**18.354/12.207 – Nonlinear Dynamics II: Continuum systems**
**Mid-term - take home.**
Due: Friday, April 25 by 1pm in E17-412.

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The above deadline is final. There will be no extensions. For this take-home exam you are allowed to use Acheson and your class notes. No other references are allowed. No collaboration or consultation with others is allowed.

**Problem 1:** DIFFUSION DRIVEN FLOWS

Diffusion driven flow in a stratified environment is an interesting example of a counterintuitive problem in fluid mechanics that finds applications in oceanography. In such systems, diffusion can drive flow along a wall inclined at angle $\alpha$ to the horizontal, placed in a stratified fluid (*note*: in a stratified fluid the density varies with height due to the presence of a component, such as salt or temperature). A schematic diagram of the problem is shown in Fig. 1.

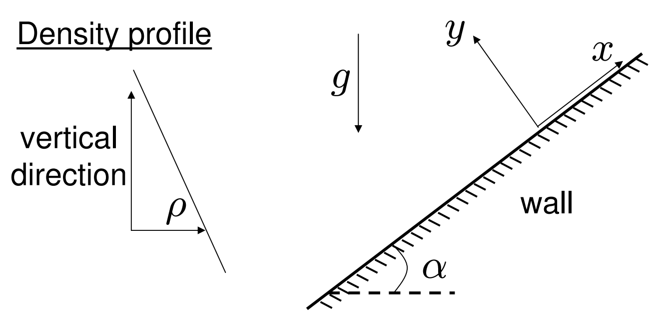



Figure 1: Setup for Problem 1: diffusion driven flow.

(i) Making the $x$- and $y$-axes parallel and perpendicular to the wall (see Fig. 1), and assuming a flow of the form $\mathbf{u} = (u(y), 0)$, show that the Navier-Stokes equations and the advection-diffusion equation reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} - \rho g \sin\alpha, \quad (1)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha, \quad (2)$$

$$u\frac{\partial \rho}{\partial x} = \kappa \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right). \quad (3)$$

where $\kappa$ is the diffusion coefficient of the quantity affecting the density. Note that the advection-diffusion equation is also called the transport equation, which you derived in problem set 2 – $Dc/Dt = \kappa\nabla^2 c$ – for the concentration $c$ of the component. You may assume $c$ is proportional to the density $\rho$, i.e. $c = a\rho$, where $a$ is a constant.

(ii) At the inclined surface *the velocity and the normal density gradient vanish*, while far away the velocity vanishes and the density distribution reduces to its undisturbed state,

$$\rho \to \rho_0 + K(x \sin\alpha + y \cos\alpha).$$

This corresponds to a linear density gradient in the vertical direction. Note that $K$ should be a negative number so that we have light fluid over heavy fluid and $\rho_0$ is a reference density far away from the wall along the line $x\sin\alpha + y\cos\alpha = 0$. Seeking a solution for the density of the form

$$\rho = \rho_0 + K(x \sin\alpha + y\cos\alpha) + \rho_0 f(y), \tag{4}$$

show that $f(y)$ satisfies

$$\frac{d^4 f}{dy^4} = -4\gamma^4 f, \tag{5}$$

where $\gamma$ is given below (see hint), and determine $f(y)$ for solutions of the type $f = e^{\delta y}$.

(*Hint : You will want to eliminate pressure from your equations. To make your solution more presentable, it will help to define* $N^2 = -Kg/\rho_0$ *and* $\gamma = (N^2\rho_0 \sin^2\alpha/4\mu\kappa)^{\frac{1}{4}}$. $N$ *is a well known quantity in fluid dynamics, called buoyancy frequency, and corresponds to the natural frequency of oscillation within a stratified fluid.* Note that $\delta$ may be a complex number, whereas you want your final result for $f(y)$ to be real).

(iii) From your solution for the density find the velocity $u(y)$. Then sketch both $u\,(y)$ and $f\,(y)$ as functions of the coordinate $y$ away from the wall. Write a sentence about what you see in the plots. Comment on anything unphysical in the analytic solution for $u\,(y)$. In case you had trouble in (ii), assume that $u(y)$ can be found from

$$u(y) = \frac{\kappa\rho_o}{K\sin\alpha}\frac{d^2 f}{dy^2}$$

**Problem 2:** MODELING A TORNADO

A horizontal slice through a tornado can be modeled by two distinct regions. The *inner* or *core region* $(0 < r < R)$ is modeled by solid body rotation – a rotating but inviscid region of flow. The *outer region* $(r > R)$ is modeled as an irrotational regions of flow. The flow is assumed to be two-dimensinoal in the $(r, \theta)$ plane, and the components of the velocity field $\mathbf{u} = (u_r, r_\theta)$ are given by

$$u_r = 0 \qquad\qquad u_\theta = \begin{cases} \omega r & \text{if } 0 < r < R \\ \frac{\omega R^2}{r} & \text{if } r > R \end{cases} \tag{6}$$

where $\omega$ is the magnitude of the angular velocity in the inner region. The ambient pressure far away from the tornado (at $r \to \infty$) is $P_\infty$ and the fluid there ($\infty$) is stationary. You can neglect gravity effects - an additional hydrostatic pressure field exists in the z-direction but does not affect the dynamics of the flow. The pressure along the z-axis is $P_o(z)$ which is constant for a particular slice. Moreover, assume that the flow is steady, incompressible and inviscid.

(i) Calculate the pressure field in the *inner region*, $P(r)$ (for $0 < r < R$), in a horizontal slice of the tornado. Although $R$ increases and $\omega$ decreases with increasing elevation, $z$, assume that both $R$ and $\omega$ are constants in a particular horizontal slice.

(ii) Calculate the pressure field in the *outer region*, $P(r)$ (for $R < r < \infty$) for the same horizontal slice of the tornado as in (i).

(iii) What is the pressure at $r = R$ and $r = 0$ (as a function of $P_\infty$). Then, re-write your results of (i) and (ii) involving $P_\infty$ only (not $P_o$) in dimensionless form, i.e.

$$\bar{u_\theta}(r) = \frac{u_\theta}{\omega R} =? \qquad \text{and} \qquad \bar{P}(r) = \frac{P(r) - P\infty}{\rho \omega^2 R^2} =? \tag{7}$$

(iv) Plot both the pressure and velocity fields in the dimensionless form you found in (iii).

**Problem 3:** BENDING OF A THIN ELASTIC SHEET UNDER GRAVITY

In class we saw that, in the linearized limit of small deformations, the elastic bending energy of a two dimensional sheet with shape $y(x)$ is

$$U[y(x)] = \frac{1}{2}\left(\frac{Yh^3}{12(1-\sigma^2)}\right)\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx, \tag{8}$$

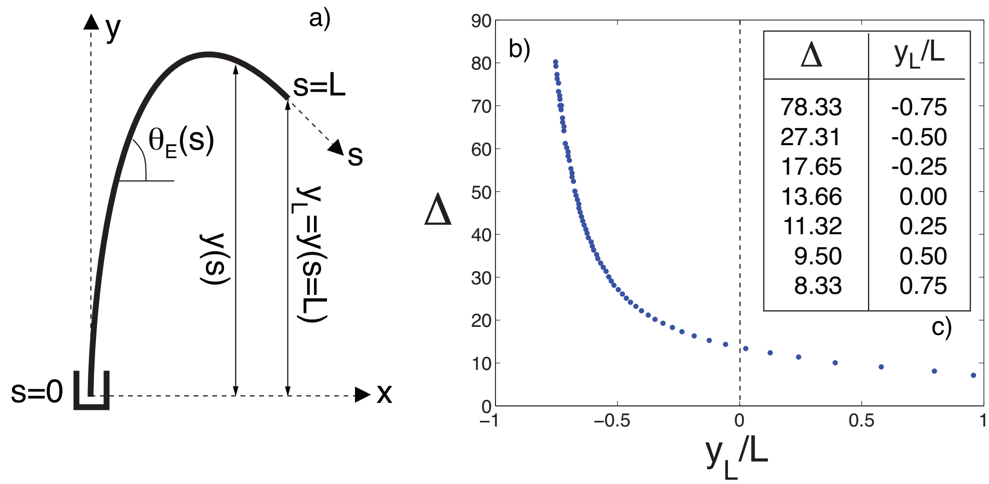
where $l$ is the projected length of the beam, $h$ is the thickness of the sheet and $Y$ is it's Young's modulus. The term $B = Yh^3/[12(1-\sigma^2)]$ is often referred to the bending modulus which measures how stiff the sheet is under bending deformations.

For large deformations, the corresponding bending energy (per unit width) is

$$U[y(s)] = \frac{1}{2}B\int_0^L \left(\frac{d\theta}{ds}\right)^2 ds, \tag{9}$$

where $L$ is the total length of the sheet, $\theta$ is the angle that the sheet makes with the horizontal and $s$ is the arclength along the neutral surface of the sheet. Eqn. (9) can be read in *english* as: "*The bending energy of a thin sheet per unit width equals one half of the bending modulus times the curvature ($d\theta/ds$) squared, integrated along its total length.*"

Consider the following configuration. A thin sheet of thickness $h$, width $b$ and total length $L$ is held vertically at $s = 0$ and is free to bend under gravity otherwise. The boundary condition are that: 1) the sheet is vertical at the clamp $\theta(s = 0) = \pi/2$ and 2) there is no curvature at the free end $(d\theta/ds)(s = L) = 0$. Assume that the sheet has a linear density (mass per unit length) given by $\rho_l = \rho hb$, where $\rho$ is the volumetric density. A schematic diagram of this configuration is given in Fig. 2a).




| Δ | $y_L/L$ |
|---|---|
| 78.33 | -0.75 |
| 27.31 | -0.50 |
| 17.65 | -0.25 |
| 13.66 | 0.00 |
| 11.32 | 0.25 |
| 9.50 | 0.50 |
| 8.33 | 0.75 |

Figure 2: a) Schematic diagram of a thin sheet clamped vertically and bending under gravity. The sheet has dimensions: length $L$, thickness $h$ and width/span $b$ Note that $y_L$ is the height of the free end of the sheet at $s = L$ b) Plot of the dimensionless parameters $\Delta$ and $y_L/L$ for equilibrium shapes $\theta_E$ that satisfy Eqn. (12). c) Table for some values of $\Delta$ v.s. $y_L/L$.

(i) Write the total energy functional, $\mathcal{E}[\theta(s)]$, for this thin strip bent under gravity. Make sure that your expression for the energy only depends on $s$, $\theta(s)$ and $d\theta/ds$. (*hint: the fact that* $dy = \sin\theta ds$ *i.e.* $y(s) = \int_0^s \sin[\theta(s)]ds$ *will help*).

(ii) Assuming small functional perturbations, $\delta\theta(s)$, on the equilibrium shape of the beam $\theta_E(s)$, use calculus of variations ($\delta\mathcal{E} \sim \mathcal{E}[\theta_E(s) + \delta\theta(s)] - \mathcal{E}[\theta_E(s)] \to 0$) to show that the equilibrium shape satisfies the following differential equation

$$Bb\frac{d^2\theta_E}{ds^2} = \rho_l g(L - s)\cos\theta_E \qquad (10)$$

(iii) Non-dimensionalize Eqn. (10) using $L$ and show that

$$\frac{d^2\theta_E}{d\bar{s}^2} = \Delta(1 - \bar{s})\cos\theta_E, \qquad (11)$$

where $\Delta = (L/L_c)^3$ and $L_c$ is often called the elasto-gravity lengthscale. What is $L_c$ in terms of the physical quantities in the problem? What is its physical significance of $L_c$?

(iv) Determine $L_c$ from scaling arguments and ensure that you get the same result as in (iii).

(v) Show that the differential equation in (11) is identical to

$$\frac{1}{2}\left(\frac{d\theta_E}{d\bar{s}}\right)^2 = \Delta\left[(1 - \bar{s})\sin\theta_E + (\bar{y} - \bar{y}_L)\right], \qquad (12)$$

where $\bar{y} = y/L$, $\bar{y}_L = y_L/L$ and $y_L$ is the height of the free end with respect to the horizontal (see the diagram in Fig. 2a). (*hint: it will be easier to go backwards from Eqn. (12) to Eqn. (11). Extra points if you do it the other way round.*)

(vi) The differential equation that describes the equilibrium shapes of the bent sheet under gravity – Eqn. (12) – is nonlinear and therefore one has to solve it numerically to find $\theta_E(s)$ (that yields the equilibrium shapes). One of your friends has done this for you (he is good with MATLAB!) and plotted $\Delta$ as a function of $y_L/L$ in Fig. 2b) (some points of this graph are given in the table of Fig. 2c). Develop a method to calculate the bending modulus $B$ of a piece of paper (*hint: you may need only one of the points in the table of Fig. 2c*).

(viii) Using your technique, calculate the numerical value of $B$ for the white paper sheet in the printers of the Athena clusters (*hint: you will need the paper density that you will find written on the label packages of the paper rims. Most likely it will be* $75g/m^2$)

**Problem 4:** A SPINNING CYLINDER

In this question, you will use a complex potential to analyze a spinning cylinder of radius $a$ in a uniform flow of speed $U$ (and density $\rho$). (This is described in section 4.5 of Acheson, although the notation is slightly different.)

(i) Verify that the complex potential

$$w(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln\frac{z}{a} \tag{13}$$

correctly describes this situation: Check that (1) the boundary of the cylinder $z = ae^{i\theta}$ is a streamline in this flow, and (2) far away from the cylinder, the flow is the uniform flow given by $\mathbf{u} = (U, 0)$.

(ii) Use Bernoulli's theorem to calculate the pressure $p$ at any location $\theta$ on the cylinder and the total lift and drag. (Use $p_0$ to refer to the pressure far away from the cylinder.)

(iii) Find a formula for the location $z$ of the stagnation points and draw a sketch of the cylinder, the stagnation points, and the streamlines for the three cases $\Gamma/4\pi aU < 1$, $\Gamma/4\pi aU = 1$, and $\Gamma/4\pi aU > 1$. (Assume $\Gamma > 0$. For $\Gamma < 0$ the problem is identical but with the cylinder moving downward instead.)

(iv) In real life, the flow around a cylinder doesn't look like this. Forgetting the spinning for now, at moderate Reynolds number, a cylinder "sheds" vortices. In this situation, what assumption(s) that we used in the first three parts of this question were incorrect? If $n$ is the frequency of vortex shedding ($n$ vortices per second, say), what dimensionless parameter(s) does this problem involve?