18.354 – Nonlinear Dynamics II: Continuum systems
Problem Set 3: SCALINGS AND CALCULUS OF VARIATIONS
Due: Monday, March 31 (by 3pm in class)

Problem 0: PROPOSAL OF COURSE PROJECT

Write a short proposal of your final project. This proposal should be one page long, include a couple of references and describe what you would like to do. While choosing a topic and implementing it, you should show application of concepts connected to the class material. A relevant experiment, a numerical implementation or a theoretical review are all valid approaches. Original work will carry extra credit. You may want to browse through the course schedule to see the topics that we will be covering after spring break (elasticity, hydrodynamic lift and drag, classic airfoil theory, boundary layers and rotating flows). I have posted project suggestions in the "Final Project" page of the course website. I will be available throughout the week after Spring Break to further discuss projects with you.

Problem 1: LEAPING ANIMALS

There is strong evidence (including the Olympic world records in power lifting) that the force which may be sustained or generated by an animal scales as its cross-sectional area. Use this fact to deduce the dependence on body size of the height which may be achieved by a leaping animal. (*Hint: assume that all dimensions are geometrically similar.*)

Problem 2: CHAMPAGNE AND BEER BUBBLES

How does the pressure inside a bubble depend on the relevant physical quantities? Assume that the surrounding fluid has density ρ and is at a pressure P_o whereas the pressure inside the bubble is $P_o + \Delta P$. Why are champagne louder than beer bubbles when they pop? (*Hint: Surface tension is important. Also, as in the pendulum example we saw in class, maybe not all physical quantities are relevant.*)

Problem 3: AN EXPERIMENT ON SURFACE TENSION

Get some paper clips of different sizes. Now, fill a glass with water and try placing the paper clip on the surface of the water. If you are very careful, you will find that some of the paper clips will sit on the surface. (check out the picture gallery in the "surface tension" of wikipedia, in case you lose faith). How is this possible? Use dimensional analysis to determine the size of the largest paper clip that can be placed on the surface of water? Following on from this, use a scaling argument to explain why it is that insects, and not

elephants, reside on the surface of lakes. (*Hint: The surface tension at an air water interface is 0.07Nm^{-1}, and you may assume that paper-clips, insects and elephants are isometric.*)

Problem 4: FERMAT'S PRINCIPLE

Fermat's principle states that *the time elapsed in the passage of light between two fixed points is an extremum with respect to possible paths connecting the points.*

(a) Consider a flat interface $(y = y_0)$ between two optically homogeneous media. The respective light velocities within the two media are u_1 and u_2. Apply Fermat's principle to the passage of light from a point in one medium to a point in the other to obtain Snell's law,

$$\frac{\sin \phi_1}{u_1} = \frac{\sin \phi_2}{u_2}, \tag{1}$$

where ϕ_i is the angle between the normal to the interface $y = y_0$ and the light path in the corresponding medium.

(b) Now consider an optically inhomogeneous medium in which the light velocity is a continuous function of the y coordinate, $u = u(y)$. Use the calculus of variations to show that

$$\frac{1}{u\sqrt{1 + y'^2}} = K, \tag{2}$$

where K is a constant.

(c) Solve for the path y if the velocity function is $u(y) = \sqrt{y - y_0}$, where y_0 is a constant. Comment on your solution.