

## 14 Surface Tension

Moving on, we shall now use our knowledge of variational principles to investigate the shape of water droplets and soap films. The surface  $S$  of a droplet has energy

$$E[S] = \gamma \int dS \quad (1)$$

with the coefficient of proportionality  $\gamma$  being the *surface tension*. The questions we now want to answer are: What are the equilibrium shapes predicted by this energy? How can we incorporate constraints, such as fixed volume, into the problem? Is the solution stable? First we will look at the problem in one dimension, and then extend our analysis to many dimensions.

### 14.1 Two-dimensional bubble

Consider a two-dimensional bubble, corresponding to an area  $B$  confined by a non-intersecting closed path  $\partial B$  in the plane. The surface energy is

$$E[y] = \gamma \int_{\partial B} ds = \gamma \int_{\partial B} \sqrt{1 + y'^2} dx. \quad (2)$$

We have already solved the problem to determine the curve that connects two points while satisfying  $\delta E/\delta y = 0$ , the solution being a straight line. Now we need to introduce the constraint of a fixed volume, and this is done by adding a Lagrange multiplier. Thus we consider the functional

$$E[y] = \gamma \int_{\partial B} \sqrt{1 + y'^2} dx - \lambda \int_B y dx \quad (3)$$

the second integral being the volume of the bubble. We consider a deformation of the surface  $y + \delta y$  and determine that

$$\delta E = E[y + \delta y] - E[y] \sim \gamma \int \delta y \left( \frac{y'}{\sqrt{1 + y'^2}} \right)' dx - \lambda \int \delta y dx. \quad (4)$$

Since  $\delta y$  is arbitrary, the condition for energy minimisation ( $\delta E = 0$ ) is

$$\left( \frac{y'}{\sqrt{1 + y'^2}} \right)' = \lambda. \quad (5)$$

The term on the left hand side is called the *mean curvature* of the surface, and the solution is readily shown to be a circle (as we expect). You can test this yourself since you know the equation of a circle is just  $x^2 + y^2 = r^2$ .

## 14.2 Soap film between two hoops

Now let's be a little more ambitious and consider a cylindrically symmetric two dimensional surface, confined by two parallel rings located at  $x = -a$  and  $x = a$ . Given the axi-symmetry of the problem, we may express the surface energy is

$$E = \gamma \int_A dA = \gamma \int_A 2\pi r ds. \quad (6)$$

The arclength  $ds$  is simply  $\sqrt{1 + r'^2}dx$ , and we see that through symmetry we have reduced this to a one dimensional problem. The problem to be solved is now

$$E[r] = 2\pi\gamma \int_{-a}^a r\sqrt{1 + r'^2} dx \stackrel{!}{=} \min. \quad (7)$$

Putting the integrand into the Euler-Lagrange equation gives

$$\sqrt{1 + r'^2} - \frac{d}{dx} \left( \frac{rr'}{\sqrt{1 + r'^2}} \right) = 0 \quad (8)$$

which can be simplified to

$$rr'' - r'^2 = 1 \quad (9)$$

We can see immediately that a particular solution of this equation is given by the *catenoid*<sup>1</sup>

$$r = a \cosh(x/a), \quad (10)$$

which corresponds to the special case  $R/a = \cosh(1)$ . For other ratios, we need to solve (9) numerically.

## 14.3 Rayleigh-Plateau Instability

In the previous section, we considered the shape of a soap film, stretched between two hoops. We know, however, that the film breaks after a certain extension, but it is not clear that there is anything in our problem to account for this. What is wrong? To answer this question we must analyze the stability of the solution we have found. This is a topic that we will address in more detail later on in the course, but shall touch upon now.

In one-dimensional calculus we check if our extremum is a maximum or a minimum, and if it is the former it is unstable. We now adopt the same approach in variational calculus, and consider the instability of a thin cylinder of fluid. This was studied by Plateau in the 1870's, who was interested in the formation of droplets from a jet. In cylindrical coordinates  $y(x) = r_0$  for the perfect cylinder. The cylinder has surface energy and we look for an extremum by considering the functional

$$E[r] = \gamma \int 2\pi r \sqrt{1 + r'^2} dx - \lambda \int \pi r^2 dx, \quad (11)$$

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<sup>1</sup>You can find more soap film solutions at <http://www.susqu.edu/brakke/evolver/examples/cat/catnooids.html>. There, you will also download a surface evolver code to investigate minimal surfaces in more detail.

where the second part is just the volume constraint.

We shall perturb the shape and show that if the wavelength of disturbance is greater than  $2\pi r_0$ , the energy of the system decreases. Using the analysis from above, we know that an extremum is required to satisfy the Euler-Lagrange equations

$$\gamma \left( \frac{1}{\sqrt{1+r'^2}} - \frac{r r''}{(1+r'^2)^{\frac{3}{2}}} \right) = \lambda r. \quad (12)$$

For the perfect cylinder  $y(x) = r_0$  is a solution with  $\lambda = \gamma/r_0$ . Note that  $\lambda$  has dimensions of force/area, and is in fact the pressure. We now consider a perturbation such that

$$r(x) = r_0 + \epsilon \cos kx. \quad (13)$$

This perturbation is in some sense arbitrary, because any perturbation can be decomposed into sines and cosines. Thus

$$E[r] = \gamma \int \left[ 2\pi(r_0 + \epsilon \cos kx) \sqrt{1 + \epsilon^2 k^2 \sin^2 kx} - \frac{\pi}{r_0} (r_0 + \epsilon \cos kx)^2 \right] dx. \quad (14)$$

Assuming that  $\epsilon$  is small, we can expand the square root, yielding

$$E[r] = \gamma \int \left[ 2\pi(r_0 + \epsilon \cos kx) \left( 1 + \frac{1}{2} \epsilon^2 k^2 \sin^2 kx \right) - \frac{\pi}{r_0} (r_0 + \epsilon \cos kx)^2 \right] dx. \quad (15)$$

Expanding everything out, we find that terms linear in  $\epsilon$  cancel out, and we are left with

$$E[r] = \gamma \int \pi r_0 dx + \epsilon^2 \gamma \pi \int \left[ r_0 k^2 \sin^2 kx - \frac{\cos^2 kx}{r_0} \right] dx. \quad (16)$$

The first term corresponds to the energy of the undeformed problem, so we are really interested in the sign of the second integral. If it is positive then the energy is increased as a result of the perturbation, and the system will be stable. However, if the integral is negative, then by perturbing the system the energy is decreased, and the system must be unstable.

Since the disturbance is periodic, we integrate over one wavelength,  $2\pi/k$ , to determine the energy per wavelength. We know that

$$\int_0^{2\pi} dx \sin^2 kx = \int_0^{2\pi} dx \cos^2 kx = \frac{\pi}{k}, \quad (17)$$

so that the energy change,  $\delta E$ , is

$$\delta E = \epsilon^2 \gamma \pi^2 \left( r_0 k - \frac{1}{r_0 k} \right). \quad (18)$$

So now we see that if  $r_0 k > 1$  the system is stable, because  $\delta E$  is positive. However, if  $r_0 k < 1$  then  $\delta E$  is negative and the system is unstable. Since  $k = 2\pi/\lambda$  this can be rearranged to say that the system is unstable to wavelengths  $\lambda$  greater  $2\pi r_0$ , the circumference of the cylinder. This was Plateau's results, which Rayleigh found to be wrong by a factor

of  $\sqrt{2}$ , due to hydrodynamic effects within the cylinder (i. e.  $\lambda = 2\sqrt{2}\pi r_0$  is the correct answer). The physical reasoning arises from the fact that by perturbing the cylinder you may be increasing the length of the surface along the cylinder, but you are reducing the cross sectional area. It is the trade off between these two that determines the stability.

So does this answer our question of why the soap film breaks? Perhaps. You see, if the distance between the two hoops is less than the  $2\pi r_0$ , then the system is not open to long wavelength perturbations, and is therefore stable. As soon as the separation exceeds the critical wavelength, instability sets in and the bubble breaks. Note, however, that a critical part of our stability analysis relied on the constraint of constant volume to determine instability. With the soap film this is not present, so perhaps an alternative mechanism is responsible (e.g., possibly the shape of the soap film between the two hoops becomes so eccentric that surfaces in the middle touch, and there is a pinch off?). What would happen if the ends of the two hoops were closed, so that the volume constraint did then apply? Our stability analysis would now seem more relevant. The only way to check is to now go do some experiments (a good course project!).