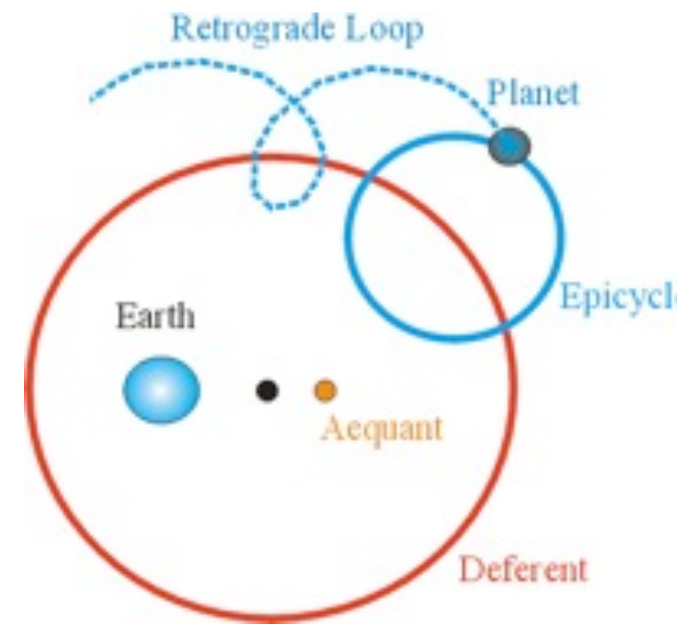
L03:
Kepler problem & Hamiltonian dynamics

18.354



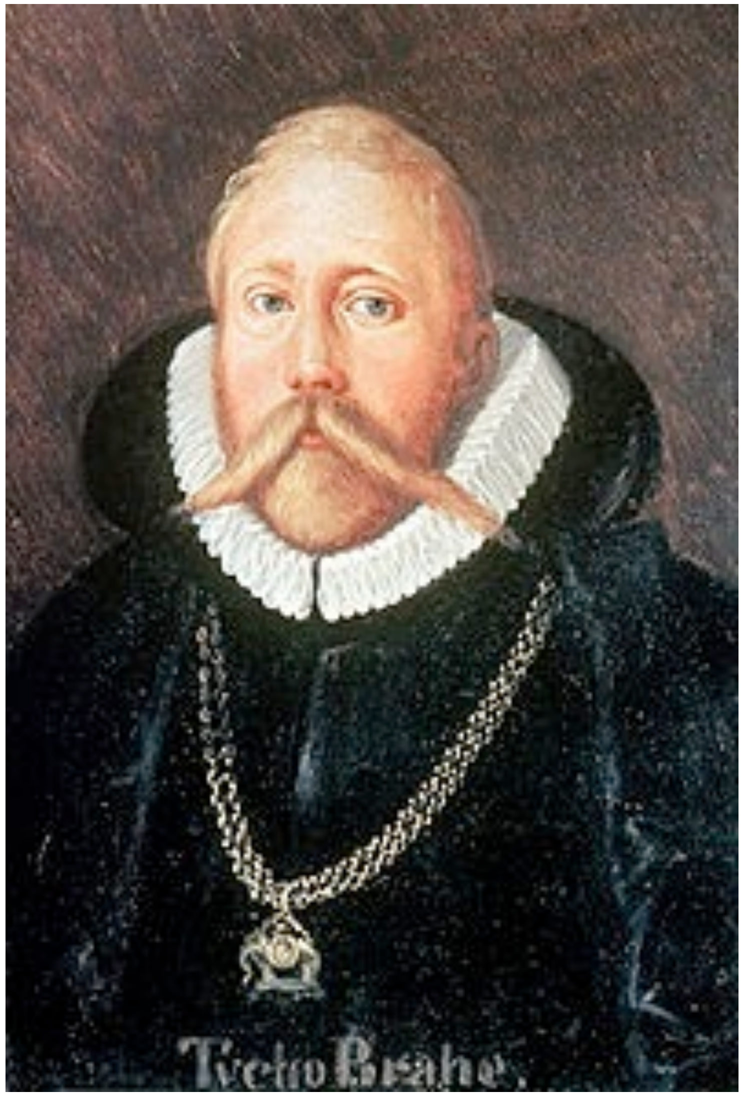
Ptolemy

circa.85 (Egypt) -165 (Alexandria)

<u>Greek geocentric view of the universe</u>

Tycho Brahe
1546 (Denmark) - 1601 (Prague)

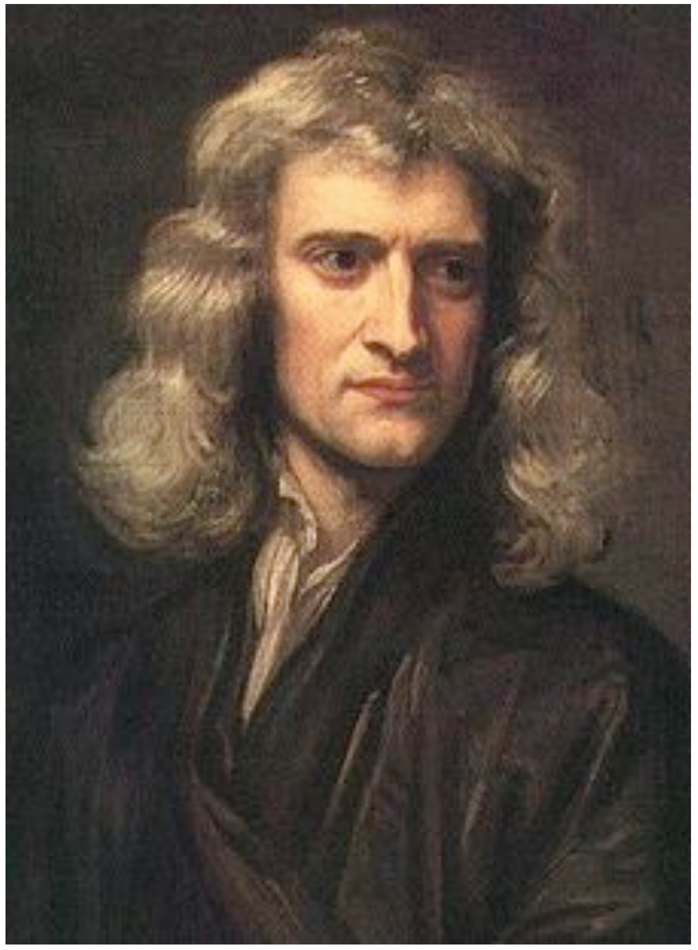
"geo-heliocentric" system



last of the major [naked eye](#) astronomers

Johannes Kepler
1571 - 1630 (Germany)

Kepler's 3 laws

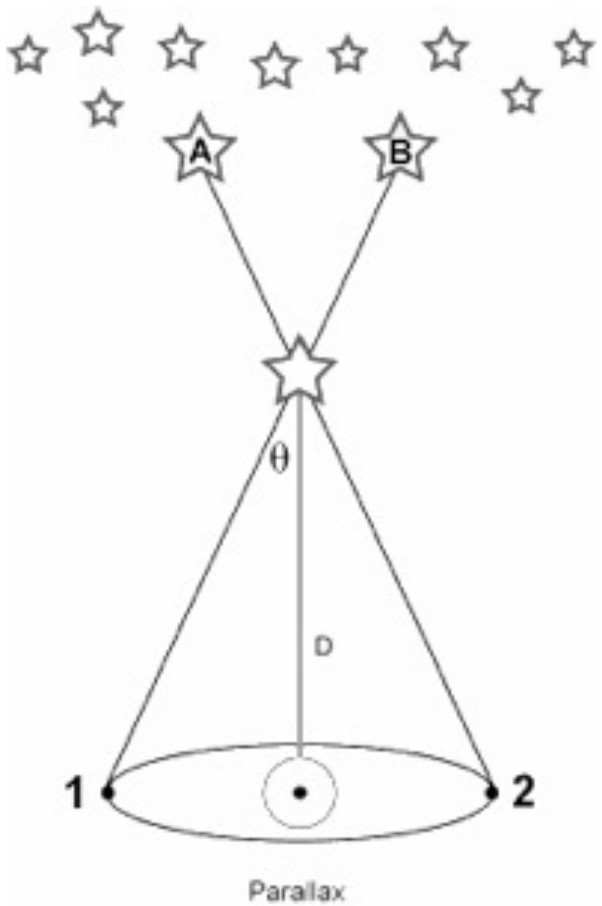


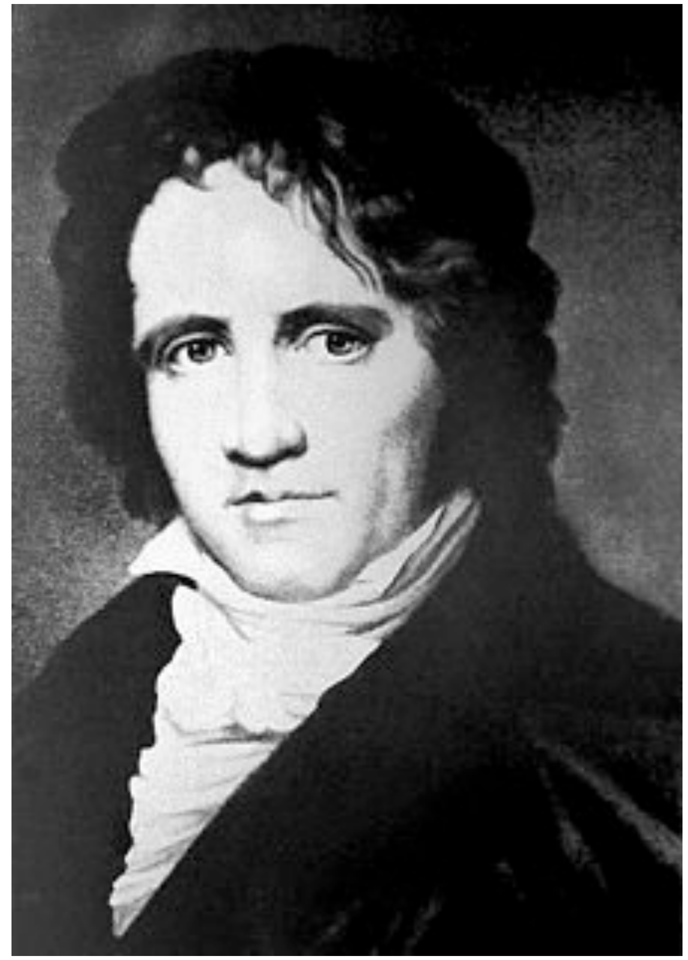
Isaac Newton
1643 - 1727 (England)

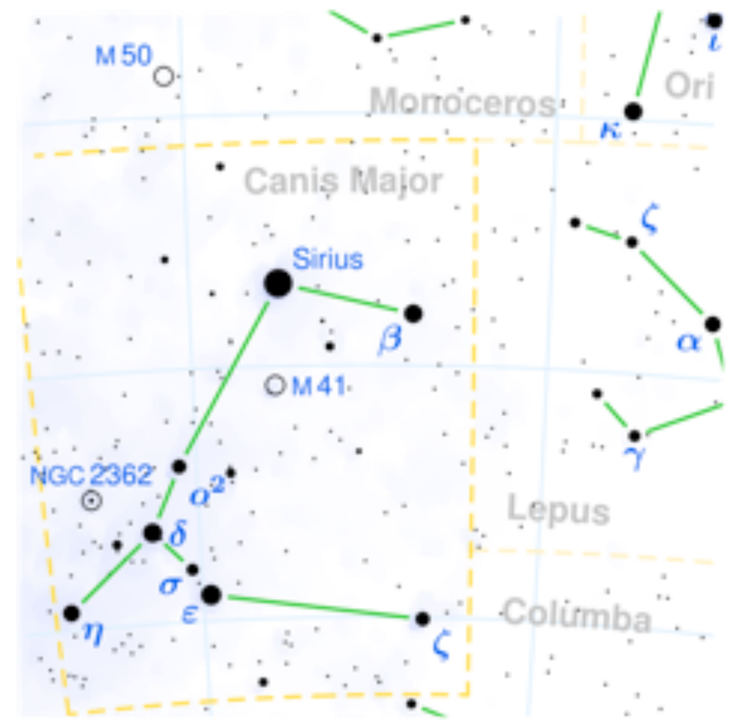
$$|\mathbf{F}_{12}| = G\frac{m_1 m_2}{r^2}$$

Friedrich Bessel

1784 (Germany) - 1846 (Prussia)





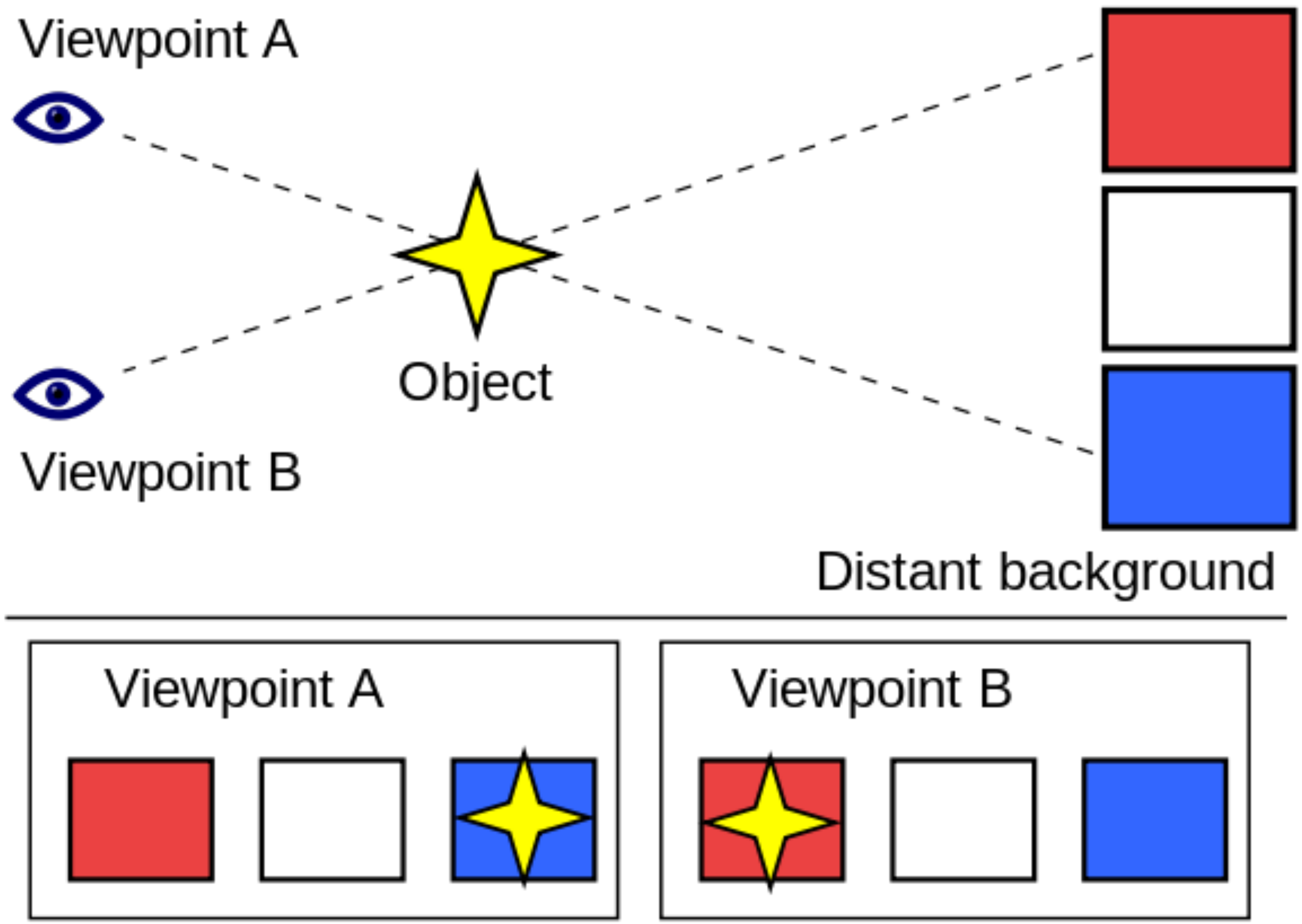


Cygni 61

d~10.3 ly

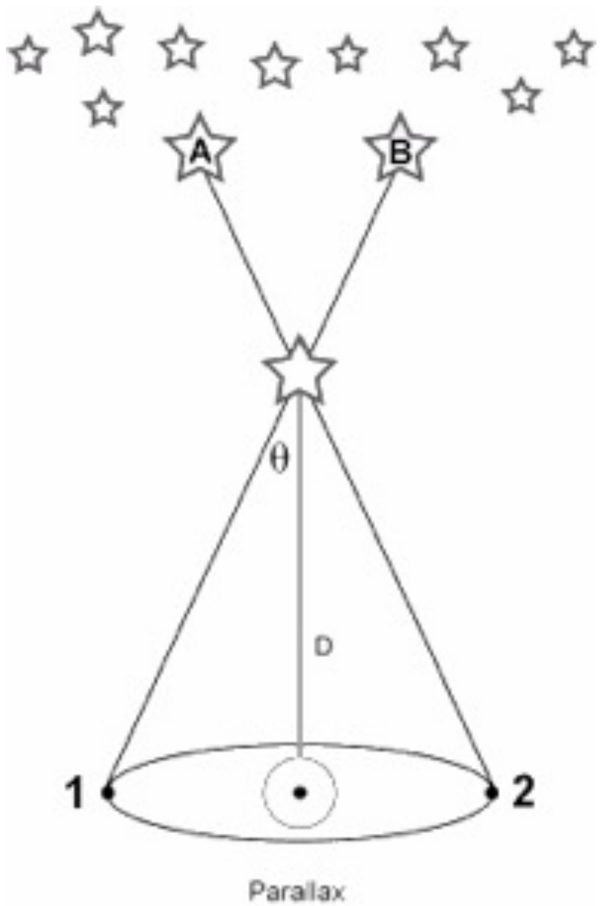
gravitational many-body problem:

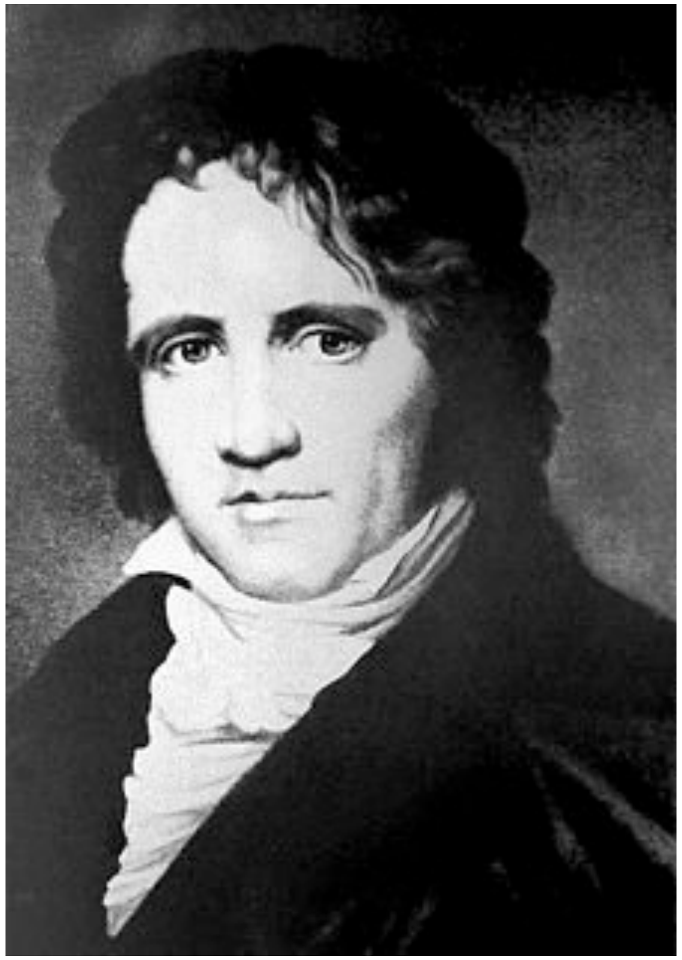
prediction of Sirius B, Bessel functions


Viewpoint A
Object
Viewpoint B
Distant background
Viewpoint A
Viewpoint B

Friedrich Bessel

1784 (Germany) - 1846 (Prussia)





Cygni 61
d~10.3 ly

gravitational many-body problem:
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4 Kepler's problem and Hamiltonian dynamics

4.1 Kepler's laws of planetary motion

In the early seventeenth century (1609-1619) Kepler proposed three laws of planetary motion

(i) The orbits of the planets are ellipses, with the Sun's centre of mass at one focus of the ellipse.

(ii) The line joining a planet and the Sun describes equal areas in equal intervals of time.

(iii) The squares of the periods of the planets are proportional to the cubes of their semi-major axes.

These laws were based on detailed observations made by Tycho Brahe, and put to rest any notion that planets move in perfectly circular orbits. However, it wasn't until Newton proposed his law of gravitation in 1687 that the origins of this motion were understood. Newton proposed that

> "Every object in the Universe attracts every other object with a force directed along a line of centres for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation of the two objects."

Based on this one statement, it is possible to derive Kepler's laws.

4.1.1 Second law

Keplers second law is the simplest to derive, and is a statement that the angular momentum of a particle moving under a central force, such as gravity, is constant. By definition, the angular momentum $\boldsymbol{L}$ of a particle with mass m and velocity $\boldsymbol{u}$ is

$$\boldsymbol{L} = \boldsymbol{r} \wedge m\frac{d\boldsymbol{r}}{dt}, \tag{53}$$

where $\boldsymbol{r}$ is the vector position of the particle. The rate of change of angular momentum is given by

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \wedge \boldsymbol{f} = \boldsymbol{r} \wedge f(r)\hat{\boldsymbol{r}} = 0, \tag{54}$$

where $f(r)\hat{\boldsymbol{r}}$ is the central force, depending only on the distance $r = |\boldsymbol{r}|$ and pointing in the direction $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. It can therefore be seen that the angular momentum of a particle moving under a central force is constant, a consequence of this being that motion takes place in a plane. The area swept out by the line joining a planet and the sun is half the area of the parallelogram formed by $\boldsymbol{r}$ and $d\boldsymbol{r}$. Thus

$$dA = \frac{1}{2}|\boldsymbol{r} \wedge d\boldsymbol{r}| = \frac{1}{2}\left|\boldsymbol{r} \wedge \frac{d\boldsymbol{r}}{dt}dt\right| = \frac{L}{2m}dt, \tag{55}$$

where $L = |\boldsymbol{L}|$ is a constant. The area swept out is therefore also constant.

4.1.2 First law

To prove Keplers first law consider the sun as being stationary (i.e., infinitely heavy), and the planets in orbit around it. The equation of motion for a planet is

$$m\frac{d^2\boldsymbol{r}}{dt^2} = f(r)\hat{\boldsymbol{r}}. \tag{56}$$

In plane polar coordinates

$$\frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \tag{57a}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \tag{57b}$$

In component form, equation (56) therefore becomes

$$m(\ddot{r} - r\dot{\theta}^2) = f(r), \tag{58a}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \tag{58b}$$

Putting (57a) into (53) gives

$$L = \left|\boldsymbol{r} \wedge m\frac{d\boldsymbol{r}}{dt}\right| = |mr^2\dot{\theta}|. \tag{59}$$

Thus

$$r^2\dot{\theta} = l, \tag{60}$$

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where $l = L/m$ is the angular momentum per unit mass. Given a radial force $f(r)$, equations (58a) and (58b) can now be solved to obtain r and θ as functions of t. A more practical result is to solve for $r(\theta)$, however, and this requires the definition of a new variable

$$r = \frac{1}{u}. \tag{61}$$

Rewriting the equations of motion in terms of the new variable requires the identities

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -l\frac{du}{d\theta}, \tag{62a}$$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}. \tag{62b}$$

Equation (58a) becomes

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f\left(\frac{1}{u}\right). \tag{63}$$

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This is the differential equation governing the motion of a particle under a central force. Conversely, if one is given the polar equation of the orbit $r = r(\theta)$, the force function can be derived by differentiating and putting the result into the differential equation. According to Newtons law of gravitation $f(r) = -k/r^2$, so that

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{ml^2}. \tag{64}$$

This has the general solution

$$u = A\cos(\theta - \theta_0) + \frac{k}{ml^2}, \tag{65}$$

where A and θ_0 are constants of integration that encode information about the initial conditions. Choosing θ_0=0 and replacing u by the original radial coordinate $r = 1/u$,

$$r = \left(A\cos\theta + \frac{k}{ml^2}\right)^{-1} \tag{66}$$

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which is the equation of a conic section with the origin at the focus. This can be rewritten in standard form

$$r = r_0 \frac{1+\epsilon}{1+\epsilon\cos\theta}, \tag{67}$$

where

$$\epsilon = \frac{Aml^2}{k}, \qquad r_0 = \frac{ml^2}{k(1+\epsilon)}. \tag{68}$$

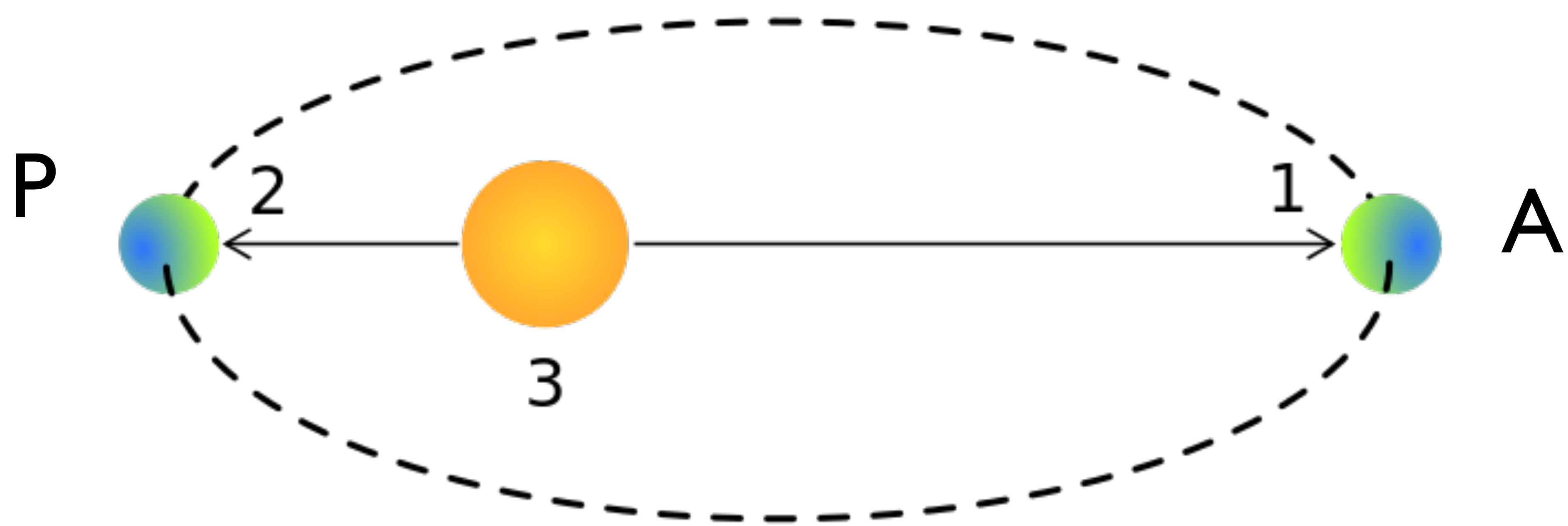
ϵ is called the eccentricity of the orbit:

- $\epsilon = 0$ is a circle,
- $\epsilon < 1$ is an ellipse,
- $\epsilon = 1$ is a parabola and
- $\epsilon > 1$ is a hyperbola.

For an elliptical orbit, r_0 is the distance of closest approach to the sun, and is called the *perihelion.* Similarly,

$$r_1 = r_0(1+\epsilon)/(1-\epsilon) \tag{69}$$

is the furthest distance from the sun and is called the *aphelion.* Orbital eccentricities are small for planets, whereas comets have parabolic or hyperbolic orbits. Interestingly though, Halley's comet has a very eccentric orbit but, according to the definition just given, is not a comet!



4.1.3 Third law

To prove Keplers third law go back to equation (55). Integrating this area law over time gives

$$A(\tau) = \int_0^\tau dA = \frac{l\tau}{2}, \tag{70}$$

where τ is the period of the orbit. The area A of an ellipse can also be written as $A = \pi ab$ where a and b are the semi-major and semi-minor axes, yielding

$$\frac{l\tau}{2} = \pi ab \tag{71}$$

The ratio of a and b can be expressed in terms of the eccentricity

$$\frac{b}{a} = \sqrt{1-\epsilon^2}. \tag{72}$$

Using this expression to substitute b in (71)

$$\tau = \frac{2\pi a^2}{l}\sqrt{1-\epsilon^2} \tag{73}$$

The length of the major axis is

$$2a = r_0 + r_1 = \frac{2ml^2}{k(1-\epsilon^2)}. \tag{74}$$

Squaring (71) and replacing gives

$$\tau^2 = \frac{4\pi^2 m}{k}a^3, \tag{75}$$

confirming Kepler's 3rd law.

4.2 Hamiltonian dynamics of many-body systems

The Kepler problem is essentially a two-body problem. In the remainder of this course, we will be interested in classical (non-quantum) systems that consist of $N \gg 2$ particles. The complete microscopic dynamics of such systems is encoded in their Hamiltonian

$$H = \sum_{n=1}^{N} \frac{\boldsymbol{p}_n^2}{2m_n} + U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \tag{76a}$$

where m_n, $\boldsymbol{p}_n(t)$ and $\boldsymbol{x}_n(t)$ denote the mass, momentum and position of the nth particle. The first contribution on the rhs. of Eq. (76a) is the kinetic energy, and U is the potential energy. For our purposes, it is sufficient to assume that we can decompose (76a) into a sum of pair interactions

$$U(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \frac{1}{2} \sum_{n,k:n\neq k} \Phi(\boldsymbol{x}_n, \boldsymbol{x}_k). \tag{76b}$$

Given H, Newton's equations can be compactly rewritten as

$$\dot{\boldsymbol{x}}_n = \nabla_{\boldsymbol{p}_n} H \ , \qquad \dot{\boldsymbol{p}}_n = -\nabla_{\boldsymbol{x}_n} H. \tag{77}$$

That this so-called Hamiltonian dynamics is indeed equivalent to Newton's laws of motion can be seen by direct insertion, which yields

$$\dot{\boldsymbol{x}}_n = \frac{\boldsymbol{p}_n}{m_n} \ , \qquad \dot{\boldsymbol{p}}_n = m_n \ddot{\boldsymbol{x}}_n = -\nabla_{\boldsymbol{x}_n} U. \tag{78}$$

Conservation laws

An important observation is that many physical systems obey certain conservation laws. For instance, the Hamiltonian (76a) itself remains conserved under the time-evolution (77)

$$\begin{aligned} \frac{d}{dt}H &= \sum_n \left[(\nabla_{\boldsymbol{p}_n} H) \cdot \dot{\boldsymbol{p}}_n + (\nabla_{\boldsymbol{x}_n} H) \cdot \dot{\boldsymbol{x}}_n \right] \\ &= \sum_n \left[(\nabla_{\boldsymbol{p}_n} H) \cdot (-\nabla_{\boldsymbol{x}_n} H) + (\nabla_{\boldsymbol{x}_n} H) \cdot \nabla_{\boldsymbol{p}_n} H \right] \equiv 0. \end{aligned} \tag{79}$$

which is just the statement of energy conservation. Other important examples of conserved quantities are total linear momentum and angular momentum,

$$\boldsymbol{P} = \sum_n \boldsymbol{p}_n \,, \qquad \boldsymbol{L} = \sum_n \boldsymbol{x}_n \wedge \boldsymbol{p}_n \tag{80}$$

if the pair potentials Φ only depend on the distance between particles.

Reflection of underlying symmetry ! (Noether's theorem)

4.3 Practical limitations

... too many particles in interesting systems

Solution: **continuum** theory ... leads to new challenges

(i) How does one write down macroscopic descriptions in terms of microscopic constants in a systematic way? It would be terrible to have to solve 10^{23} coupled differential equations!

(ii) Forces and effects that *a priori* appear to be small are not always negligible. This turns out to be of fundamental importance, but was not recognised universally until the 1920's.

(iii) The mathematics of how to solve 'macroscopic equations', which are nonlinear partial differential equations, is non-trivial. We will need to introduce many new ideas.