Rotating flows

Sir Geoffrey Ingram Taylor

(7 March 1886 – 27 June 1975)

Experiments on the Motion of Solid Bodies in Rotating Fluids.

By G. I. Taylor, F.R.S.

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one. On the other hand, if an attempt is made to produce a slow steady motion by moving a three-dimensional body† with a small uniform velocity (relative to axes which rotate with the fluid) three possibilities present them-‐elves :—

(*a*) The motion in the liquid may never become steady, however long the ody goes on moving.

(*b*) The motion may be steady but it may not be small in the neighbourhood f the body.

(*c*) The motion may be steady and two-dimensional.

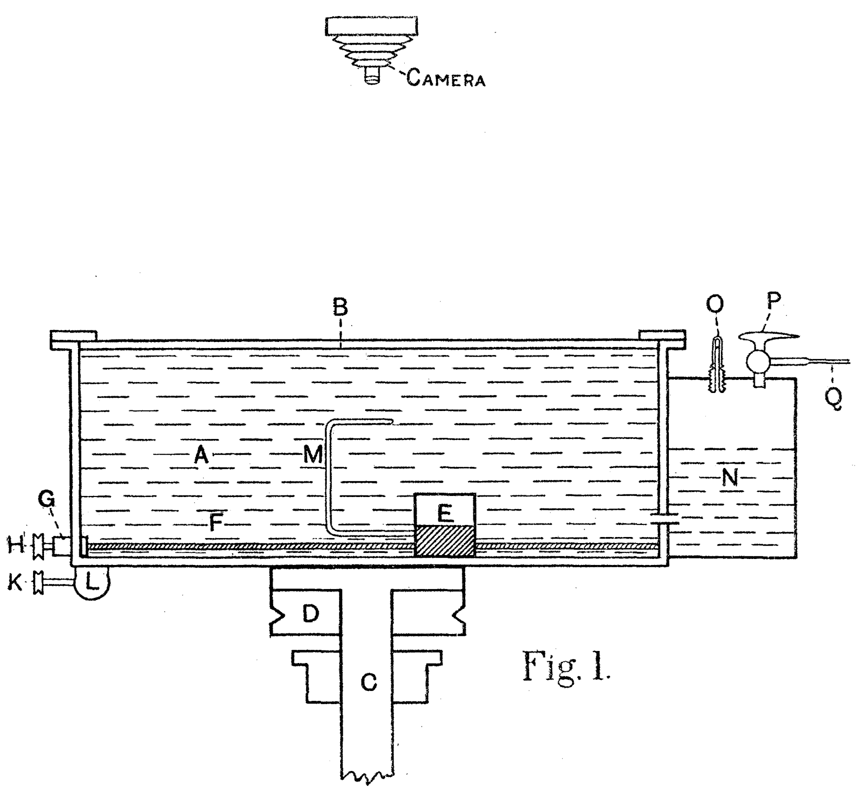

Fig. 1.

There remains the third possibility (*c*). In this case the motion would be a very remarkable one. If the liquid were contained between parallel planes perpendicular to the axis of rotation, the only possible two-dimensional motion satisfying the required conditions is one in which a cylinder of fluid moves as if fixed to the body. The boundary of such a cylinder would act as a solid body, and the liquid outside would behave as though a solid cylindrical body were being moved through it. No fluid would cross this boundary, and the liquid inside it would, in general, be at rest relative the solid body. This idea appears fantastic, but the experiments now to be described show that the true motion does, in fact, approximate to this curious type.

$$\mathrm{Ro} = \frac{U}{\Omega L} \ll 1.$$

Taylor - Proudman

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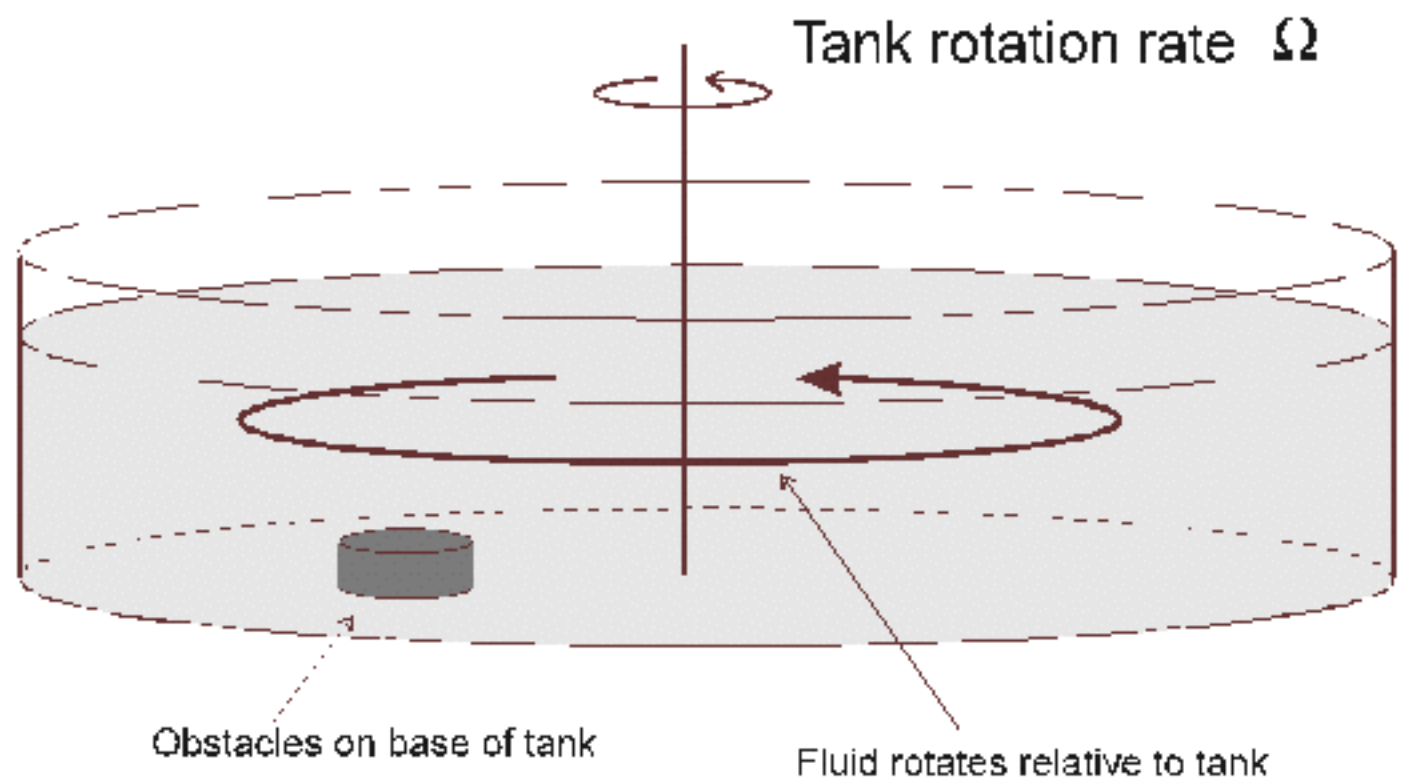

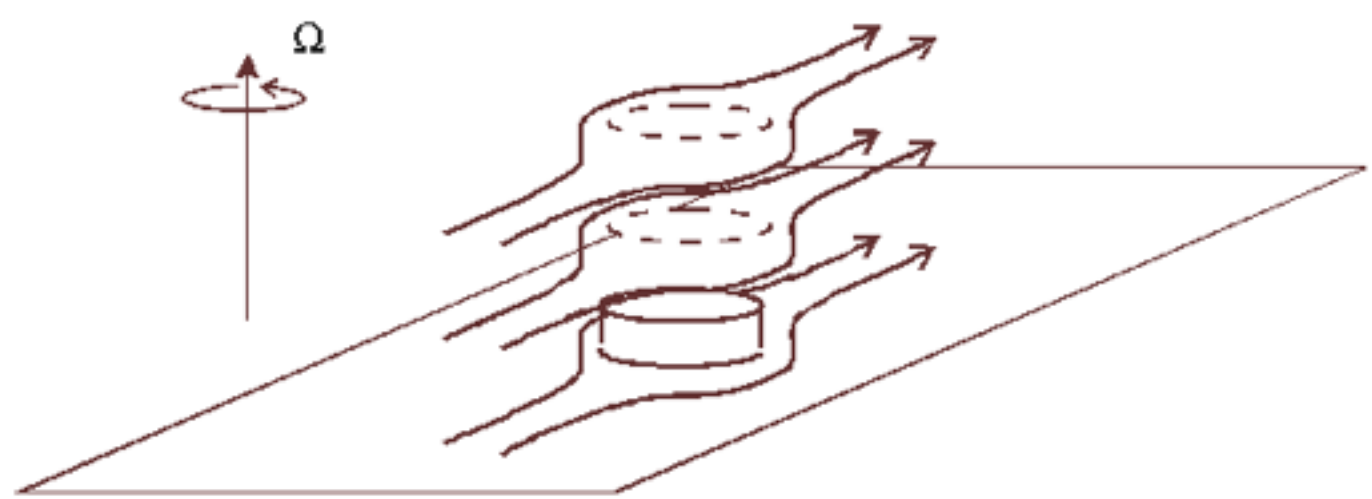

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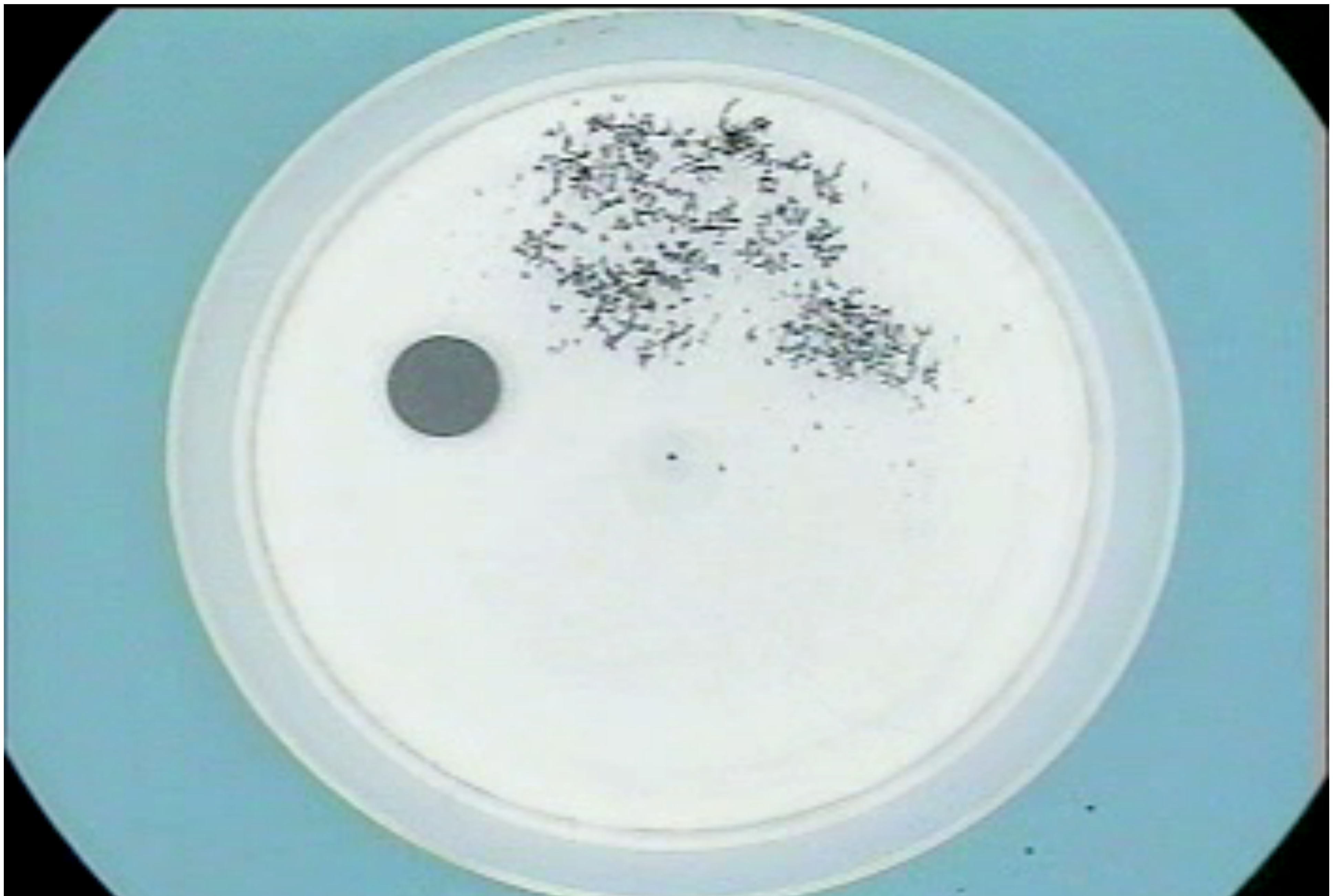
Taylor - column



Ekman-Layer

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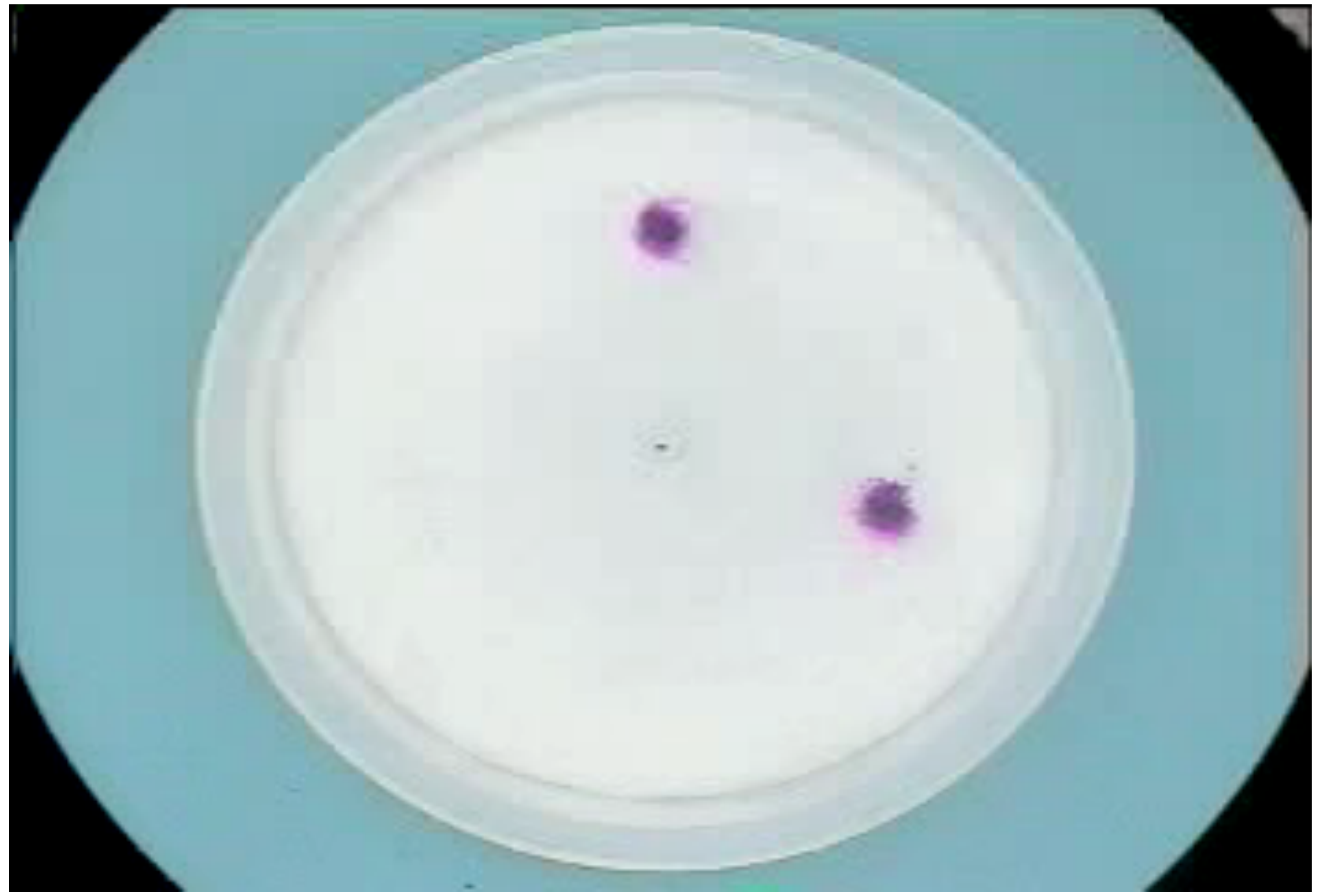
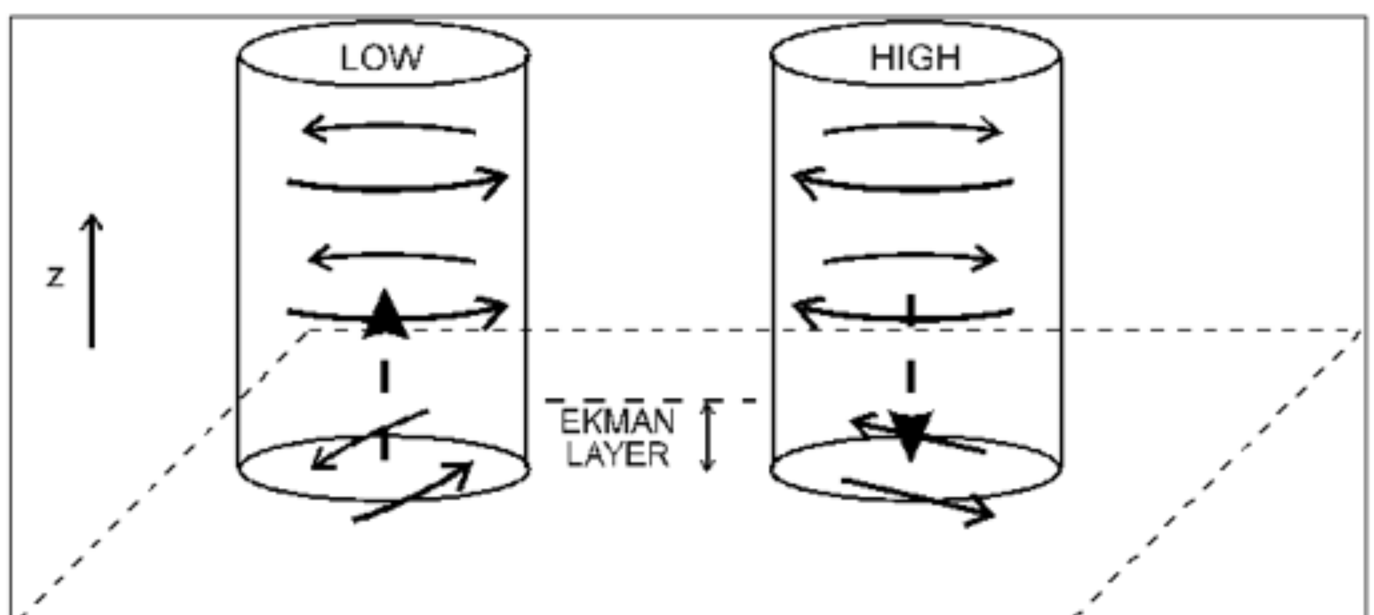
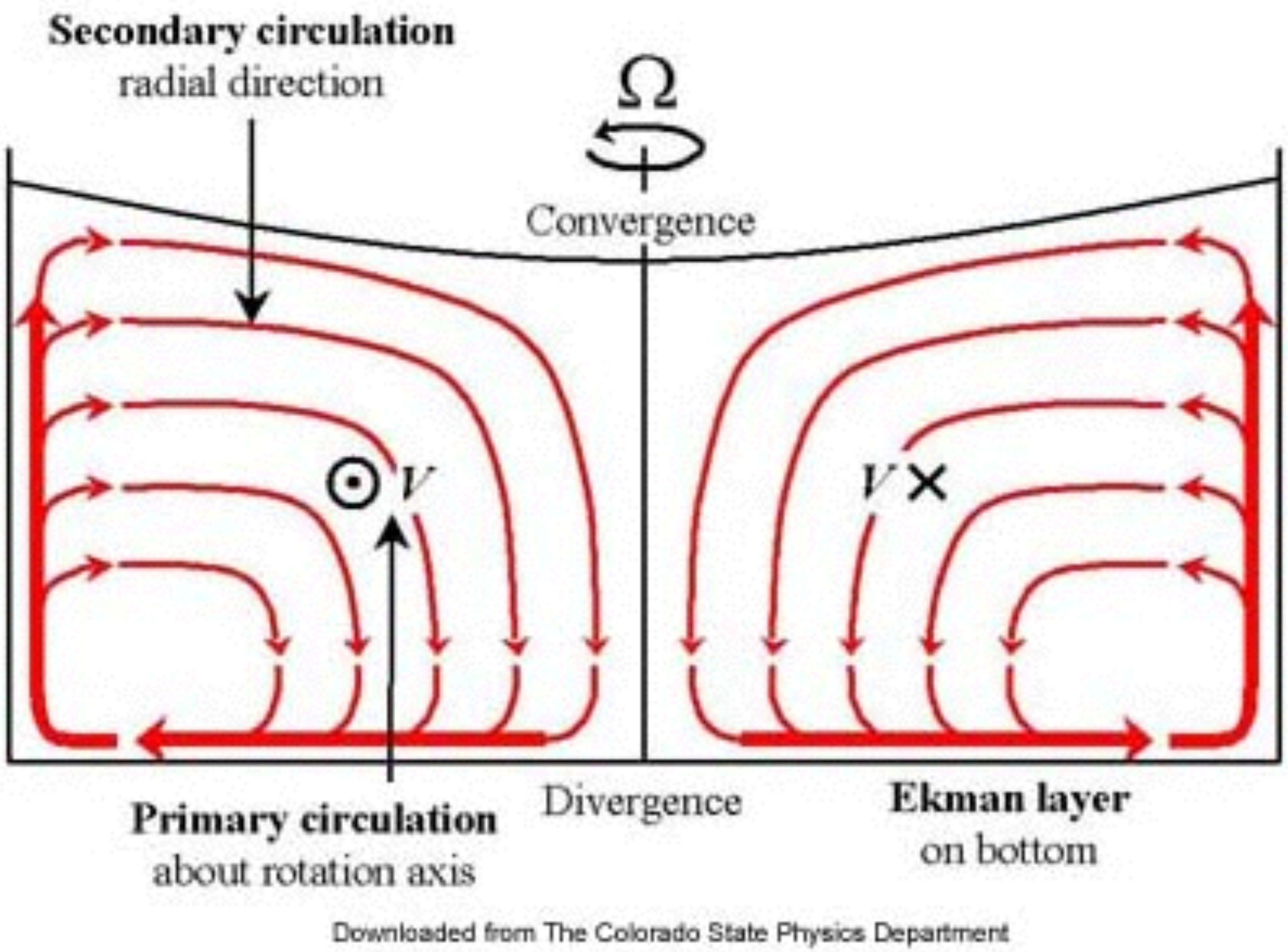






A viscous force between the Ekman layer and the stationary fluid causes fluid adjacent to the boundary layer to accelerate and a Coriolis force propels it outward.

The result is a flow circulating along the boundary layers and downward along the center of the cylinder.

22 The Ekman layer

Inner

$$
\begin{aligned}
-2\Omega v_I &= -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \\
2\Omega u_I &= -\frac{1}{\rho}\frac{\partial p_I}{\partial y}, \\
0 &= \frac{1}{\rho}\frac{\partial p_I}{\partial z}.
\end{aligned}
$$

Boundary

$$
\begin{aligned}
-2\Omega v &= -\frac{1}{\rho}\frac{\partial p_I}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \\
2\Omega u &= -\frac{1}{\rho}\frac{\partial p_I}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \\
0 &= \frac{1}{\rho}\frac{\partial p_I}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}, \\
\nabla \cdot \boldsymbol{u} &= 0.
\end{aligned}
$$

Here we have made the boundary layer approximation that $\partial/\partial z \gg \partial/\partial x, \partial/\partial y$.

From the continuity equation we deduce that w is much smaller than the velocity components parallel to the boundary so that $\partial p_I/\partial z = 0$, and the equations become

$$-2\Omega(v - v_I) = \nu\frac{\partial^2 u}{\partial z^2}, \tag{528a}$$

$$2\Omega(u - u_I) = \nu\frac{\partial^2 v}{\partial z^2}. \tag{528b}$$

Solution

$$-2\Omega(v - v_I) = \nu \frac{\partial^2 u}{\partial z^2}, \tag{528a}$$

$$2\Omega(u - u_I) = \nu \frac{\partial^2 v}{\partial z^2}. \tag{528b}$$

These are the equations we must solve. Acheson has a good trick. Multiplying the second equation by i and adding the two yields

$$\nu \frac{\partial^2 f}{\partial z^2} = 2\Omega i f, \tag{529a}$$

where

$$f = u - u_I + i(v - v_I). \tag{529b}$$

The solution is obtained by guessing $f \sim \mathrm{e}^{\alpha z}$, which yields $\alpha^2 = 2\Omega i/\nu$. Hence,

$$f = Ae^{(1+i)z^*} + Be^{-(1+i)z^*}, \qquad z^* = z\sqrt{\Omega/\nu}. \tag{530}$$

We require that as $z^* \to \infty$, $f \to 0$. This implies that $A = 0$. We are in the frame of reference moving with the bottom plate, so the no slip boundary condition at $z = 0$ requires that $f(z = 0) = -u_I - iv_I$. Splitting f into its real an imaginary parts implies

$$u = u_I - \mathrm{e}^{-z^*}(u_I \cos(z/\delta) + v_I \sin(z/\delta)), \tag{531}$$

$$v = v_I - \mathrm{e}^{-z^*}(v_I \cos(z/\delta) - u_I \sin(z/\delta)). \tag{532}$$

This is the velocity profile in the boundary layer.

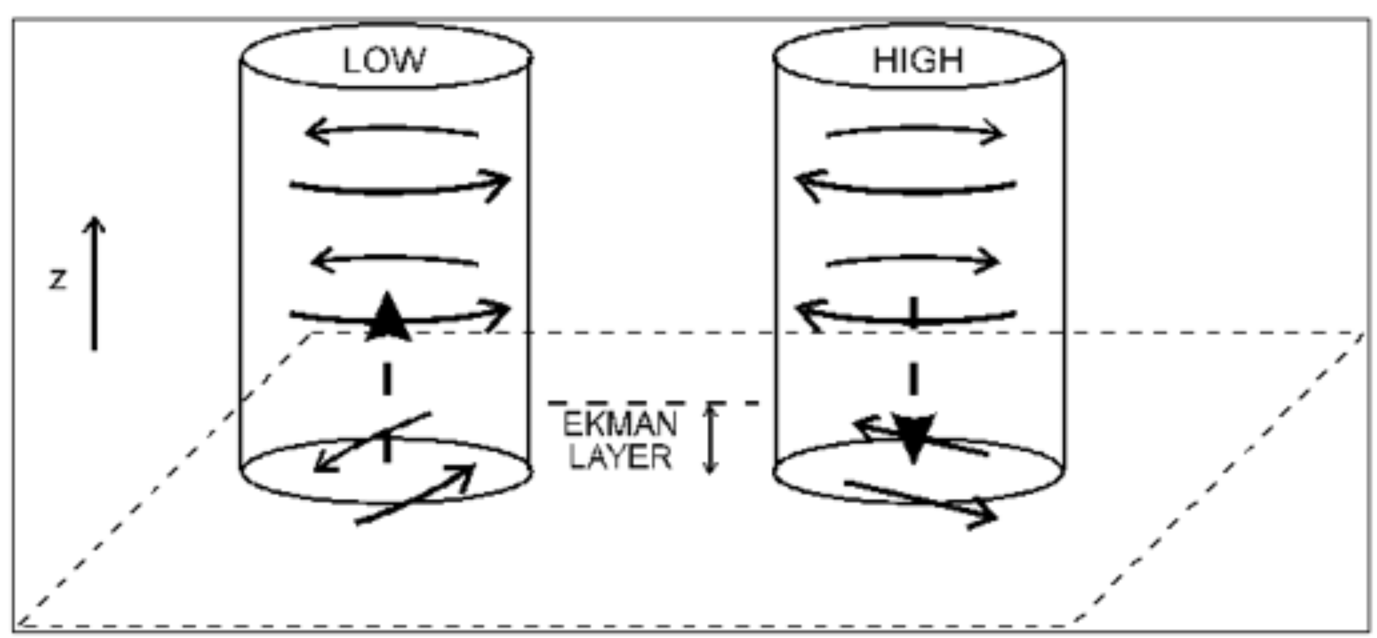
What about the z-component? From the divergence free condition, we have

$$\left(\frac{\Omega}{\nu}\right)^{1/2}\frac{\partial w}{\partial z^*}=\frac{\partial w}{\partial z}=-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\left(\frac{\partial v_I}{\partial x}-\frac{\partial u_I}{\partial y}\right)e^{-z^*}\sin z^*. \tag{533}$$

Integrating from $z^*=0$ to ∞ gives

$$w=\frac{1}{2}\left(\frac{\Omega}{\nu}\right)^{-1/2}\left(\frac{\partial v_I}{\partial x}-\frac{\partial u_I}{\partial y}\right)=\frac{\hat{\omega}_I}{2}\sqrt{\frac{\nu}{\Omega}}, \tag{534}$$

where $\hat{\omega}_I$ is the vorticity in the inviscid flow. Thus, if $\hat{\omega}_I>0$ (i.e., the bottom boundary is moving slower than the main body of fluid) then there is flow from the boundary layer into the fluid.



22.2 Matching

Now we have these Ekman layers at the top and the bottom. What we just assumed was that the boundary is moving at frequency Ω. If it is not, but instead moving at an angular frequency Ω_B relative to the rotating frame, then we need to change the boundary conditions a little in the rotating frame. In this case

$$w = \left(\frac{\nu}{\Omega_B}\right)^{1/2}\left(\frac{\hat{\omega}_I}{2} - \Omega_B\right). \tag{535a}$$

We could derive this, but it is intuitive since $(\hat{\omega}_I - 2\Omega_B)$ is the vorticity of the interior flow relative to the moving lower boundary. Similarly, if Ω_T denotes the angular velocity of the rigid upper boundary relative to the rotating frame, then there is a small z-component of velocity up into the boundary layer

$$w = \left(\frac{\nu}{\Omega_T}\right)^{1/2}\left(\Omega_T - \frac{\hat{\omega}_I}{2}\right). \tag{535b}$$

Now in our container both are happening. Since u_I, v_I and w_I are all independent of z then so is ω_I. Thus, the only way the experiment could work is if the induced value of ω_I from both cases matches. This implies that

$$\omega_I = \Omega_T + \Omega_B. \tag{536}$$

With $\Omega_B = 0$ and $\Omega_T = \epsilon$ we have that $\omega_I = \epsilon$. Thus, the flow in the inner region has a velocity which is entirely set by the boundary layers. Note that there is no viscosity in this formula, but viscosity plays a role in determining the flow. We have completely different behaviour for $\nu = 0$ and in the limit $\nu \to 0$.

22.3 Spin-down of this apparatus

We now want to finally solve the spin-down of our coffee cup. To do so we assume the coffee cup to be a cylinder with a top and a bottom both rotating with angular velocity $\Omega + \epsilon$. At $t = 0$ the angular velocity of the boundaries is reduced to Ω. How long does it take to reach a steady state?

We use the time-dependent formula

$$\frac{\partial u_I}{\partial t} - 2\Omega v_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{537a}$$

$$\frac{\partial v_I}{\partial t} + 2\Omega u_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial y}. \tag{537b}$$

Now in our container both are happening. Since u_I, v_I and w_I are all independent of z then so is $\hat{\omega}_I$. Thus, the only way the experiment could work is if the induced value of $\hat{\omega}_I$ from both cases matches. This implies that

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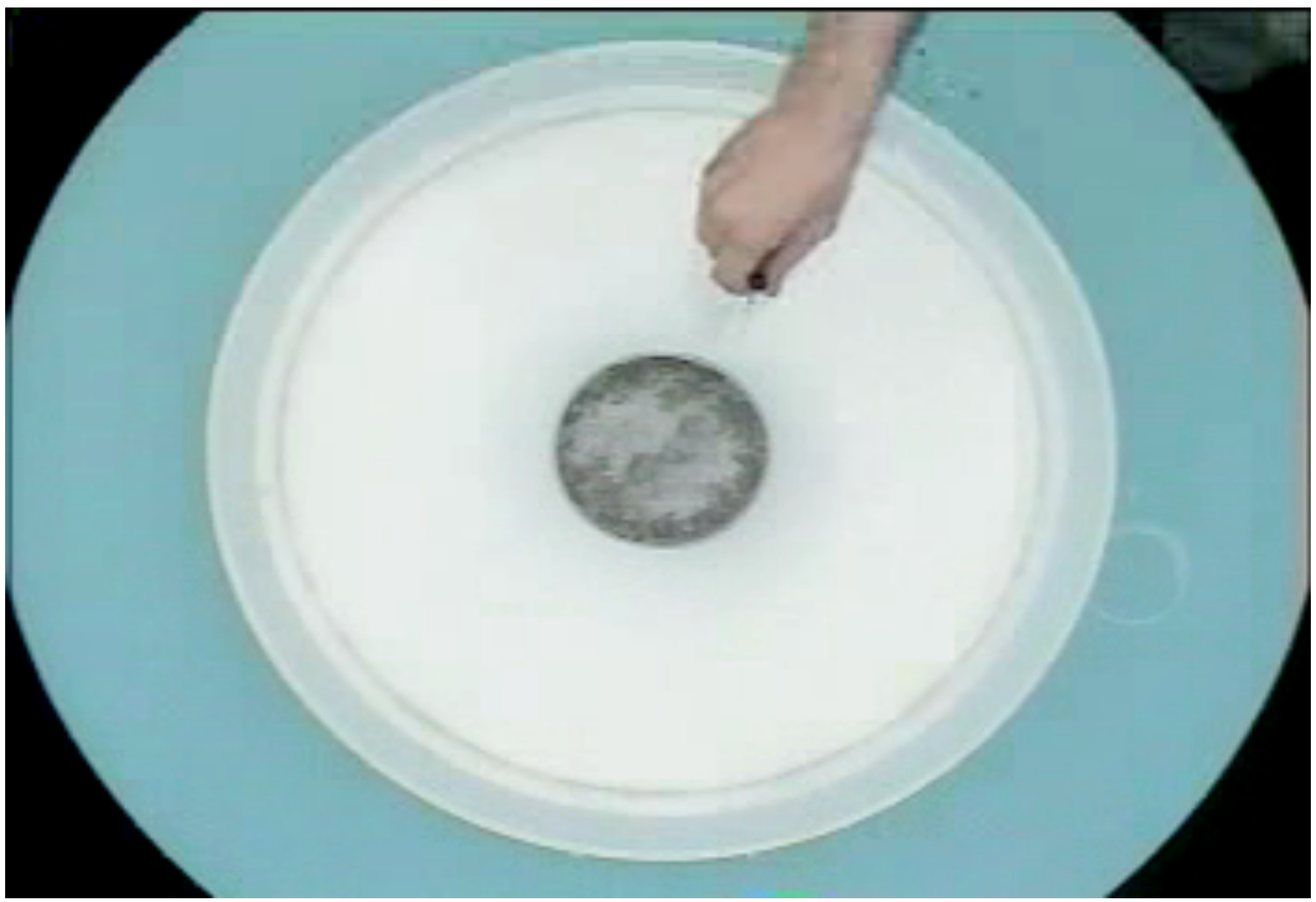
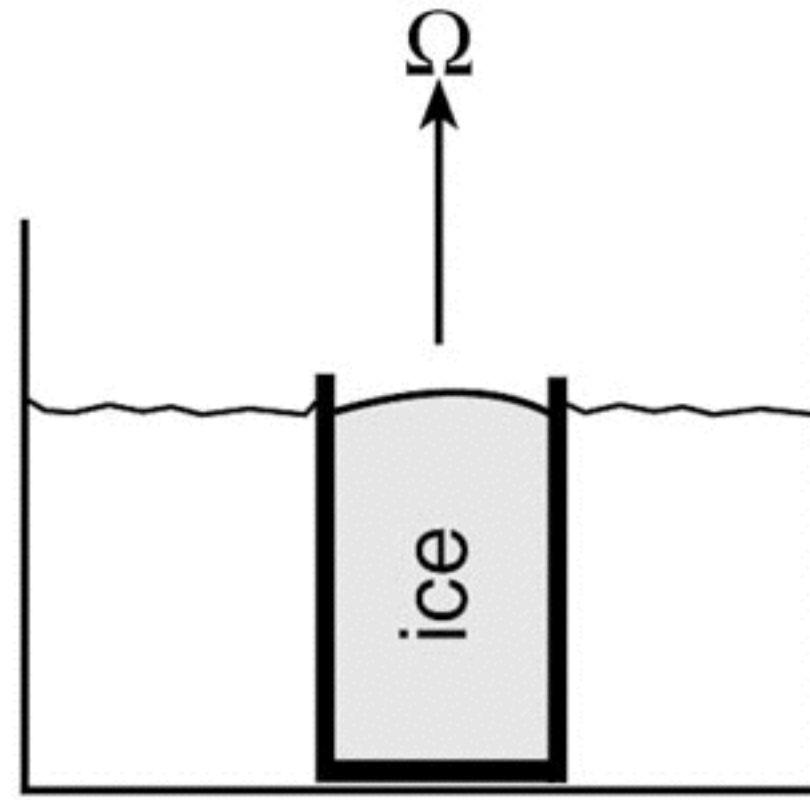
Atmospheric general circulation





23 Water waves

The flow is assumed to be inviscid, and as it is initially irrotational it must remain so. Fluid motion is therefore described by the velocity potential $(u, v) = \nabla\phi$, and satisfies Laplace's equation (incompressibility condition)

$$\nabla^2\phi = 0. \tag{541a}$$

The momentum equation becomes

$$\frac{\partial \nabla\phi}{\partial t} + \frac{1}{2}\nabla(\nabla\phi)^2 = -\frac{1}{\rho}\nabla p - \nabla\chi, \tag{541b}$$

where χ is the gravitational potential such that $\mathbf{g} = -\nabla\chi$. This can be integrated to give the unsteady Bernoulli relation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + \chi = C(t). \tag{542}$$

Here, $C(t)$ is a time dependent constant that does not affect the flow, which is related to ϕ only through spatial gradients. The surface is $h(x,t)$ and we have the kinematic condition

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v \tag{543}$$

on $y = h(x,t)$. This simply states that if you choose an element of fluid on the surface, the rate at which that part of the surface rises or falls is, by definition, the vertical velocity. Finally, we require that the pressure be atmospheric, p_0 at the surface. From the unsteady Bernoulli relation we get

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + gh = 0 \tag{544}$$

on $h(x, t)$, where we have chosen the constant $C(t)$ appropriately to simplify things.

Include surface tension

$$p = p_0 - \gamma \frac{\partial^2 h(x,t)}{\partial x^2}. \tag{545}$$

Including surface tension in our pressure condition at the surface, we have that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + gh - \frac{\gamma}{\rho} \frac{\partial^2 h}{\partial x^2} = 0 \tag{546}$$

at $y = h(x, t)$.

Linearize

We now follow the same procedure as in the last lecture and assume all the variables to be small, so that we can linearise the equations. The linearised system of equations consists of Laplace's equation and the boundary conditions at $y = 0$:

$$\nabla^2\phi = 0 \tag{547a}$$

$$\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y}(x, 0, t), \tag{547b}$$

$$\frac{\partial \phi}{\partial t} = -gh + \frac{\gamma}{\rho}\frac{\partial^2 h}{\partial x^2}, \tag{547c}$$

These conditions arise because we have Taylor expanded terms such as

$$v(x, h, t) = v(x, 0, t) + hv_y(x, 0, t), \tag{548}$$

and then ignored nonlinear terms. We guess solutions of the form

$$\phi = Ae^{ky}\sin(kx - \omega t)\ , \qquad h = \epsilon e^{ky}\cos(kx - \omega t), \tag{549}$$

knowing that these satisfy Laplace's equation (we have ignored terms of the form e^{-ky}, as the surface is at $y = 0$ and we need all terms to disappear as $y \to -\infty$). Putting these into the surface boundary conditions (8) and (9) gives

Insert

$$\omega\epsilon = Ak, \tag{550a}$$

$$\omega A = g\epsilon + \frac{\gamma k^2 \epsilon}{\rho}. \tag{550b}$$

Eliminating A we get the dispersion relation

$$\omega^2 = gk + \frac{\gamma k^3}{\rho}. \tag{551}$$

What are the consequences of this relation? On the simplest level we know that the *phase speed*, c, of a disturbance is given by the relation $c = \omega/k$. Thus

$$c^2 = \frac{g}{k} + \frac{\gamma k}{\rho}. \tag{552}$$

The relative importance of surface tension and gravity in determining wave motion is given by the *Bond number* $B_o = \gamma k^2/\rho g$. If $B_o < 1$ then we have *gravity waves*, for which longer wavelengths travel faster. If $B_o > 1$ then we have *capillary waves*, for which shorter wavelengths travel faster. For water, the Bond number becomes unity for wavelengths of about 2 cm, and this accounts for the different ring patterns you can observe when a stone and a raindrop fall into water.

23.2.1 The wake of an airplane

From our previous work with sound waves we know that the equation governing the propagation of a 2D disturbance in air is the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi = c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right), \tag{555}$$

where ϕ is some scalar quantity representing the disturbance (e.g., the velocity potential, the density or the pressure). For an airplane moving through the air we anticipate a solution that is constant in the frame of reference of the plane. Thus

$$\phi(x, y, t) = \tilde{\phi}(x - Ut, y), \tag{556}$$

and we have

$$U^2 \frac{\partial^2 \tilde{\phi}}{\partial x^2} = c^2 \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \right). \tag{557}$$

$$U^2 \frac{\partial^2 \tilde{\phi}}{\partial x^2} = c^2 \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \right)$$

Defining the *Mach number* $M = U^2/c^2$, the above equation becomes

$$(M^2 - 1)\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} = 0. \tag{558}$$

If $M < 1$ we can make a simple change of variables $X = x/\sqrt{1 - M^2}$ and regain Laplace's equation. Thus everything can be solved using our conformal mapping techniques. However, if $M > 1$ then the original equation now looks like a wave equation, with y replacing t, yielding solutions of the form

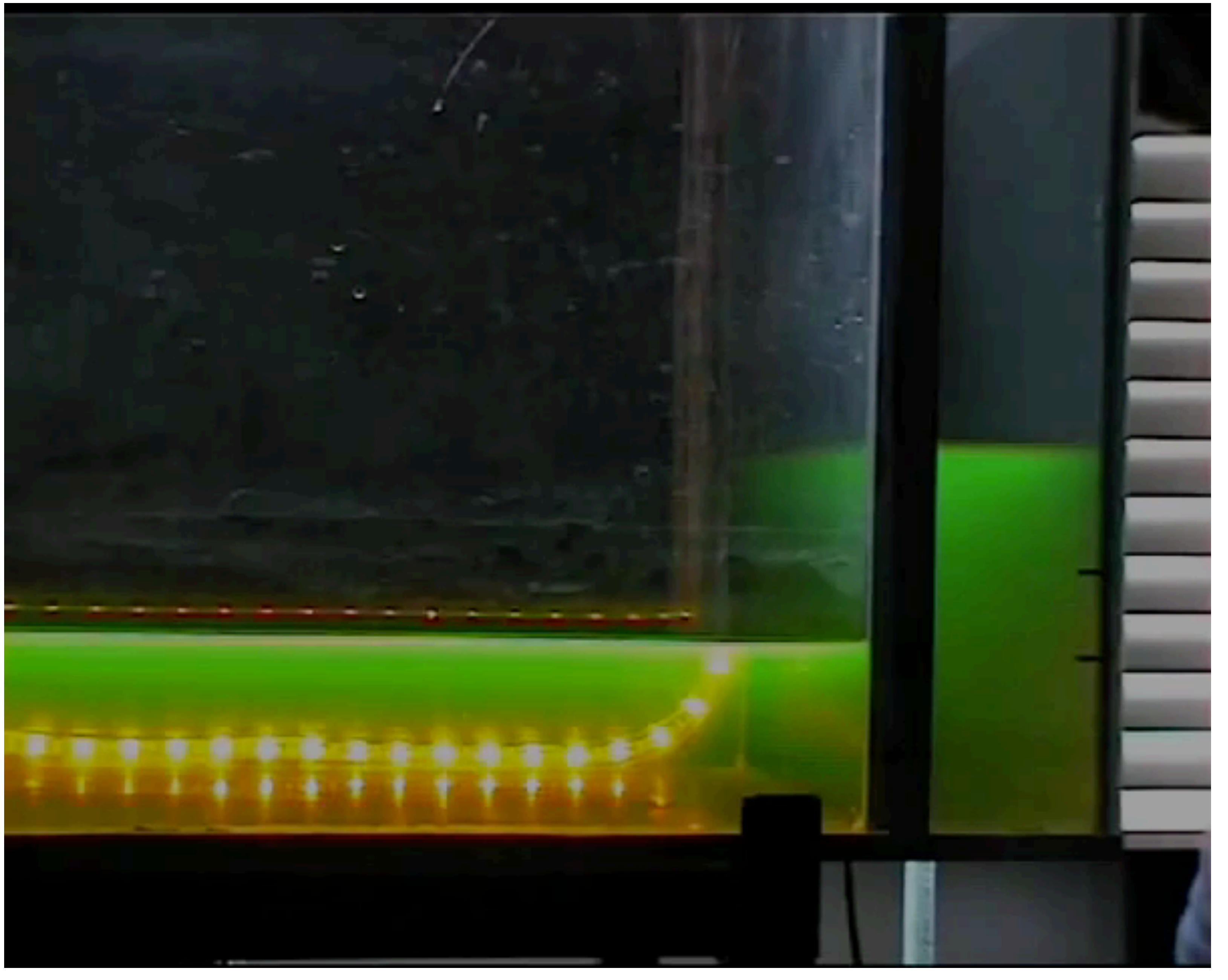
$$\tilde{\phi}(x, y) = \Phi(x - y\sqrt{M^2 - 1}) \tag{559}$$

Thus disturbances are confined to a wake whose half angle is given by

$$\tan\theta = \frac{1}{\sqrt{M^2 - 1}}. \tag{560}$$

Only a narrow region behind the plane knows it exists, and the air ahead doesn't know what's coming!

Solitons



credit: Christophe Finot