From Newton's law to hydrodynamic equations

18.354 - L14

Goal: derive

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{-\nabla p}{\rho} + \boldsymbol{g}.$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla)$$

13.2 From Newton's laws to hydrodynamic equations

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i \ , \qquad m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}_i,$$

$$\boldsymbol{F}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \boldsymbol{G}(\boldsymbol{x}_i) + \sum_{j \neq i} \boldsymbol{H}(\boldsymbol{x}_i - \boldsymbol{x}_j) = -\nabla_{\boldsymbol{x}_i} \Phi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$\boldsymbol{H}(\boldsymbol{r}) = -\boldsymbol{H}(-\boldsymbol{r})$$

We define the fine-grained phase-space density

$$f(t, \boldsymbol{x}, \boldsymbol{v}) = \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i(t))\delta(\boldsymbol{v} - \boldsymbol{v}_i(t))$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_i) = \delta(x - x_i)\delta(y - y_i)\delta(z - z_i)$

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$$\begin{aligned}
\frac{\partial}{\partial t} f &= \sum_{i=1}^{N} \frac{d}{dt}\left[\delta(\boldsymbol{x}-\boldsymbol{x}_i)\delta(\boldsymbol{v}-\boldsymbol{v}_i)\right] \\
&= \sum_{i}^{N} \{\delta(\boldsymbol{v}-\boldsymbol{v}_i)\nabla_{\boldsymbol{x}_i}\delta(\boldsymbol{x}-\boldsymbol{x}_i)\cdot\dot{\boldsymbol{x}}_i + \delta(\boldsymbol{x}-\boldsymbol{x}_i)\nabla_{\boldsymbol{v}_i}\delta(\boldsymbol{v}-\boldsymbol{v}_i)\cdot\dot{\boldsymbol{v}}_i\} \\
&= -\nabla_{\boldsymbol{x}} \sum_{i=1}^{N} \delta(\boldsymbol{v}-\boldsymbol{v}_i)\delta(\boldsymbol{x}-\boldsymbol{x}_i)\cdot\boldsymbol{v}_i - \nabla_{\boldsymbol{v}} \sum_{i=1}^{N} \delta(\boldsymbol{x}-\boldsymbol{x}_i)\delta(\boldsymbol{v}-\boldsymbol{v}_i)\cdot\frac{\boldsymbol{F}_i}{m}
\end{aligned}$$

where, in the last step, we inserted Newton's equations and used that

$$\frac{\partial}{\partial x_i}\delta(x-x_i) = -\frac{\partial}{\partial x}\delta(x-x_i)$$

$$\begin{aligned}
\frac{\partial}{\partial t} f &= -\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \sum_{i=1}^{N} \delta(\boldsymbol{v} - \boldsymbol{v}_i)\delta(\boldsymbol{x} - \boldsymbol{x}_i) - \nabla_{\boldsymbol{v}} \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\delta(\boldsymbol{v} - \boldsymbol{v}_i) \cdot \frac{\boldsymbol{F}_i}{m} \\
&= -\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f - \frac{1}{m} \nabla_{\boldsymbol{v}} \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\delta(\boldsymbol{v} - \boldsymbol{v}_i) \cdot \boldsymbol{F}_i. \qquad (313)
\end{aligned}$$

Writing $\nabla = \nabla_{\boldsymbol{x}}$ and inserting (309) for the forces, we may rewrite

$$\begin{aligned}
m\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) f &= -\nabla_{\boldsymbol{v}} \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\delta(\boldsymbol{v} - \boldsymbol{v}_i) \cdot \left[\boldsymbol{G}(\boldsymbol{x}_i) + \sum_{j \neq i} \boldsymbol{H}(\boldsymbol{x}_i - \boldsymbol{x}_j)\right] \\
&= -\nabla_{\boldsymbol{v}} \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\delta(\boldsymbol{v} - \boldsymbol{v}_i) \cdot \left[\boldsymbol{G}(\boldsymbol{x}) + \sum_{\boldsymbol{x}_j \neq \boldsymbol{x}} \boldsymbol{H}(\boldsymbol{x} - \boldsymbol{x}_j)\right] \\
&= -\left[\boldsymbol{G}(\boldsymbol{x}) + \sum_{\boldsymbol{x}_j \neq \boldsymbol{x}} \boldsymbol{H}(\boldsymbol{x} - \boldsymbol{x}_j)\right] \cdot \nabla_{\boldsymbol{v}} f
\end{aligned}$$

$$m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)f \quad = \quad -\left[\boldsymbol{G}(\boldsymbol{x})+\sum_{\boldsymbol{x}_j\neq\boldsymbol{x}}\boldsymbol{H}(\boldsymbol{x}-\boldsymbol{x}_j)\right]\cdot\nabla_{\boldsymbol{v}}f \tag{314}$$

To obtain the hydrodynamic equations from (314), two additional reductions will be necessary:

- We need to replace the fine-grained density $f(t, \boldsymbol{x}, \boldsymbol{v})$, which still depends implicitly on the (unknown) solutions $\boldsymbol{x}_j(t)$, by a *coarse-grained* density $\langle f(t, \boldsymbol{x}, \boldsymbol{v})\rangle$.
- We have to construct the relevant field variables, the mass density $\rho(t, \boldsymbol{r})$ and velocity field $\boldsymbol{u}$, from the coarse-grained density $\bar{f}$.

$$\{\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_N(t)\}$$

$$\{\boldsymbol{x}_1(0), \ldots, \boldsymbol{x}_N(0); \boldsymbol{v}_1(0), \ldots, \boldsymbol{v}_N(0)\} =: \Gamma_0$$

$$\langle f(t, \boldsymbol{x}, \boldsymbol{v})\rangle = \int \mathrm{d}\mathbb{P}(\Gamma_0)\, f(t, \boldsymbol{x}, \boldsymbol{v}). \tag{315}$$

$$m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\langle f\rangle \quad = \quad -\nabla_{\boldsymbol{v}}\cdot[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}] \tag{316}$$

where the collision-term

$$C(t,\boldsymbol{x},\boldsymbol{v}) := \sum_{\boldsymbol{x}_j\neq\boldsymbol{x}}\langle\boldsymbol{H}(\boldsymbol{x}-\boldsymbol{x}_j)f(t,\boldsymbol{x},\boldsymbol{v})\rangle \tag{317}$$

We now define the mass density ρ, the velocity field $\boldsymbol{u}$, and the specific kinetic energy tensor $\boldsymbol{\Sigma}$ by

$$\rho(t,\boldsymbol{x}) \quad = \quad m\int d^3v\,\langle f(t,\boldsymbol{x},\boldsymbol{v})\rangle, \tag{318a}$$

$$\rho(t,\boldsymbol{x})\,\boldsymbol{u}(t,\boldsymbol{x}) \quad = \quad m\int d^3v\,\langle f(t,\boldsymbol{x},\boldsymbol{v})\rangle\;\boldsymbol{v}. \tag{318b}$$

$$\rho(t,\boldsymbol{x})\,\boldsymbol{\Sigma}(t,\boldsymbol{x}) \quad = \quad m\int d^3v\,\langle f(t,\boldsymbol{x},\boldsymbol{v})\rangle\;\boldsymbol{v}\boldsymbol{v}. \tag{318c}$$

We now define the mass density ρ, the velocity field $\boldsymbol{u}$, and the specific kinetic energy tensor $\boldsymbol{\Sigma}$ by

$$\rho(t, \boldsymbol{x}) = m \int d^3v \, \langle f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle, \tag{318a}$$

$$\rho(t, \boldsymbol{x}) \, \boldsymbol{u}(t, \boldsymbol{x}) = m \int d^3v \, \langle f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \, \boldsymbol{v}. \tag{318b}$$

$$\rho(t, \boldsymbol{x}) \, \boldsymbol{\Sigma}(t, \boldsymbol{x}) = m \int d^3v \, \langle f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \, \boldsymbol{v}\boldsymbol{v}. \tag{318c}$$

The tensor $\boldsymbol{\Sigma}$ is, by construction, symmetric as can be seen from the definition of its individual components

$$\rho(t, \boldsymbol{x}) \, \Sigma_{ij}(t, \boldsymbol{x}) = m \int d^3v \, \langle f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \, v_i v_j,$$

and the trace of $\boldsymbol{\Sigma}$ defines the local *kinetic energy density*

$$\epsilon(t, \boldsymbol{x}) := \frac{1}{2} \mathrm{Tr}(\rho \boldsymbol{\Sigma}) = \frac{m}{2} \int d^3v \, \langle f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \, |\boldsymbol{v}|^2. \tag{319}$$

Mass conservation

$$m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\langle f\rangle \quad = \quad -\nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}\right] \tag{316}$$

Integrating Eq. (316) over $\boldsymbol{v}$, we get

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\boldsymbol{u}) \quad = \quad -\int dv^3\ \nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}\right], \tag{320}$$

but the rhs. can be transformed into a surface integral (in velocity space) that vanishes since for physically reasonable interactions $[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}]\to\mathbf{0}$ as $|\boldsymbol{v}|\to\infty$. We thus recover the mass conservation equation

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\boldsymbol{u})=0. \tag{321}$$

Momentum conservation

$$m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\langle f\rangle \quad = \quad -\nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}\right] \tag{316}$$

To obtain the momentum conservation law, lets multiply (316) by $\boldsymbol{v}$ and subsequently integrate over $\boldsymbol{v}$,

$$\int dv^3\, m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\langle f\rangle\boldsymbol{v} \quad = \quad -\int dv^3\, \boldsymbol{v}\nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle+\boldsymbol{C}\right]. \tag{322}$$

$$\int dv^3\, m\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\langle f\rangle\boldsymbol{v} \;=\; -\int dv^3\, \boldsymbol{v}\nabla_{\boldsymbol{v}}\cdot\left[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle + \boldsymbol{C}\right]. \tag{322}$$

The lhs. can be rewritten as

$$\begin{aligned}
\int dv^3\, m\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\langle f\rangle\boldsymbol{v} &= \frac{\partial}{\partial t}(\rho\boldsymbol{u}) + \nabla\cdot\int dv^3\, m\langle f\rangle\boldsymbol{v}\boldsymbol{v} \\
&= \frac{\partial}{\partial t}(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{\Sigma}) \\
&= \frac{\partial}{\partial t}(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) + \nabla\cdot[\rho(\boldsymbol{\Sigma}-\boldsymbol{u}\boldsymbol{u})] \\
&= \rho\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\frac{\partial}{\partial t}\rho + \boldsymbol{u}\nabla\cdot(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\cdot[\rho(\boldsymbol{\Sigma}-\boldsymbol{u}\boldsymbol{u})] \\
&\overset{(321)}{=} \rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} + \nabla\cdot[\rho(\boldsymbol{\Sigma}-\boldsymbol{u}\boldsymbol{u})]
\end{aligned} \tag{323}$$

The rhs. of (322) can be computed by partial integration, yielding

$$\begin{aligned}
-\int dv^3\, \boldsymbol{v}\nabla_{\boldsymbol{v}}\cdot[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle + \boldsymbol{C}] &= \int dv^3\, \cdot[\boldsymbol{G}(\boldsymbol{x})\langle f\rangle + \boldsymbol{C}] \\
&= \rho\boldsymbol{g} + \boldsymbol{c}(t,\boldsymbol{x}),
\end{aligned} \tag{324}$$

where $\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{G}(\boldsymbol{x})/m$ is the force per unit mass (acceleration) and the last term

$$\boldsymbol{c}(t,\boldsymbol{x}) = \int dv^3\boldsymbol{C} = \int dv^3 \sum_{\boldsymbol{x}_j\neq\boldsymbol{x}} \langle \boldsymbol{H}(\boldsymbol{x}-\boldsymbol{x}_j)f(t,\boldsymbol{x},\boldsymbol{v})\rangle \tag{325}$$

encodes the mean pair interactions. Combining (323) and (324), we find

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} \quad = \quad -\nabla\cdot[\rho(\boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u})] + \rho\boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{c}(t,\boldsymbol{x}). \tag{326}$$

The symmetric tensor

$$\boldsymbol{\Pi} := \boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u} \tag{327}$$

measures the covariance of the local velocity fluctuations of the molecules

related to their temperature. To estimate $\boldsymbol{c}$, let us assume that the pair interaction force $\boldsymbol{H}$ can be derived from a pair potential φ, which means that $\boldsymbol{H}(\boldsymbol{r}) = -\nabla_{\boldsymbol{r}}\varphi(\boldsymbol{r})$. Assuming further that $\boldsymbol{H}(\mathbf{0}) = \mathbf{0}$, we may write

$$\boldsymbol{c}(t,\boldsymbol{x}) = -\int dv^3 \sum_{\boldsymbol{x}_j(t)} \langle[\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x} - \boldsymbol{x}_j)] f(t,\boldsymbol{x},\boldsymbol{v})\rangle \tag{328}$$

Replacing for some function $\zeta(\boldsymbol{x})$ the sum over all particles by the integral

$$\sum_{\boldsymbol{x}_j} \zeta(\boldsymbol{x}_j) \simeq \frac{1}{m}\int d^3y\ \rho(t,\boldsymbol{y})\ \zeta(\boldsymbol{y}) \tag{329}$$

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we have

$$\begin{aligned} \boldsymbol{c}(t, \boldsymbol{x}) &\simeq -\frac{1}{m} \int dv^3 \int d^3y \; \rho(t, \boldsymbol{y}) \, \langle [\nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x} - \boldsymbol{y})] f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \\ &= -\frac{1}{m} \int dv^3 \int d^3y \; \rho(t, \boldsymbol{y}) \, \langle [-\nabla_{\boldsymbol{y}} \varphi(\boldsymbol{x} - \boldsymbol{y})] f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \\ &= -\frac{1}{m} \int dv^3 \int d^3y \; [\nabla \rho(t, \boldsymbol{y})] \, \langle \varphi(\boldsymbol{x} - \boldsymbol{y}) f(t, \boldsymbol{x}, \boldsymbol{v}) \rangle \end{aligned} \tag{330}$$

$$\boldsymbol{c}(t,\boldsymbol{x}) \simeq -\frac{1}{m}\int dv^3 \int d^3y\, [\nabla\rho(t,\boldsymbol{y})]\, \langle \varphi(\boldsymbol{x}-\boldsymbol{y}) f(t,\boldsymbol{x},\boldsymbol{v})\rangle$$

In general, it is impossible to simplify this further without some explicit assumptions about initial distribution $\mathbb{P}$ that determines the average $\langle \cdot \rangle$. There is however one exception, namely, the case when interactions are very short-range so that we can approximate the potential by a delta-function,

$$\varphi(\boldsymbol{r}) = \phi_0 a^3 \delta(\boldsymbol{r}), \tag{331}$$

where φ_0 is the interaction energy and a^3 the effective particle volume. In this case,

$$\begin{aligned}
\boldsymbol{c}(t,\boldsymbol{x}) &= -\frac{\varphi_0 a^3}{m}\int dv^3 \int d^3y\, [\nabla\rho(t,\boldsymbol{y})]\, \langle \delta(\boldsymbol{x}-\boldsymbol{y}) f(t,\boldsymbol{x},\boldsymbol{v})\rangle \\
&= -\frac{\varphi_0 a^3}{m}[\nabla\rho(t,\boldsymbol{x})] \int dv^3 \langle f(t,\boldsymbol{x},\boldsymbol{v})\rangle \\
&= -\frac{\varphi_0 a^3}{m^2}[\nabla\rho(t,\boldsymbol{x})]\rho(t,\boldsymbol{x}) \\
&= -\frac{\varphi_0 a^3}{2m^2}\nabla\rho(t,\boldsymbol{x})^2
\end{aligned} \tag{332}$$

Inserting this into (326), we have thus derived the following hydrodynamic equations

$$\begin{aligned}\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{u}) &= 0 &\quad (333a)\\ \rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} &= \nabla\cdot\boldsymbol{\Xi} + \rho\boldsymbol{g}(\boldsymbol{x}), &\quad (333b)\end{aligned}$$

where

$$\boldsymbol{\Xi} := -\left[\rho(\boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u}) + \frac{\varphi_0 a^3}{2m^2}\rho^2\boldsymbol{I}\right] \quad (333c)$$

is the *stress tensor* with $\boldsymbol{I}$ denoting unit matrix.

Closure problem

a commonly adopted closure condition is the ideal isotropic gas approximation

$$\boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u} = \frac{kT}{m}\boldsymbol{I}, \quad (334)$$

where T is the temperature and k the Boltzmann constant. For this closure condition, Eqs. (333a) and (333b) become to a closed system for ρ and $\boldsymbol{u}$.

$$\boldsymbol{\Xi} := -\left[\rho(\boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u}) + \frac{\varphi_0 a^3}{2m^2}\rho^2 \boldsymbol{I}\right]$$

$$\boldsymbol{\Sigma} - \boldsymbol{u}\boldsymbol{u} = \frac{kT}{m}\boldsymbol{I},$$

Traditionally, and in most practical applications, one does not bother with microscopic derivations of $\boldsymbol{\Xi}$; instead one merely postulates that

$$\boldsymbol{\Xi} = -p\boldsymbol{I} + \mu(\nabla^\top \boldsymbol{u} + \nabla \boldsymbol{u}^\top) - \frac{2\mu}{3}(\nabla \cdot \boldsymbol{u}), \tag{335}$$

where $p(t, \boldsymbol{x})$ is the pressure field and μ the dynamic viscosity, which can be a function of pressure, temperature etc. depending on the fluid. Equations (333a) and (333b) combined with the empirical ansatz (335) are the famous *Navier-Stokes equations*. The second summand in Eq. (335) contains the *rate-of-strain* tensor

$$\boldsymbol{E} = \frac{1}{2}(\nabla^\top \boldsymbol{u} + \nabla \boldsymbol{u}^\top) \tag{336}$$

and $(\nabla \cdot \boldsymbol{u})$ is the *rate-of-expansion* of the flow.

For incompressible flow, defined by $\rho = const.$, the Navier-Stokes equations simplify to

$$\nabla \cdot \boldsymbol{u} = 0 \tag{337a}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g}. \tag{337b}$$

In this case, one has to solve for $(p, \boldsymbol{u})$.

14 The Navier-Stokes Equations

$$\begin{aligned}\rho\frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} + 2\mu\sum\frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\\ &= -\nabla_i p + \mu\nabla_i(\nabla\cdot\mathbf{u}) + \mu\nabla^2 u_i.\end{aligned}$$

When the fluid density doesn't change very much we have seen that $\nabla\cdot\mathbf{u} = 0$, and under these conditions the *Navier-Stokes equations* for fluid motion are

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u}. \tag{347}$$

14.2 The Reynolds number

For an incompressible flow, we have established that the equations of motion are

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}_{ext}, \tag{348}$$

An important parameter that indicates the relative importance of viscous and inertial forces in a given situation is the *Reynolds number*. Suppose we are looking at a problem where the characteristic velocity scale is U_0, and the characteristic length scale for variation of the velocity is L. Then the size of the terms in the equation are

$$\frac{\partial \mathbf{u}}{\partial t} \sim \frac{U_0^2}{L}, \qquad \mathbf{u}\cdot\nabla\mathbf{u} \sim \frac{U_0^2}{L}, \qquad \mu\nabla^2\mathbf{u} \sim \frac{\mu U_0}{L^2}. \tag{349}$$

The ratio of the inertial terms to the viscous term is

$$\frac{\rho U_0^2/L}{\mu U_0/L^2} = \frac{\rho U_0 L}{\mu} = Re, \tag{350}$$