9 The coffee cup (Acheson, pp. 42-46)

Let's try and apply our knowledge of fluid dynamics to a real observation, to test whether the theory actually works. We shall consider the problem you have in the second problem set: how long does it take a cup of coffee (or glass of water) to spin down if you start by stirring it vigorously? To proceed, we need a model of a coffee cup. For mathematical simplicity, let's just take it to be an infinite cylinder occupying $r \le R$. Suppose that at $t = 0$ the fluid and cylinder are spinning at an angular frequency ω, and then the cylinder is suddenly brought to rest. Assuming constant density, we expect the solution to be cylindrically symmetric

$$p(\boldsymbol{x}, t) = p(r, t) \;, \qquad \boldsymbol{u}(\boldsymbol{x}, t) = u_\phi(r, t)\boldsymbol{e}_\phi, \tag{1}$$

with only a component of velocity in the angular direction $\boldsymbol{e}_\phi$. This component will only depend on r, not the angular coordinate or the distance along the axis of the cylinder (because the cylinder is assumed to be infinite).

We shall just plug the assumed functional form into the Navier-Stokes equations and see what comes out. Before tackling this, we need to deal with one mathematical complication; how to write the equations in cylindrical coordinates. These equations are written out completely in Acheson (p. 42). We will discuss their various features in due course. Before, let us briefly recall how vector fields and derivatives can be decomposed in Cartesian and cylindrical coordinate systems.

Cartesian coordinates In a global orthornormal Cartesian frame $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$, the position vector is given by $\boldsymbol{x} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$, and accordingly the flow field $\boldsymbol{u}(\boldsymbol{x})$ can be represented in the form

$$\boldsymbol{u}(\boldsymbol{x}) = u_x(x, y, z)\,\boldsymbol{e}_x + u_y(x, y, z)\,\boldsymbol{e}_y + u_z(x, y, z)\,\boldsymbol{e}_z. \tag{2a}$$

The gradient vector is given by

$$\nabla = \boldsymbol{e}_x\partial_x + \boldsymbol{e}_y\partial_y + \boldsymbol{e}_z\partial_z, \tag{2b}$$

and, using the orthonormality $\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}$, the Laplacian is obtained as

$$\Delta = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{2c}$$

One therefore finds for the vector-field divergence

$$\nabla \cdot \boldsymbol{u} \;=\; \partial_i u_i \;=\; \partial_x u_x + \partial_y u_y + \partial_z u_z \tag{2d}$$

and the vector-Laplacian

$$\Delta\boldsymbol{u} \;=\; \partial_i\partial_i\boldsymbol{u} \;=\; \begin{pmatrix} \partial_x^2 u_x + \partial_y^2 u_x + \partial_z^2 u_x \\ \partial_x^2 u_y + \partial_y^2 u_y + \partial_z^2 u_y \\ \partial_x^2 u_z + \partial_y^2 u_z + \partial_z^2 u_z \end{pmatrix}. \tag{2e}$$

Cylindrical coordinates The local cylindrical basis vectors $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ are defined by

$$\boldsymbol{e}_r = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y \ , \qquad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y \ , \qquad \phi \in [0, 2\pi) \tag{3a}$$

and they form a orthonormal system $\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}$, where now $i, j = r, \phi, z$. The volume element is given by

$$dV = r\ \sin\phi\, dr\, d\phi\, dz. \tag{3b}$$

In terms of cylindrical basis system, the position vector $\boldsymbol{x}$ can be expressed as

$$\boldsymbol{x} = r\boldsymbol{e}_r + z\boldsymbol{e}_z \ , \qquad r = \sqrt{x^2 + y^2} \tag{3c}$$

and the flow field $\boldsymbol{u}(\boldsymbol{x})$ can be decomposed in the form

$$\boldsymbol{u}(\boldsymbol{x}) = u_r(r, \phi, z)\, \boldsymbol{e}_r + u_\phi(r, \phi, z)\, \boldsymbol{e}_\phi + u_z(r, \phi, z)\, \boldsymbol{e}_z. \tag{3d}$$

The gradient vector takes the form

$$\nabla = \boldsymbol{e}_r\partial_r + \boldsymbol{e}_\phi\frac{1}{r}\partial_\phi + \boldsymbol{e}_z\partial_z, \tag{3e}$$

yielding the divergence

$$\nabla \cdot \boldsymbol{u} \quad = \quad \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\phi u_\phi + \partial_z u_z. \tag{3f}$$

The Laplacian of a scalar function $f(r, \phi, z)$ is given by

$$\nabla^2 f \quad = \quad \frac{1}{r}\partial_r(r\partial_r f) + \frac{1}{r^2}\partial_\phi^2 f + \partial_z^2 f \tag{3g}$$

and the Laplacian of a vector field $\boldsymbol{u}(r, \phi, z)$ by

$$\nabla^2\boldsymbol{u} = L_r\boldsymbol{e}_r + L_\phi\boldsymbol{e}_\phi + L_z\boldsymbol{e}_z \tag{3h}$$

where

$$L_r \quad = \quad \frac{1}{r}\partial_r(ru_r) + \frac{1}{r^2}\partial_\phi^2 u_r + \partial_z^2 u_r - \frac{2}{r^2}\partial_\phi u_\phi - \frac{1}{r^2}u_r \tag{3i}$$

$$L_\phi \quad = \quad \frac{1}{r}\partial_r(ru_\phi) + \frac{1}{r^2}\partial_\phi^2 u_\phi + \partial_z^2 u_\phi + \frac{2}{r^2}\partial_\phi u_r - \frac{1}{r^2}u_\phi \tag{3j}$$

$$L_z \quad = \quad \frac{1}{r}\partial_r(ru_z) + \frac{1}{r^2}\partial_\phi^2 u_z + \partial_z^2 u_z \tag{3k}$$

Compared with the scalar Laplacian, the additional terms in the vector Laplacian arise from the coordinate dependence of the basis vectors.

Similarly, one finds that, the r-component of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is not simply $(\boldsymbol{u}\cdot\nabla)u_r$, but instead

$$e_r \cdot [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] = (\boldsymbol{u} \cdot \nabla)u_r - \frac{1}{r}u_\phi^2. \tag{4}$$

Physically, the term u_ϕ^2/r corresponds to the centrifugal force, and it arises because $\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\phi\boldsymbol{e}_\phi + u_z\boldsymbol{e}_z$ and some of the unit vectors change with ϕ (e.g., $\partial_\phi\boldsymbol{e}_\phi = -\boldsymbol{e}_r$).

Returning to the problem of the coffee cup, let's put our ansatz $p = p(r,t)$ and $\boldsymbol{u} = (0, u_\phi(r,t), 0)$, which satisfies $\nabla \cdot \boldsymbol{u} = 0$, into the cylindrical Navier-Stokes equations. The radial equation for $\boldsymbol{e}_r$-component becomes

$$\frac{u_\phi^2}{r} = \frac{\partial p}{\partial r}. \tag{5a}$$

Physically this represents the balance between pressure and centrifugal force. The angular equation to be satisfied by the $\boldsymbol{e}_\phi$-component is

$$\frac{\partial u_\phi}{\partial t} = \nu \left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} \right), \tag{5b}$$

and the vertical equation is

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{5c}$$

The last equation of these three is directly satisfied by our solution ansatz, and the first equation can be used to compute p by simple integration over r once we have found u_ϕ.

We want to solve these equation (5b) with the initial condition

$$u_\phi(r,0) = \omega r, \tag{6a}$$

and the boundary conditions that, for all $t > 0$,

$$u_\phi(0,t) = 0 \ , \qquad u_\phi(R,t) = 0. \tag{6b}$$

This is done using *separation of variables.* Since the lhs. of Eq. (5b) features first-order time derivative, let's guess a solution of the form

$$u_\phi = e^{-k^2 t} F(r). \tag{7}$$

Putting this into the governing equation (5b) gives the ODE

$$-k^2 F = \nu \left(F'' + \frac{F'}{r} - \frac{F}{r^2} \right) \tag{8}$$

This equation looks complicated. However, note that if the factors of r weren't in this equation we would declare victory. The equation would just be $F'' + k^2/\nu F = 0$, which has solutions that are sines and cosines. The general solution would be $A\sin(k/\sqrt{\nu} r) + B\cos(k/\sqrt{\nu} r)$. We would then proceed by requiring that (a) the boundary conditions were satisfied, and (b) the initial conditions were satisfied.

We rewrite the above equation as

$$r^2 F'' + rF' + \left(\frac{k^2}{\nu} r^2 - 1 \right) F = 0, \tag{9}$$

and make a change of variable,

$$\xi = kr/\sqrt{\nu}.$$

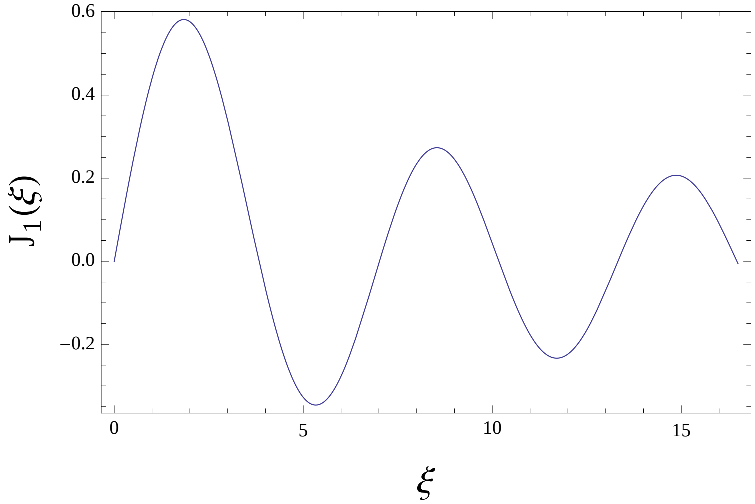

Figure 1: The Bessel function of the first order $J_1(\xi)$.

The equation becomes

$$\xi^2 F'' + \xi F' + \left(\xi^2 - 1\right) F = 0. \tag{10}$$

Even with the factors of ξ included, this problem is not more conceptually difficult, though it does require knowing solutions to the equation. It turns out that the solutions are called *Bessel Functions*. You should think of them as more complicated versions of sines and cosines. There is close form of the solution in terms of elementary functions. However, people usually denote the solution to the Eq. (10) as $J_1(\xi)$, named the *Bessel function of first order*[1]. This function is plotted in Fig. 1, satisfies the inner boundary conditions $J_1(0) = 0$. For more information, see for example the book *Elementary Applied Partial Differential Equations*, by Haberman (pp. 218-224).

Now let's satisfy the boundary condition $u_\phi(R, t) = 0$. Since we have that

$$u_\phi = AJ_1(\xi) = AJ_1(kr/\sqrt{\nu}), \tag{11}$$

this implies that $AJ_1(kR/\sqrt{\nu}) = 0$. We can't have $A = 0$ since then we would have nothing left. Thus it must be that $J_1(kR/\sqrt{\nu}) = 0$. In other words, $kR/\sqrt{\nu} = \lambda_n$, where λ_n is the n^{th} zero of J_1 (morally, J_1 is very much like a sine function, and so has a countably infinite number of zeros.) Our solution is therefore

$$u_\phi(r, t) = \sum_{n=1}^{\infty} A_n e^{-\nu\lambda_n^2 t/R^2} J_1(\lambda_n r/R). \tag{12}$$

[1]Bessel functions $J_\alpha(x)$ of order α are solutions of

$$x^2 J'' + xJ' + \left(x^2 - \alpha\right) J = 0$$

To determine the A_n's we require that the initial conditions are satisfied. The initial condition is that

$$u_\phi(r) = \omega r. \tag{13}$$

Again, we now think about what we would do if the above sum had sines and cosines instead of J_1's. We would simply multiply by sine and integrate over a wavelength. Here, we do the same thing. We multiply by $rJ_1(\lambda_m r/R)$ and integrate from 0 to R. This gives the formula

$$A_n \int_0^R rJ_1(\lambda_n r/R)J_1(\lambda_m r/R)\, dr = \int_0^R \omega r^2 J_1(\lambda_m r/R) dr. \tag{14}$$

Using the identities

$$\int_0^R rJ_1(\lambda_n r/R)J_1(\lambda_m r/R)\, dr = \frac{R^2}{2}J_2(\lambda_n)^2\delta_{nm}, \tag{15}$$

and

$$\int_0^R \omega r^2 J_1(\lambda_m r/R) dr = \frac{\omega R^3}{\lambda_m}J_2(\lambda_m), \tag{16}$$

we get

$$A_n = -\frac{2\omega R}{\lambda_n J_0(\lambda_n)}, \tag{17}$$

where we have used the identity $J_0(\lambda_n) = -J_2(\lambda_n)$. Our final solution is therefore

$$u_\phi(r,t) = -\sum_{n=1}^{\infty}\frac{2\omega R}{\lambda_n J_0(\lambda_n)}e^{-\nu\lambda_n^2 t/R^2}J_1(\lambda_n r/R). \tag{18}$$

Okay, so this is the answer. Now lets see how long it should take for the spin down to occur. Each of the terms in the sum is decreasing exponentially in time. The smallest value of λ_n decreases the slowest. It turns out that this value is $\lambda_1 = 3.83$. Thus the spin down time should be when the argument of the exponential is of order unity, or

$$t \sim \frac{R^2}{\nu\lambda_1^2}. \tag{19}$$

This is our main result, and we should test its various predictions. For example, this says that if we increase the radius of the cylinder by 4, the spin down time increases by a factor of 16. If we increase the kinematic viscosity ν by a factor of 100 (roughly the difference between water and motor oil) then it will take roughly a factor of 100 shorter to spin down. Note that for these predictions to be accurate, one must start with the *same angular velocity* for each case.

In your second problem set you are asked to look at the spin down of a coffee cup. From our theory we have a rough estimate of the spin down time, which you can compare with your experiment. Do you get agreement between the two?