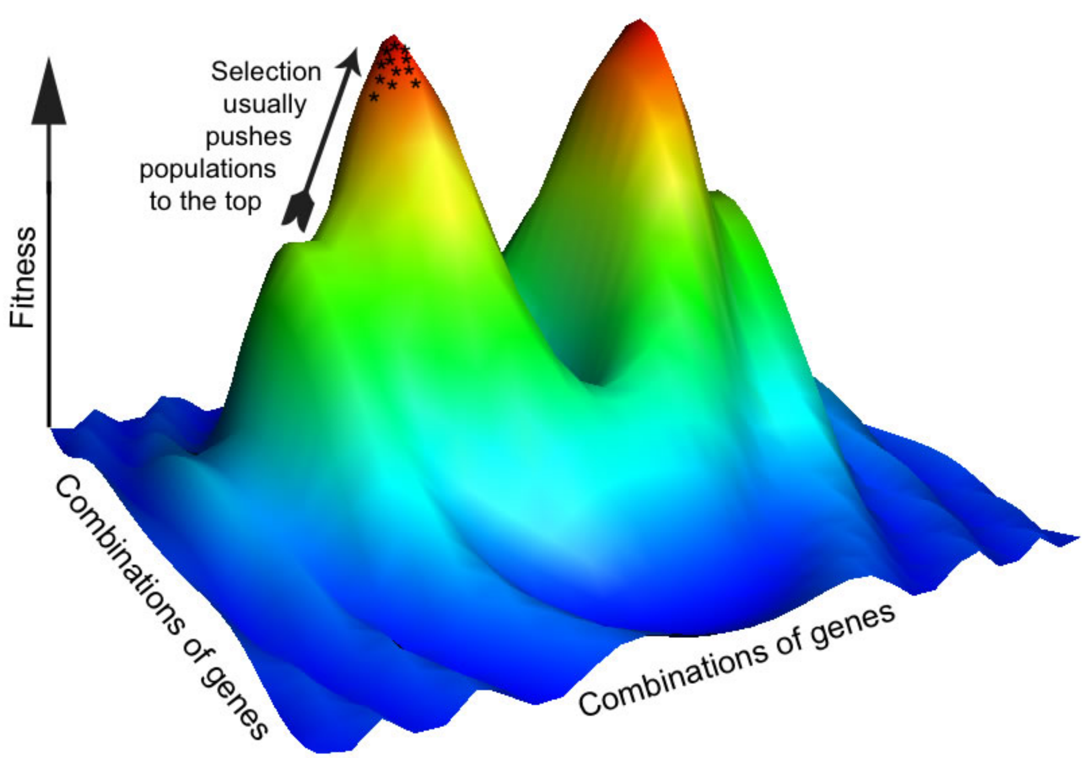
dunkel@mit.edu

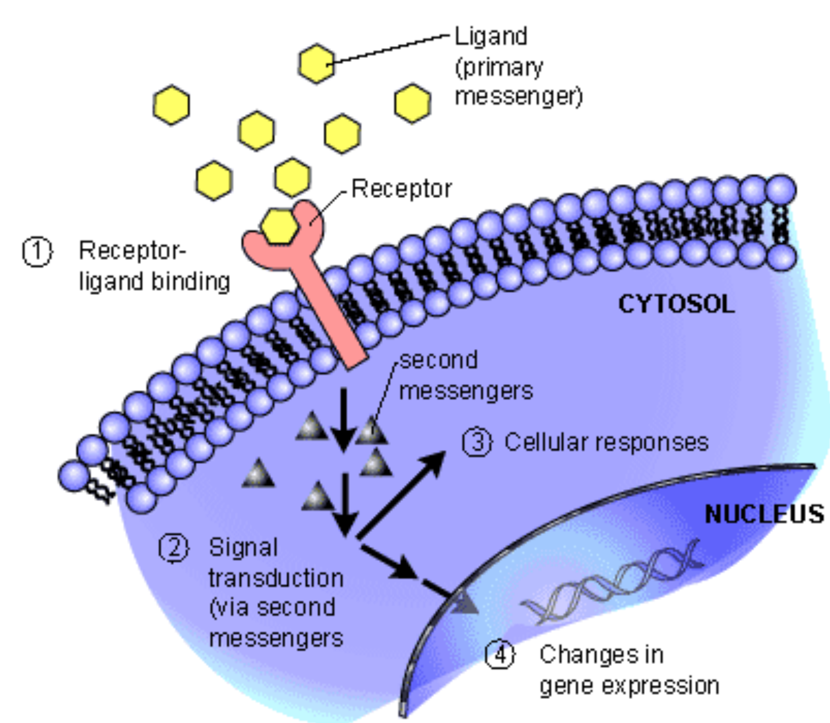
Examples

transport & evolution:
stochastic
escape problems

Illustration by J.P. Cartailler. Copyright 2007, Symmation LLC.







http://evolutionarysystemsbiology.org/intro/

dunkel@math.mit.edu

Reviews of Modern Physics, Vol. 62, No. 2, April 1990

Reaction-rate theory: fifty years after Kramers

Peter Hänggi

Lehrstuhl für Theoretische Physik, University of Augsburg, D-8900 Augsburg, Federal Republic of Germany

Peter Talkner*

Department of Physics, University of Basel, CH-4056 Basel, Switzerland

Michal Borkovec

Institut für Lebensmittelwissenschaft, ETH-Zentrum, CH-8092 Zürich, Switzerland

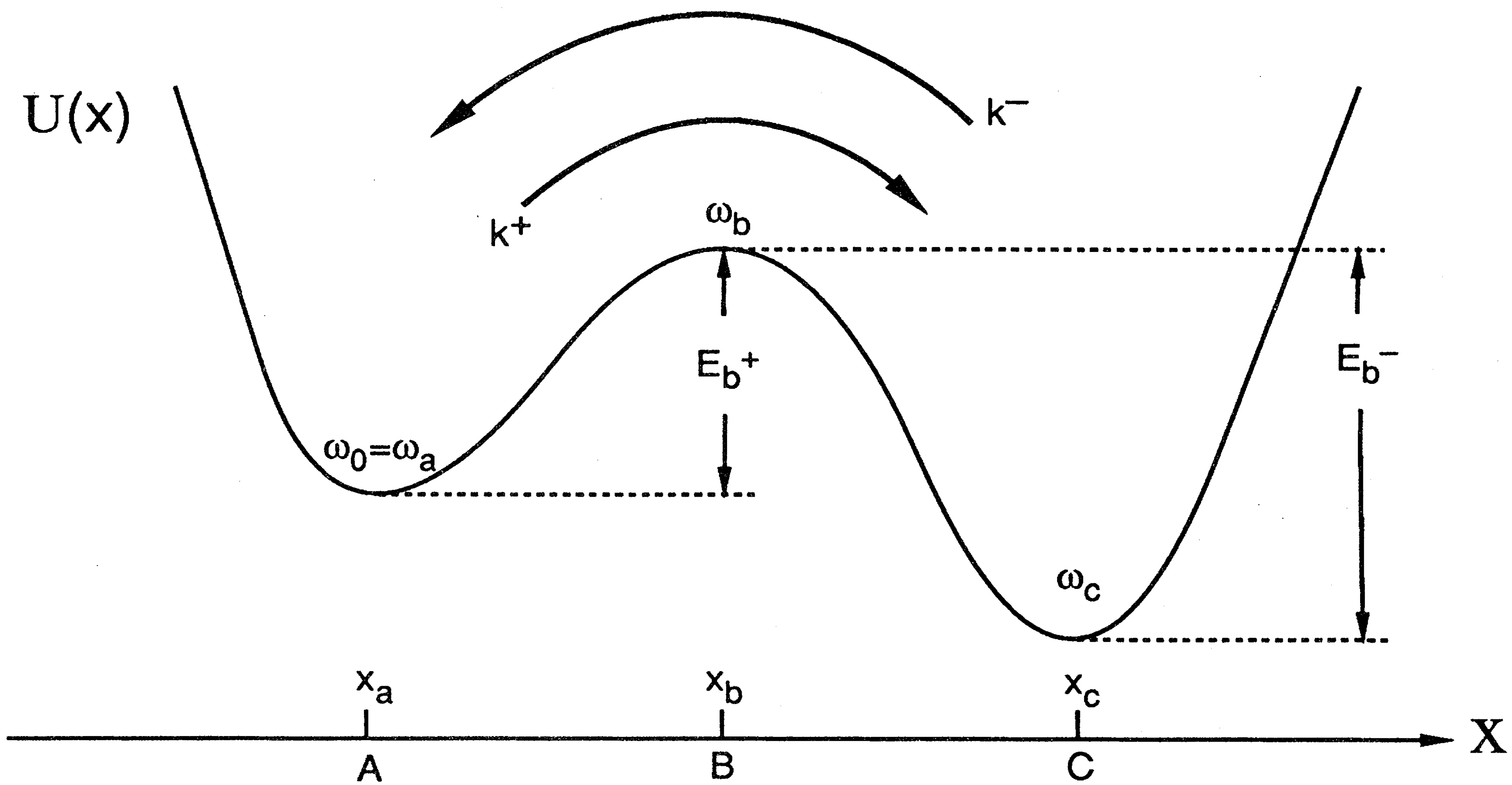


FIG. 3. Potential $U(x)$ with two metastable states A and C. Escape occurs via the forward rate k^+ and the backward rate k^-, respectively, and $E_b^{\pm}$ are the corresponding activation energies.

Arrhenius law

$$k = \nu \exp(-\beta E_b)$$

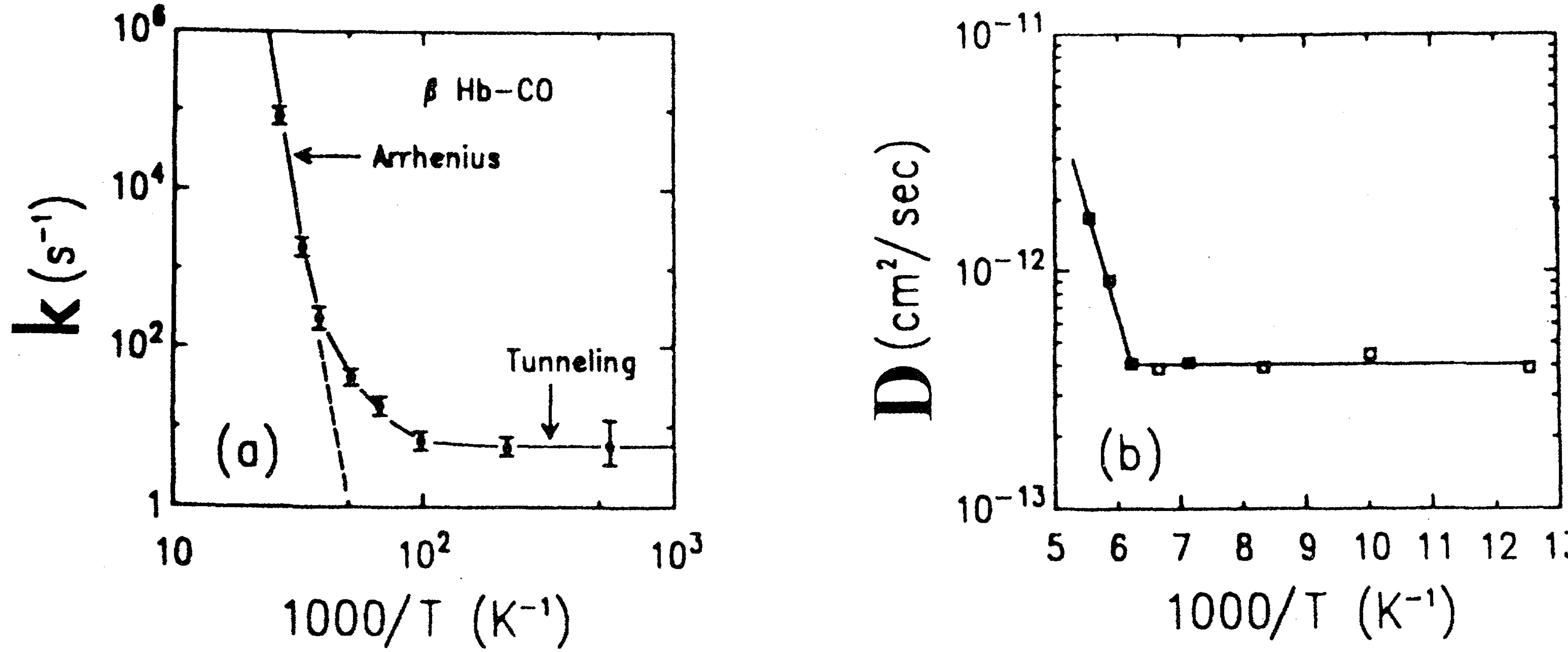


FIG. 2. Van't Hoff-Arrhenius plots of reaction-rate data for two different physical systems in which both thermal activation and tunneling events occur: (a) Rate of CO migration to a separated β chain of hemoglobin (Alberding *et al.*, 1976; Frauenfelder, 1979); (b) diffusion coefficient D of atomic hydrogen moving on the (110) plane of tungsten at a relative H-coverage of 0.1 (data taken from DiFoggio and Gomer, 1982). The diffusion D is directly proportional to the hopping rate k.

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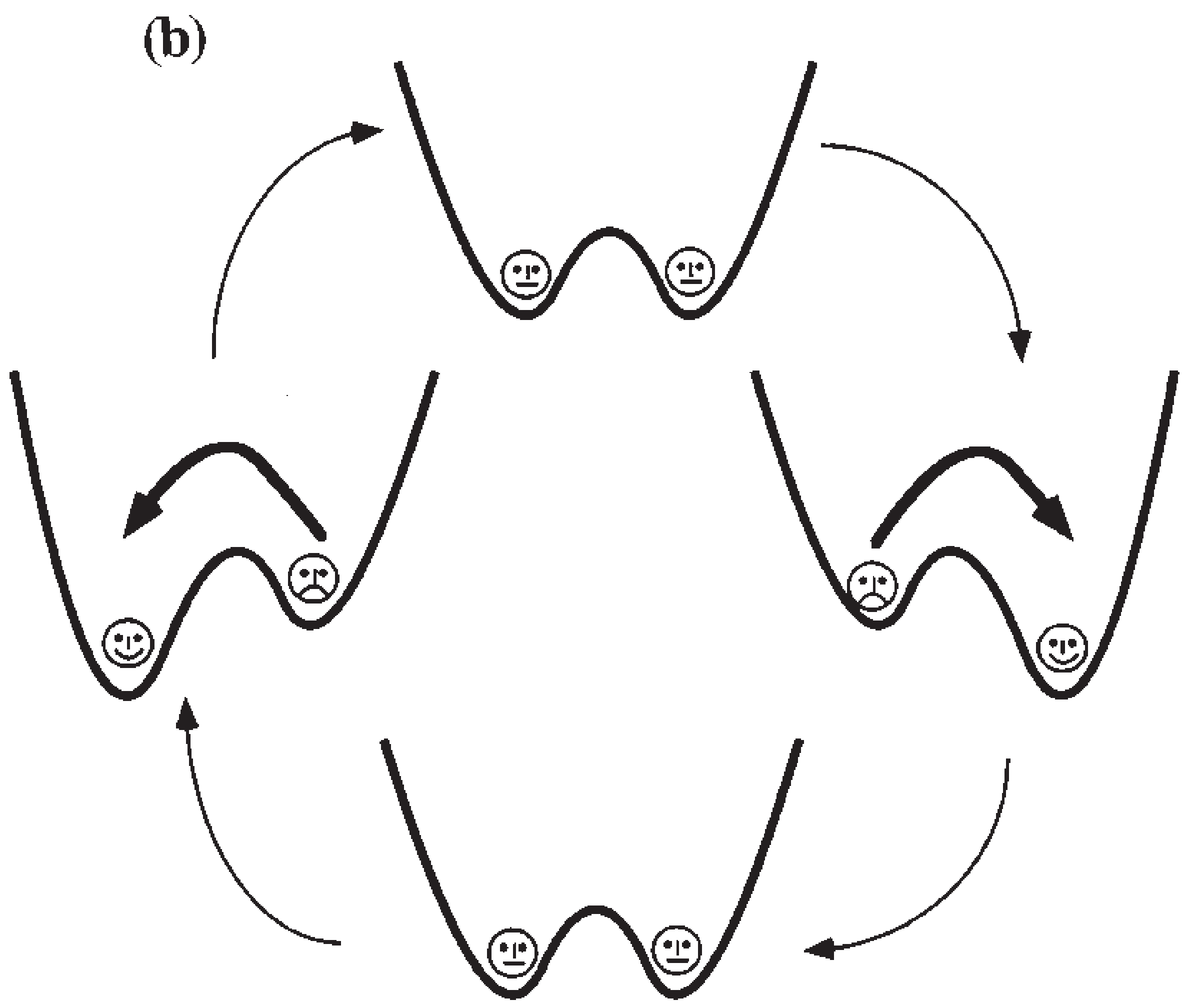
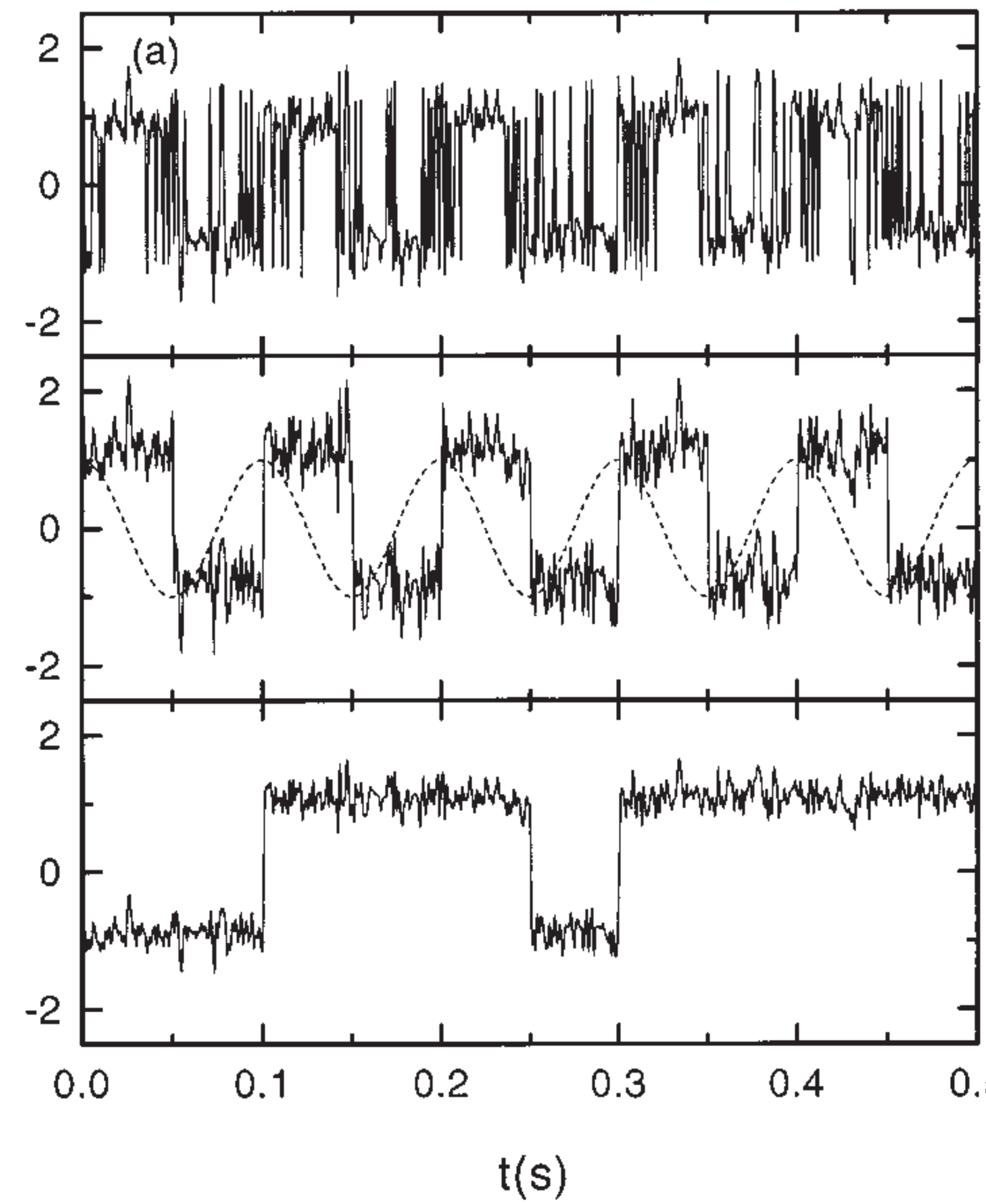
Stochastic resonance



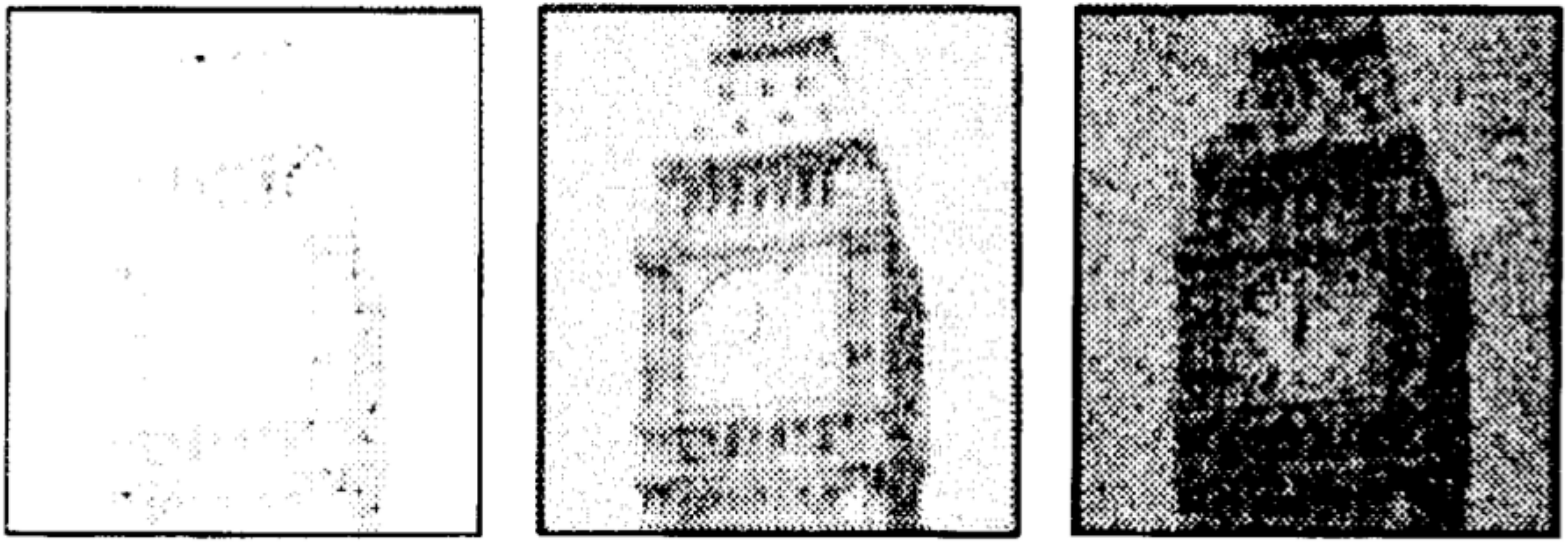


dunkel@math.mit.edu

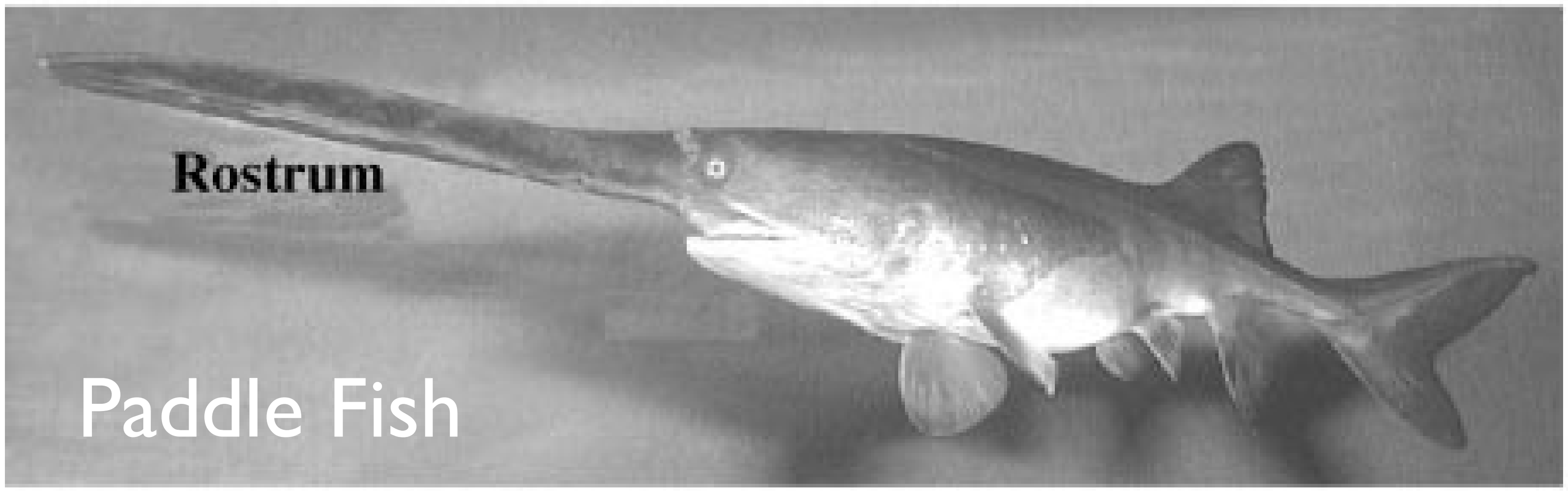
Using fitness landscapes to visualize evolution in action

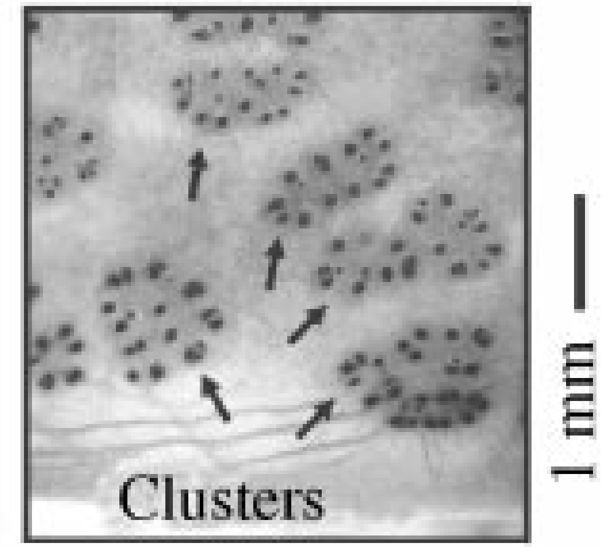
A film by Randy Olson and Bjørn Østman

https://www.youtube.com/watch?v=4pdiAneMMhU

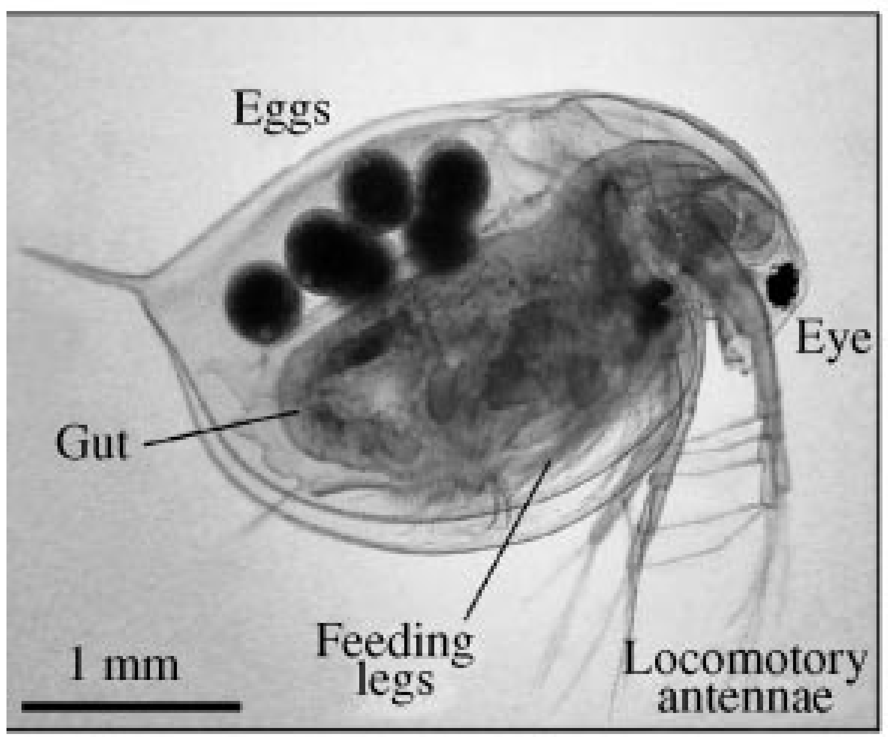


Freund et al (2002) J Theor Biol



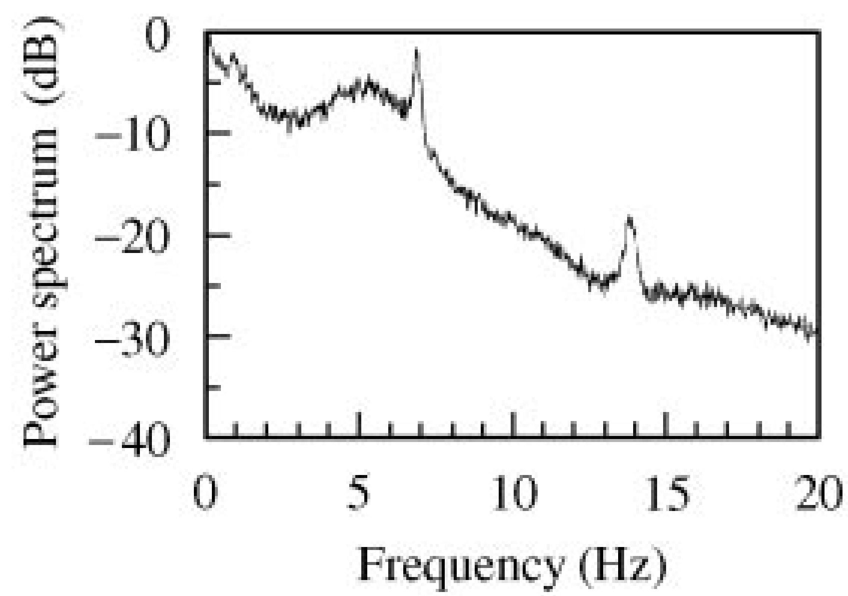


(a)



(b)

(c)



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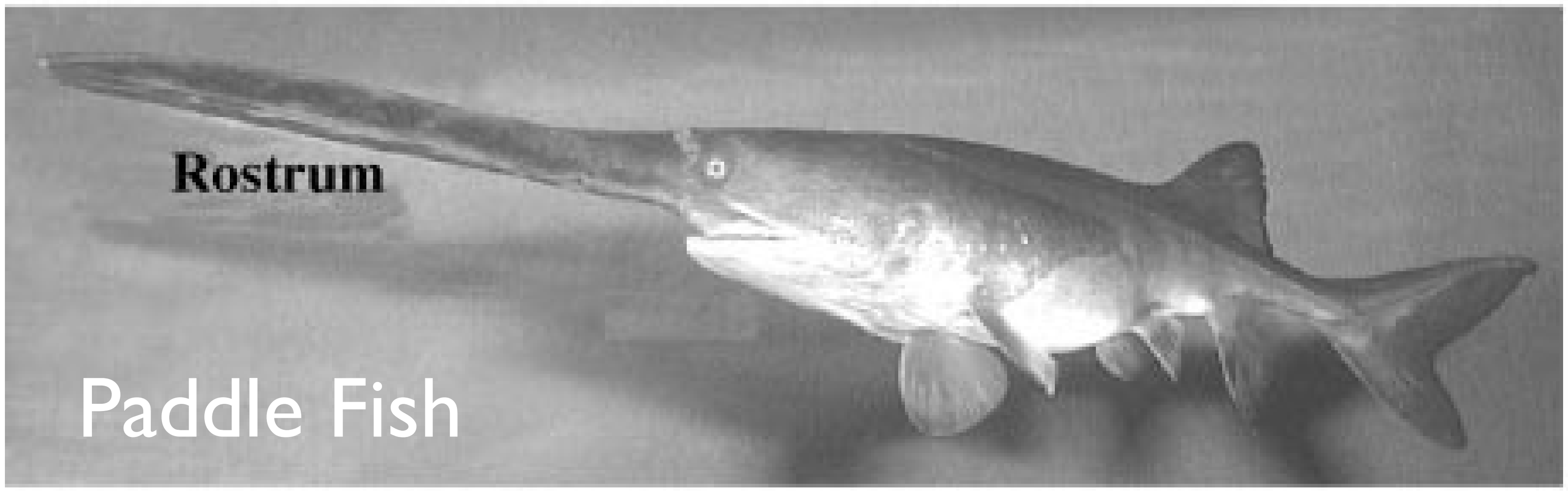
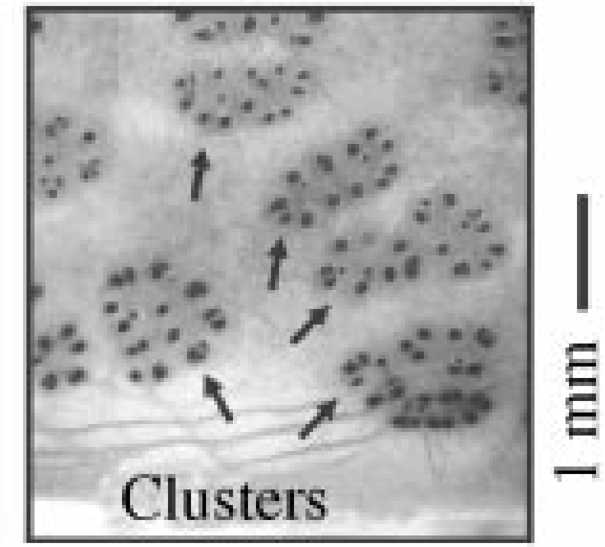
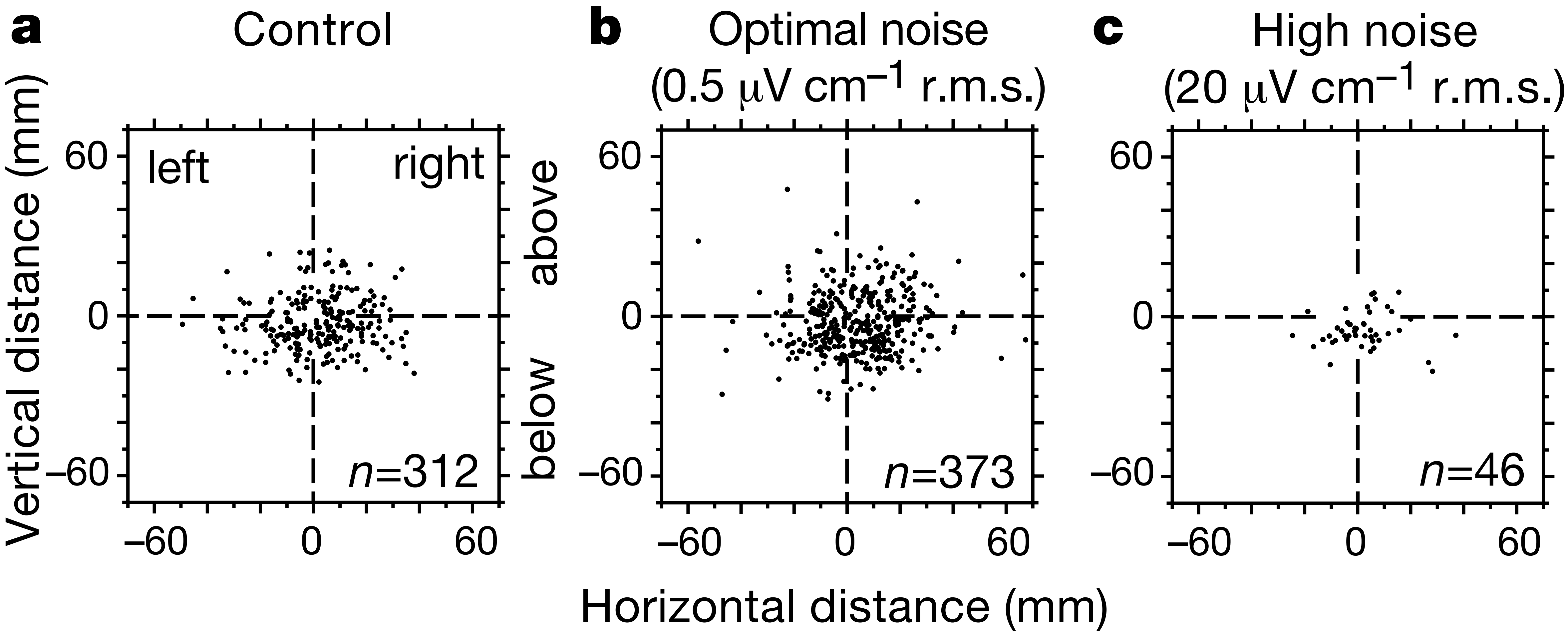
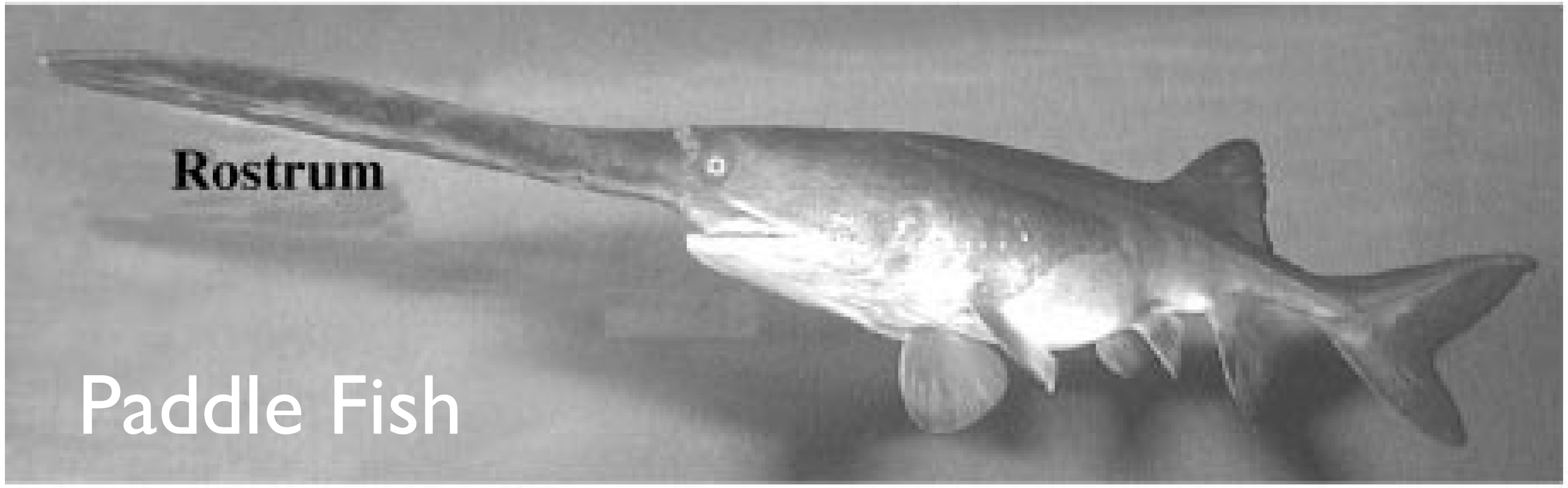


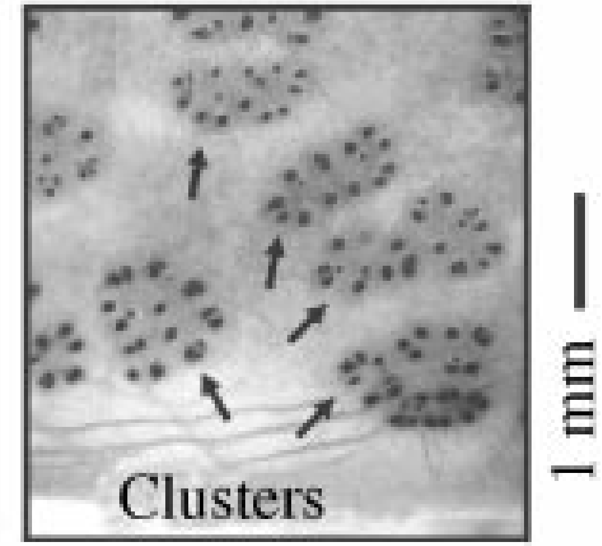




Russel et al (1999) Nature

Freund et al (2002) J Theor Biol





sensory system. However, stochastic resonance requires an external source of electrical noise in order to function. A swarm of plankton, for example *Daphnia*, can provide the required noise. We hypothesize that juvenile paddlefish can detect and attack single *Daphnia* as outliers in the vicinity of the swarm by using noise from the swarm itself. From the power spectral density of the noise plus the weak signal from a single *Daphnia*, we calculate the signal-to-noise ratio, Fisher information and discriminability at the surface of the paddlefish's rostrum. The results predict a specific attack pattern for the paddlefish that appears to be experimentally testable.

1.4.1 Generic minimal model

Consider the over-damped SDE

$$dx(t) = -\partial_x U dt + \sqrt{2D} * dB(t) \tag{1.68a}$$

with a confining potential $U(x)$

$$\lim_{x \to \pm\infty} U(x) \to \infty \tag{1.68b}$$

that has two (or more) minima and maxima. A typical example is the bistable quartile double-well

$$U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 , \qquad a, b > 0 \tag{1.68c}$$

with minima at $\pm\sqrt{a/b}$.

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with minima at $\pm\sqrt{a/b}$.

Generally, we are interested in characterizing the transitions between neighboring minima in terms of a rate k (units of time^{-1}) or, equivalently, by the typical time required for escaping from one of the minima. To this end, we shall first dicuss the general structure of the time-dependent solution of the FPE[14] for the corresponding PDF $p(t, x)$, which reads

$$\partial_t p = -\partial_x j \ , \qquad j(t, x) = -[(\partial_x U)p + D\partial_x p], \tag{1.68d}$$

and has the stationary zero-current ($j \equiv 0$) solution

$$p_s(x) = \frac{e^{-U(x)/D}}{Z} \ , \qquad Z = \int_{-\infty}^{+\infty} dx \ e^{-U(x)/D}. \tag{1.69}$$

General time-dependent solution

$$\partial_t p = -\partial_x j \ , \qquad j(t,x) = -[(\partial_x U)p + D\partial_x p], \tag{1.68d}$$

To find the time-dependent solution, we can make the ansatz

$$p(t,x) = \varrho(t,x)\, e^{-U(x)/(2D)}, \tag{1.70}$$

which leads to a Schrödinger equation in imaginary time

$$-\partial_t \varrho = \left[-D\partial_x^2 + W(x)\right] \varrho =: \mathcal{H}\varrho, \tag{1.71a}$$

with an effective potential

$$W(x) = \frac{1}{4D}(\partial_x U)^2 - \frac{1}{2}\partial_x^2 U. \tag{1.71b}$$

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Assuming the Hamilton operator $\mathcal{H}$ has a discrete non-degenerate spectrum, $\lambda_0 < \lambda_1 < \ldots$, the general solution $p(t,x)$ may be written as

$$p(t,x) = e^{-U(x)/(2D)} \sum_{n=0}^{\infty} c_n\, \phi_n(x)\, e^{-\lambda_n t}, \tag{1.72a}$$

where the eigenfunctions ϕ_n of $\mathcal{H}$ satisfy

$$\int dx\ \phi_n^*(x)\, \phi_m(x) = \delta_{nm}, \tag{1.72b}$$

and the constants c_n are determined by the initial conditions

$$c_n = \int dx\ \phi_n^*(x)\, e^{U(x)/(2D)}\ p(0,x). \tag{1.72c}$$

General time-dependent solution

$$p(t, x) = e^{-U(x)/(2D)} \sum_{n=0}^{\infty} c_n \, \phi_n(x) \, e^{-\lambda_n t}$$

At large times, $t \to \infty$, the solution (1.72a) must approach the stationary solution (1.69), implying that

$$\lambda_0 = 0 \, , \qquad c_0 = \frac{1}{\sqrt{Z}} \, , \qquad \phi_0(x) = \frac{e^{-U(x)/(2D)}}{\sqrt{Z}} . \tag{1.73}$$

Note that $\lambda_0 = 0$ in particular means that the first non-zero eigenvalue $\lambda_1 > 0$ dominates the relaxation dynamics at large times and, therefore,

$$\tau_* = 1/\lambda_1 \tag{1.74}$$

is a natural measure of the escape time. In practice, the eigenvalue λ_1 can be computed by various standard methods (WKB approximation, Ritz method, techniques exploiting supersymmetry, etc.) depending on the specifics of the effective potential W.

1.4.2 Two-state approximation

We next illustrate a commonly used simplified description of escape problems, which can be related to (1.74). As a specific example, we can again consider the escape of a particle from the left well of a symmetric quartic double well-potential

$$U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \,, \qquad p(0,x) = \delta(x - x_-) \tag{1.75a}$$

where

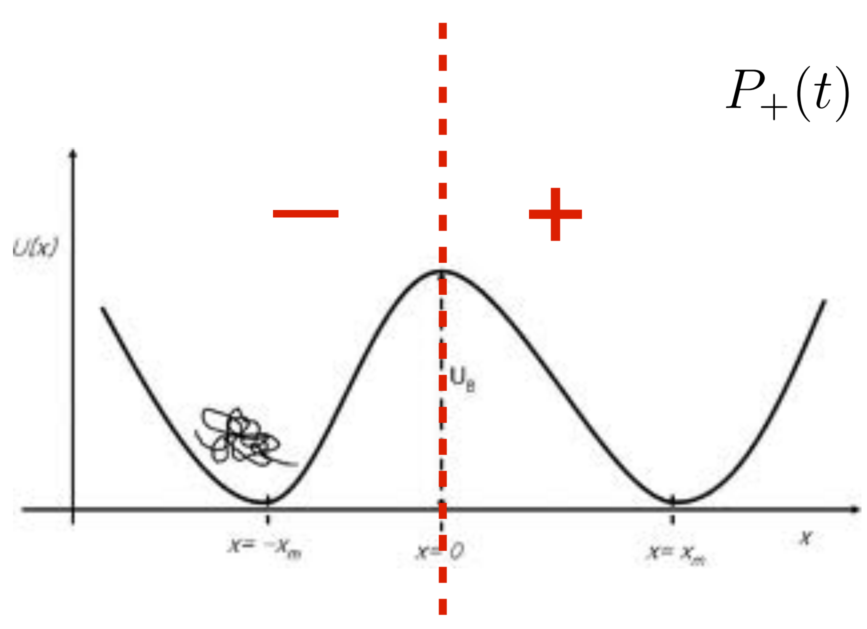
$$x_- = -\sqrt{a/b} \tag{1.75b}$$

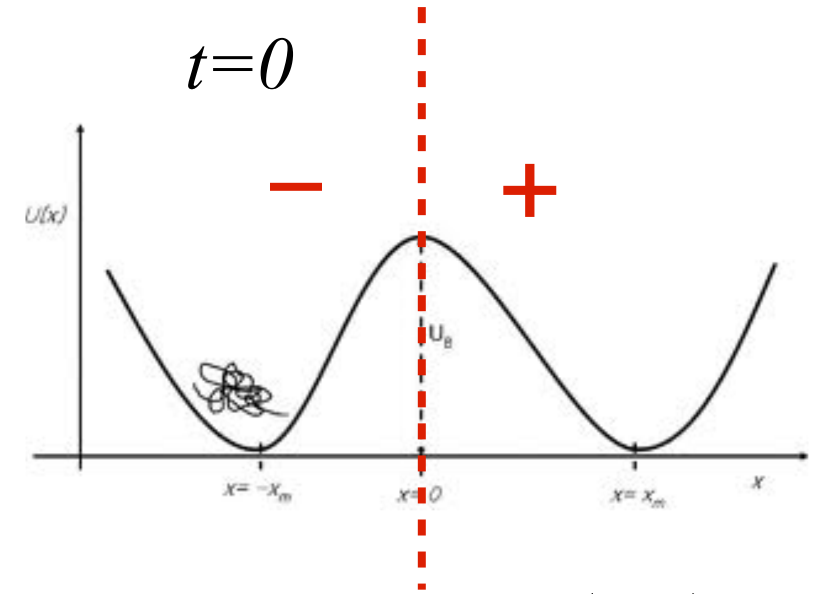
is the location of the left minimum, but the general approach is applicable to other types of potentials as well.

The basic idea of the two-state approximation is to project the full FPE dynamics onto simpler set of master equations by considering the probabilities $P_\pm(t)$ of the coarse-grained particle-states 'left well' $(-)$ and 'right well' $(+)$, defined by

$$P_-(t) \;=\; \int_{-\infty}^{0} dx\, p(t,x), \tag{1.76a}$$

$$P_+(t) \;=\; \int_{0}^{\infty} dx\, p(t,x). \tag{1.76b}$$





If all particles start in the left well, then

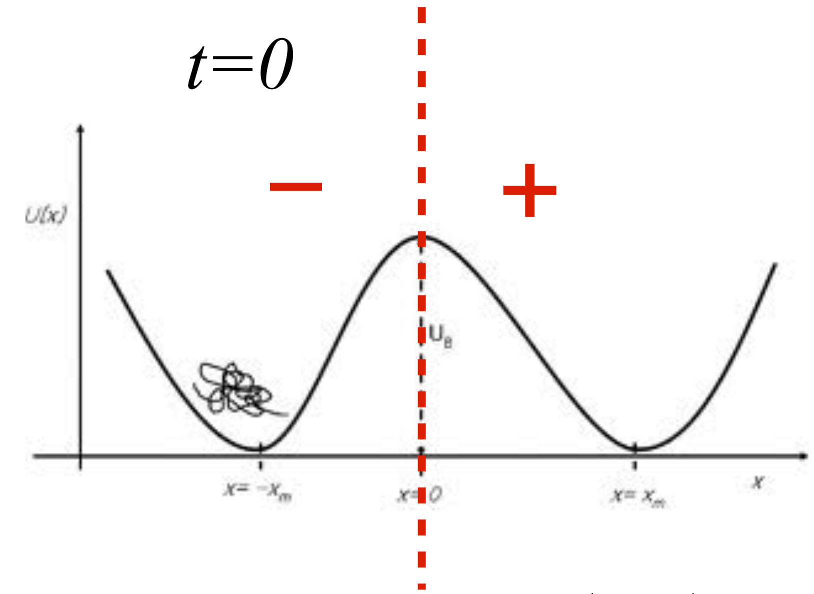
$$P_{-}(0) = 1 \; , \qquad P_{+}(0) = 0. \tag{1.77}$$

Whilst the exact dynamics of $P_{\pm}(t)$ is governed by the FPE (1.68d), the two-state approximation assumes that this dynamics can be approximated by the set of master equations[15]

$$\dot{P}_{-} = -k_{+} \, P_{-} + k_{-} \, P_{+} \; , \qquad \dot{P}_{+} = k_{+} \, P_{-} - k_{-} \, P_{+}. \tag{1.78}$$

For a symmetric potential, $U(x) = U(-x)$, forward and backward rates are equal, $k_{+} = k_{-} = k$, and in this case, the solution of Eq. (1.78) is given by

$$P_{\pm}(t) = \frac{1}{2} \mp \frac{1}{2} \, e^{-2k\,t}. \tag{1.79}$$



If all particles start in the left well, then

$$P_-(0) = 1\ , \qquad P_+(0) = 0. \tag{1.77}$$

Whilst the exact dynamics of $P_\pm(t)$ is governed by the FPE (1.68d), the two-state approximation assumes that this dynamics can be approximated by the set of master equations[15]

$$\dot{P}_- = -k_+ P_- + k_- P_+\ , \qquad \dot{P}_+ = k_+ P_- - k_- P_+. \tag{1.78}$$

For a symmetric potential, $U(x) = U(-x)$, forward and backward rates are equal, $k_+ = k_- = k$, and in this case, the solution of Eq. (1.78) is given by

$$P_\pm(t) = \frac{1}{2} \mp \frac{1}{2}\, e^{-2k\,t}. \tag{1.79}$$

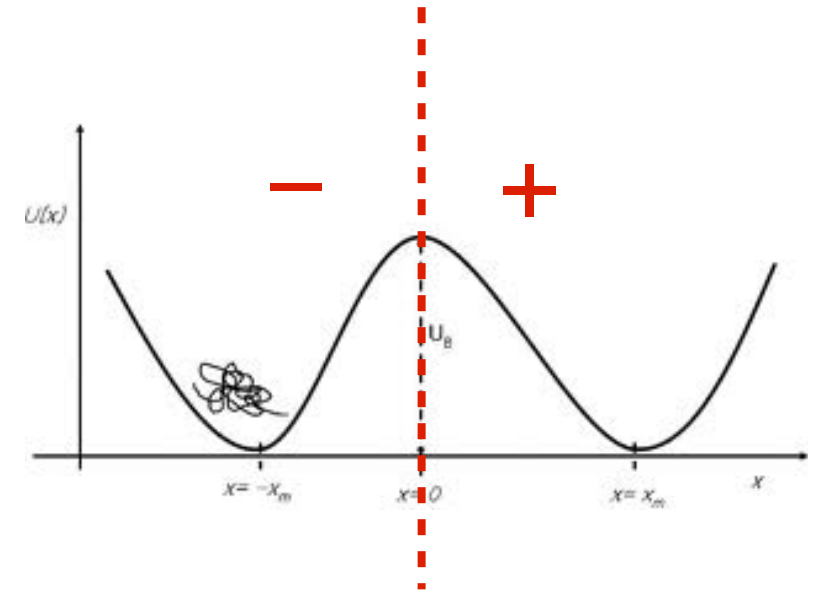
For comparison, from the FPE solution (1.72a), we find in the long-time limit

$$p(t,x) \simeq p_s(x) + c_1 e^{-U(x)/2D} \phi_1(x)\ e^{-\lambda_1 t}, \tag{1.80}$$

Due to the symmetry of $p_s(x)$, we then have

$$P_-(t) \simeq \frac{1}{2} + C_1\, e^{-\lambda_1 t} \tag{1.81a}$$

[15] Note that Eqs. (1.78) conserve the total probability, $P_-(t) + P_-(t) = 1$.



Solution

$$P_-(t) \simeq \frac{1}{2} + C_1\, e^{-\lambda_1 t}$$

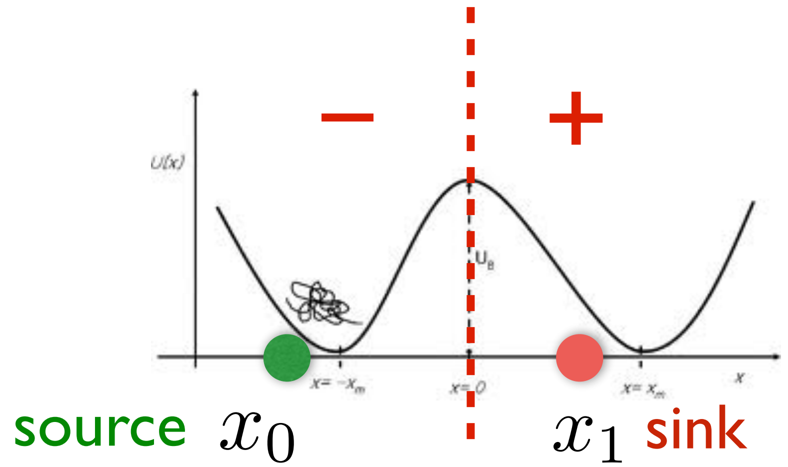
where

$$C_1 = c_1 \int_{-\infty}^{0} e^{-U(x)/2D} \phi_1(x) \;, \qquad c_1 = \phi_1^*(x_-)\, e^{U(x_-)/(2D)}. \tag{1.81b}$$

Since Eq. (1.81a) neglects higher-order eigenfunctions, C_1 is in general not exactly equal but usually close to $1/2$. But, by comparing the time-dependence of (1.81a) and (1.79), it is natural to identify

$$k \simeq \frac{\lambda_1}{2} = \frac{1}{2\tau_*}. \tag{1.82}$$

We next discuss, by considering in a slightly different setting, how one can obtain an explicit result for the rate k in terms of the parameters of the potential U.



1.4.3 Constant-current solution

Consider a bistable potential as in Eq. (1.75), but now with a particle source at $x_0 < x_- < 0$ and a sink[16] at $x_1 > x_b = 0$. Assuming that particles are injected at x_0 at constant flux $j(t,x) \equiv J = const$, the escape rate can be defined by

$$k := \frac{J}{P_-}, \tag{1.83}$$

with P_- denoting the probability of being in the left well, as defined in Eq. (1.76a) above. To compute the rate from Eq. (1.83), we need to find a stationary constant flux solution $p_J(x)$ of Eq. (1.68d), satisfying $p_J(x_1) = 0$ and

$$J = -(\partial_x U) p_J - D \partial_x p_J \tag{1.84}$$

for some constant J. This solution is given by [HTB90]

$$p_J(x) = \frac{J}{D} e^{-U(x)/D} \int_x^{x_1} dy\; e^{U(y)/D}, \tag{1.85}$$

as one can verify by differentiation

$$\begin{aligned} -(\partial_x U) p_J - D \partial_x p_J &= -(\partial_x U) p_J - D \partial_x \left[\frac{J}{D} e^{-U(x)/D} \int_x^{x_1} dy\; e^{U(y)/D} \right] \\ &= -(\partial_x U) p_J - J \left[-\frac{(\partial_x U)}{D} e^{-U(x)/D} \int_x^{x_1} dy\; e^{U(y)/D} - 1 \right] \\ &= J. \end{aligned} \tag{1.86}$$

Therefore, the inverse rate k^{-1} from Eq. (1.83) can be formally expressed as

$$k^{-1} = \frac{P_-}{J} = \frac{1}{D} \int_{-\infty}^{x_1} dx \; e^{-U(x)/D} \int_{x}^{x_1} dy \; e^{U(y)/D}, \tag{1.87}$$

and a partial integration yields the equivalent representation

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Assuming a sufficiently steep barrier, the integrals in Eq. (1.88) may be evaluated by adopting steepest descent approximations near the potential minimum at x_- and near the maximum at the barrier position x_b. More precisely, taking into account that $U'(x_-) = U'(x_b) = 0$, one can replace the potentials in the exponents by the harmonic approximations

$$\begin{aligned} U(x) &\simeq U(x_b) - \frac{1}{2\tau_b}(x - x_b)^2, \\ U(y) &\simeq U(x_-) + \frac{1}{2\tau_-}(y - x_-)^2, \end{aligned}$$

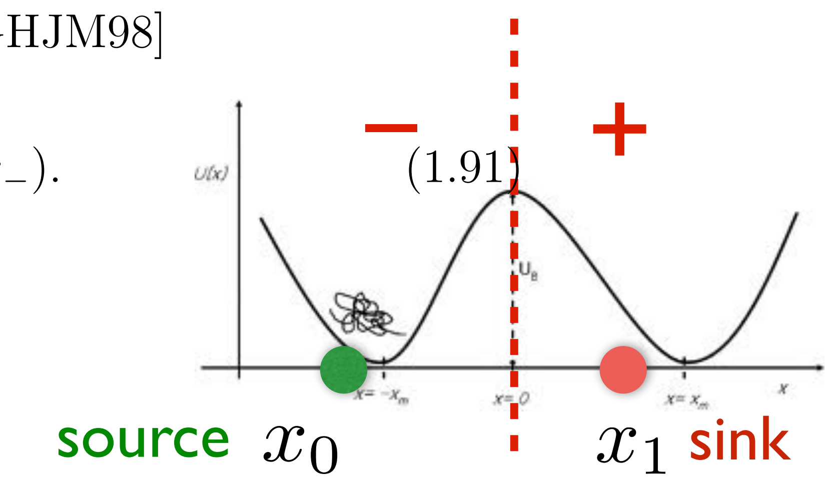
where

$$\tau_- = -U''(x_0) > 0\;, \qquad \tau_b = U''(x_b) > 0 \tag{1.90}$$

carry units of time. Inserting (1.89) into (1.88) and replacing the upper integral boundaries by $+\infty$, one thus obtains the so-called Kramers rate [HTB90, GHJM98]

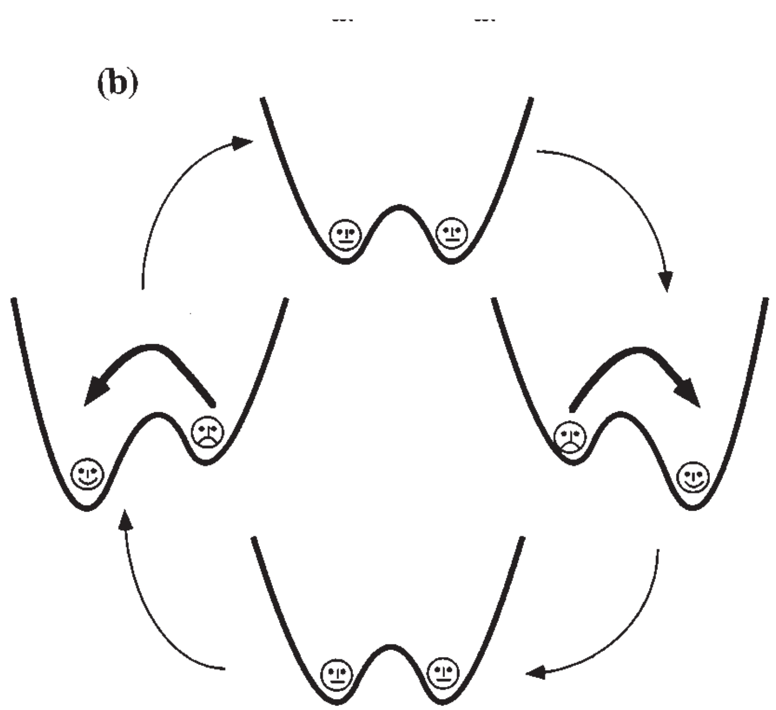
$$\boxed{k \simeq \frac{e^{-\Delta U/D}}{2\pi\sqrt{\tau_- \tau_b}} =: k_{\mathrm{K}}\;,} \qquad \Delta U = U(x_b) - U(x_-). \tag{1.91}$$

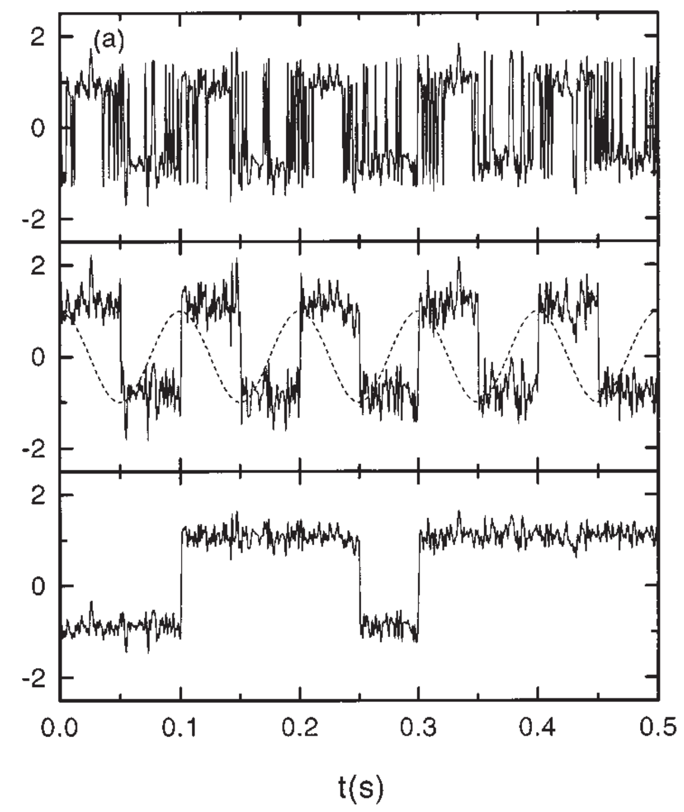
Arrhenius law



Stochastic resonance

1. a nonlinear measurement device[17],
2. a periodic signal weaker than the threshold of measurement device,
3. additional input noise, uncorrelated with the signal of interest.





$$k \simeq \frac{e^{-\Delta U/D}}{2\pi\sqrt{\tau_-\tau_b}} =: k_{\mathrm{K}} \sim \Omega$$

dunkel@math.mit.edu

1.5 Stochastic resonance

1.5.1 Generic model

To illustrate SR more quantitatively, consider the periodically driven SDE

$$dX(t) = -\partial_x U\,dt + A\cos(\Omega t)\,dt + \sqrt{2D} * dB(t), \tag{1.93a}$$

where A is the signal amplitude and

$$U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \tag{1.93b}$$

a symmetric double-well potential with minima at $\pm x_* = \pm\sqrt{a/b}$ and barrier height $\Delta U = a^2/(4b)$. Introducing rescaled variables

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$$x' = x/x_*\,, \qquad t' = at\,, \qquad A' = A/(ax_*)\,, \qquad D' = D/(ax_*^2)\,, \qquad \Omega' = \Omega/a.$$

and dropping primes. we can rewrite (1.93a) in the dimensionless form

$$dX(t) = (x - x^3)\, dt + A\cos(\Omega t)\, dt + \sqrt{2D} * dB(t), \tag{1.93c}$$

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$$dX(t) = (x - x^3)\, dt + A\cos(\Omega t)\, dt + \sqrt{2D} * dB(t), \tag{1.93c}$$

with a rescaled barrier height $\Delta U = 1/4$. The associated FPE reads

$$\partial_t p \;=\; -\partial_x\{[-(\partial_x U) + A\cos(\Omega t)]p - D\partial_x p\}. \tag{1.94}$$

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$$x' = x/x_* \ , \qquad t' = at \ , \qquad A' = A/(ax_*) \ , \qquad D' = D/(ax_*^2) \ , \qquad \Omega' = \Omega/a.$$

and dropping primes. we can rewrite (1.93a) in the dimensionless form

$$dX(t) = (x - x^3)\, dt + A\cos(\Omega t)\, dt + \sqrt{2D} * dB(t), \tag{1.93c}$$

with a rescaled barrier height $\Delta U = 1/4$. The associated FPE reads

$$\partial_t p \ = \ -\partial_x\{[-(\partial_x U) + A\cos(\Omega t)]p - D\partial_x p\}. \tag{1.94}$$

For our subsequent discussion, it is useful to rearrange terms on the rhs. as

$$\partial_t p = \partial_x[(\partial_x U)p + D\partial_x p] - A\cos(\Omega t)\partial_x p. \tag{1.95}$$

Perturbation theory

$$\partial_t p = \partial_x[(\partial_x U)p + D\partial_x p] - A\cos(\Omega t)\partial_x p. \tag{1.95}$$

To solve Eq. (1.95) perturbatively, we insert the series ansatz

$$p(t,x) = \sum_{n=0}^{\infty} A^n p_n(t,x), \tag{1.96}$$

which gives

$$\sum_{n=0}^{\infty} A^n \partial_t p_n = \sum_{n=0}^{\infty} \left\{ A^n \partial_x[(\partial_x U)p_n + D\partial_x p_n] - A^{n+1}\cos(\Omega t)\partial_x p_n \right\} \tag{1.97}$$

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Focussing on the liner response regime, corresponding to powers A^0 and A^1, we find

$$\partial_t p_0 = \partial_x[(\partial_x U)p_0 + D\partial_x p_0] \tag{1.98a}$$

$$\partial_t p_1 = \partial_x[(\partial_x U)p_1 + D\partial_x p_1] - \cos(\Omega t)\partial_x p_0 \tag{1.98b}$$

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$$\begin{aligned}
\partial_t p_0 &= \partial_x[(\partial_x U)p_0 + D\partial_x p_0] && \text{(1.98a)}\\
\partial_t p_1 &= \partial_x[(\partial_x U)p_1 + D\partial_x p_1] - \cos(\Omega t)\partial_x p_0 && \text{(1.98b)}
\end{aligned}$$

Equation (1.98a) is just an ordinary time-independent FPE, and we know its stationary solution is just the Boltzmann distribution

$$p_0(x) = \frac{e^{-U(x)/D}}{Z_0}, \qquad Z_0 = \int dx\, e^{-U(x)/D} \tag{1.99}$$

First-order correction

$$\partial_t p_1 \quad = \quad \partial_x[(\partial_x U)p_1 + D\partial_x p_1] - \cos(\Omega t)\partial_x p_0 \tag{1.98b}$$

To obtain a formal solution to Eq. (1.98b), we make use of the following ansatz

$$p_1(t,x) = e^{-U(x)/(2D)} \sum_{m=1}^{\infty} a_{1m}(t)\, \phi_m(x), \tag{1.100}$$

where $\phi_m(x)$ are the eigenfunctions of the unperturbed effective Hamiltonian, cf. Eq. (1.71),

$$\mathcal{H}_0 = -D\partial_x^2 + \frac{1}{4D}(\partial_x U)^2 - \frac{1}{2}\partial_x^2 U. \tag{1.101}$$

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Inserting (1.100) into Eq. (1.98b) gives

$$\sum_{m=1}^{\infty} \dot{a}_{1m}\phi_m = -\sum_{m=1}^{\infty} \lambda_m a_{1m}\, \phi_m - \cos(\Omega t)\; e^{U(x)/(2D)}\; \partial_x p_0. \tag{1.102}$$

First-order correction

$$\partial_t p_1 \quad = \quad \partial_x[(\partial_x U)p_1 + D\partial_x p_1] - \cos(\Omega t)\partial_x p_0 \qquad (1.98\text{b})$$

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Multiplying this equation by $\phi_n(x)$, and integrating from $-\infty$ to $+\infty$ while exploiting the orthonormality of the system $\{\phi_m\}$, we obtain the coupled ODEs

$$\dot{a}_{1m} = -\lambda_m a_{1m} - M_{m0}\cos(\Omega t), \qquad (1.103)$$

with 'transition matrix' elements

$$M_{m0} = \int dx\;\phi_m\,e^{U(x)/(2D)}\partial_x p_0. \qquad (1.104)$$

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with 'transition matrix' elements

$$M_{m0} = \int dx \; \phi_m \, e^{U(x)/(2D)} \partial_x p_0. \tag{1.104}$$

The asymptotic solution of Eq. (1.103) reads

$$a_{1m}(t) = -M_{m0} \left[\frac{\Omega}{\lambda_m^2 + \Omega^2} \sin(\Omega t) + \frac{\lambda_m}{\lambda_m^2 + \Omega^2} \cos(\Omega t) \right]. \tag{1.105}$$

Note that, because $\partial_x p_0$ is an antisymmetric function, the matrix elements M_{m0} vanish[18] for even values $m = 0, 2, 4, \ldots$, so that only the contributions from odd values $m = 1, 3, 5 \ldots$ are asymptotically relevant.

Linear response

Focussing on the leading order contribution, $m = 1$, and noting that $p_0(x) = p_0(-x)$, we can estimate the position mean value

$$\mathbb{E}[X(t)] = \int dx\, p(t,x)\, x \tag{1.106}$$

from

$$\begin{aligned}
\mathbb{E}[X(t)] &\simeq A \int dx\, p_1(t,x)\, x \\
&\simeq A \int dx\, e^{-U(x)/(2D)}\, a_{11}(t)\, \phi_1(x)\, x \\
&= -AM_{10} \left[\frac{\Omega}{\lambda_1^2 + \Omega^2} \sin(\Omega t) + \frac{\lambda_1}{\lambda_1^2 + \Omega^2} \cos(\Omega t) \right] \int dx\, e^{-U(x)/(2D)}\, \phi_1(x)\, x
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\end{aligned}$$

Using $\lambda_1 = 2k_\mathrm{K}$, where k_K is the Kramers rate from Eq. (1.91), we can rewrite this more compactly as

$$\mathbb{E}[X(t)] = \overline{X} \cos(\Omega t - \overline{\varphi}) \tag{1.107a}$$

with phase shift

$$\overline{\varphi} = \arctan\left(\frac{\Omega}{2k_\mathrm{K}} \right) \tag{1.107b}$$

and amplitude

$$\overline{X} = -A \frac{M_{10}}{(4k_\mathrm{K}^2 + \Omega^2)^{1/2}} \int dx\, e^{-U(x)/(2D)}\, \phi_1(x)\, x. \tag{1.107c}$$

Linear response

$$\overline{X} = -A\frac{M_{10}}{(4k_{\mathrm{K}}^2+\Omega^2)^{1/2}}\int dx\, e^{-U(x)/(2D)}\,\phi_1(x)\,x. \tag{1.107c}$$

The amplitude $\overline{X}$ depends on the noise strength D through k_{K}, through the integral factor and also through the matrix element

$$M_{10} \quad = \quad \int dx\,\phi_1\, e^{U(x)/(2D)}\partial_x p_0. \tag{1.108}$$

To compute $\overline{X}$, one first needs to determine the eigenfunction ϕ_1 of $\mathcal{H}_0$ as defined in Eq. (1.101). For the quartic double-well potential (1.93b), this cannot be done analytically but there exist standard techniques (e.g., Ritz method) for approximating ϕ_1 by functions that are orthogonal to $\phi_0 = \sqrt{p_0/Z_0}$. Depending on the method employed, analytical

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$$\overline{X} \simeq \frac{Aa}{Db} \left(\frac{4k_{\mathrm{K}}^2}{4k_{\mathrm{K}}^2 + \Omega^2} \right)^{1/2}, \tag{1.109}$$

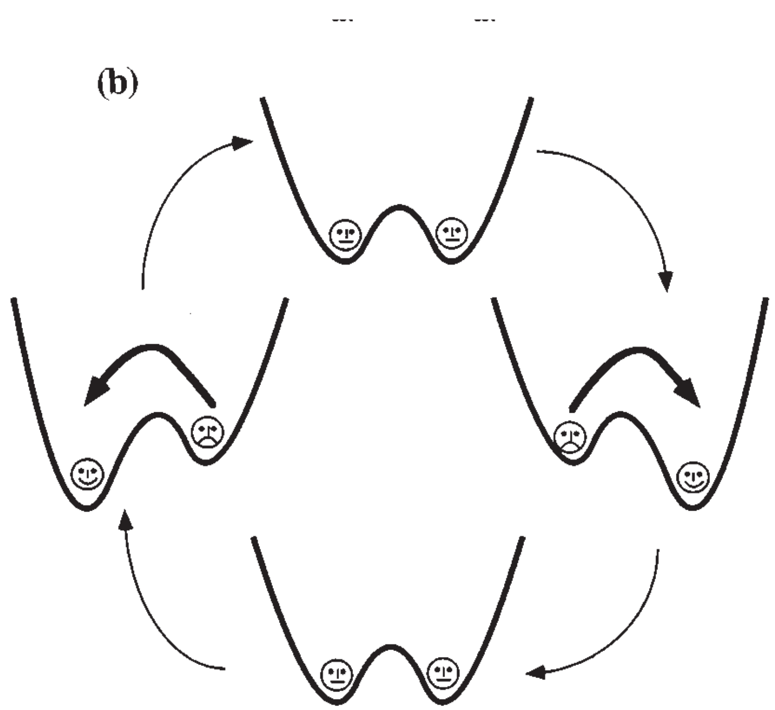
which exhibits a maximum for a critical value D_* determined by

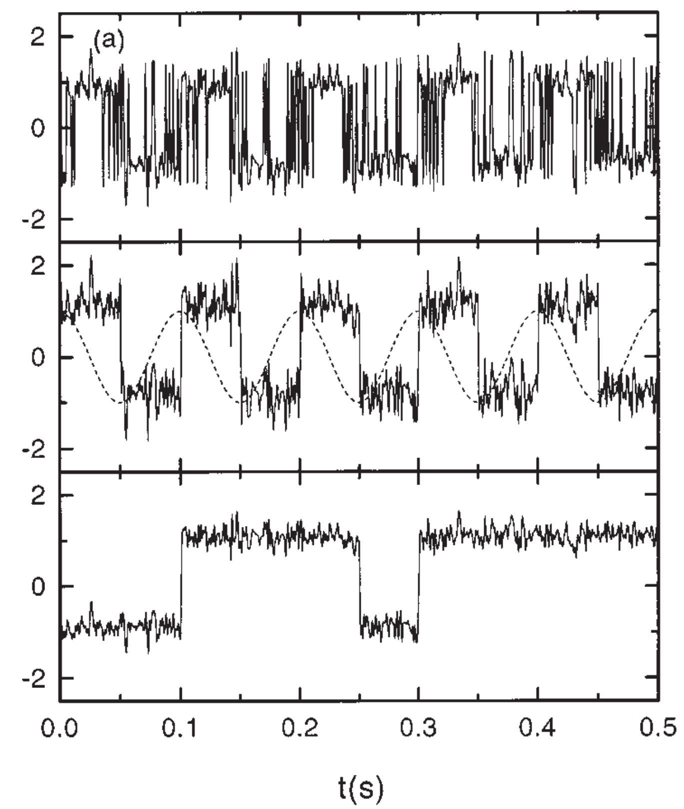
$$4k_{\mathrm{K}}^2 = \Omega^2 \left(\frac{\Delta U}{D_*} - 1 \right). \tag{1.110}$$

That is, the value D_* corresponds to the optimal noise strength, for which the mean value $\mathbb{E}[X(t)]$ shows maximal response to the underlying periodic signal – hence the name ‘stochastic resonance’ (SR).

Stochastic resonance

1. a nonlinear measurement device[17],
2. a periodic signal weaker than the threshold of measurement device,
3. additional input noise, uncorrelated with the signal of interest.





$$\boxed{k \simeq \frac{e^{-\Delta U/D}}{2\pi\sqrt{\tau_- \tau_b}} =: k_{\mathrm{K}} \sim \Omega}$$

dunkel@math.mit.edu

1.5.2 Master equation approach

Similar to the case of the escape problem, one can obtain an alternative description of SR by projecting the full FPE dynamics onto a simpler set of master equations for the probabilities $P_\pm(t)$ of the coarse-grained particle-states 'left well' ($-$) and 'right well' ($+$), as defined by Eq. (1.76). This approach leads to the following two-state master equations with time-dependent rates

$$\dot{P}_-(t) = -k_+(t)\,P_- + k_-(t)\,P_+, \tag{1.111a}$$

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The general solution of this pair of ODEs is given by [GHJM98]

$$P_\pm(t) = g(t)\left[P_\pm(t_0) + \int_{t_0}^{t} ds\,\frac{k_\pm(s)}{g(s)}\right] \tag{1.112a}$$

where

$$g(t) = \exp\left\{-\int_{t_0}^{t} ds\,[k_+(s) + k_-(s)]\right\}. \tag{1.112b}$$

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where

$$g(t) \;=\; \exp\left\{-\int_{t_0}^{t} ds\,[k_+(s)+k_-(s)]\right\}. \tag{1.112b}$$

To discuss SR within this framework, it is plausible to postulate time-dependent Arrhenius-type rates,

$$k_\pm(t) = k_{\mathrm{K}}\exp\left[\pm\frac{Ax_*}{D}\cos(\Omega t)\right]. \tag{1.113}$$

Considering the asymptotic limit $t_0 \to -\infty$, one can Taylor-expand the rates for $Ax_* \ll D$ to obtain

$$k_\pm(t) = k_\mathrm{K}\left[1 \pm \frac{Ax_*}{D}\cos(\Omega t) + \left(\frac{Ax_*}{D}\right)^2 \cos^2(\Omega t) \pm \ldots\right]$$

These approximations are valid for slow driving (adiabatic regime), and they allow us to compute expectation values to leading order in Ax_*/D. To first order, one then finds for the conditional probability

$$\begin{aligned} P_+(t|x_0,t_0) &= 1 - P_-(t|x_0,t_0) \\ &= \frac{1}{2}\left\{e^{-2k_\mathrm{K}(t-t_0)}\left[2\delta_{x_0,x_*} - 1 - \kappa(t_0)\right] + 1 + \kappa(t)\right\} \end{aligned} \tag{1.114a}$$

where

$$\kappa(t) = \frac{Ax_*}{D}\cos(\Omega t - \overline{\varphi})\left(\frac{4k_\mathrm{K}^2}{4k_\mathrm{K}^2 + \Omega^2}\right)^{1/2}, \qquad \overline{\varphi} = \arctan\left(\frac{2\Omega}{k_\mathrm{K}}\right). \tag{1.114b}$$

Note that the conditional probability $P_+(t|x_0, t_0)$ satisfies the initial condition

$$P_+(t_0|x_0, t_0) = \delta_{x_0,x_*} = \begin{cases} 1, & x_0 = x_* \\ 0, & \text{otherwise} \end{cases}, \tag{1.115}$$

where $x_* = x_\pm$ depending on whether the particle starts in the left or right well. Furthermore, one then finds for the mean position the asymptotic linear response result [GHJM98]

$$\mathbb{E}[X(t)] = \overline{X} \cos(\Omega t - \overline{\varphi}) \tag{1.116a}$$

where

$$\overline{X} = \frac{A x_*^2}{D} \left(\frac{4k_{\mathrm{K}}^2}{4k_{\mathrm{K}}^2 + \Omega^2} \right)^{1/2}, \qquad \overline{\varphi} = \arctan\left(\frac{\Omega}{2k_{\mathrm{K}}} \right). \tag{1.116b}$$

Note that Eqs. (1.116) are consistent with our earlier result (1.107).