METRIC REALIZATION OF FUZZY SIMPLICIAL SETS

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Abstract. We discuss fuzzy simplicial sets, and their relationship to something like metric spaces. Namely, we present an adjunction between the categories: a metric realization functor and fuzzy singular complex functor.

The following document is a rough draft and may have (substantial) errors.

1. Fuzzy simplicial sets

Let $I$ denote the Grothendieck site whose objects are initial open intervals contained in the half-open unit interval $[0, 1) \in \mathbb{R}$, whose morphisms are inclusions of open subsets, and whose covers are open covers. In other words, as a category, $I$ is equivalent to the partially ordered set $(0, 1]$ under the relation $\leq$.

A sheaf $S \in \text{Shv}(I)$ on $I$ is a functor $S : I^{op} \to \text{Sets}$ satisfying the sheaf condition. Explicitly, $S$ consists of a set $S([0, a))$ for all $a \in (0, 1]$, which we choose to denote by $S_{\geq a}$, and restriction maps $\rho_{b,a} : S_{\geq b} \to S_{\geq a}$ for all $b \geq a$, such that if $c \geq b \geq a$ then $\rho_{b,a} \circ \rho_{c,b} = \rho_{c,a}$, and such that for all $a \in I$, one has

$$S_{\geq a} \cong \lim_{a'<a} S_{\geq a'}.$$ 

A sheaf $S$ is called a fuzzy set if for each $b \geq a$ in $(0, 1]$, the restriction map $\rho_{b,a}$ is injective. Let $\text{Fuz}$ denote the full subcategory of $\text{Shv}(I)$ spanned by the fuzzy sets. This definition is slightly different than Goguen’s $\text{Fuz}$, but is closely related. See [Bar]. The difference between fuzzy sets $T$ and arbitrary sheaves $S \in \text{Shv}(I)$ is that, in $T$ two elements are either equal or they are not, whereas two elements $x \neq y \in S_{\leq a}$ may be equal to a certain degree, $\rho_{a,c}(x) = \rho_{a,c}(y)$ for some $c < a$.

Suppose $S \in \text{Shv}(I)$ is a sheaf. For $a \in (0, 1]$, let $S(a) = S_{\geq a} - \text{colim}_{b>a} S_{\geq b}$, and note that $S_{\geq a} = \text{colim}_{b \geq a} \rho_{b,a}[S(b)]$. If $T$ is a fuzzy set, we can make this easier on the eyes:

$$T_{\geq a} = \prod_{b \geq a} T(b).$$

We write $x \in S$ and say that $x$ is an element of $S$, if there exists $a \in (0, 1]$ such that $x \in S(a)$; in this case we may say that $x$ is an element of $S$ with strength $a$.

The following lemma says that, under a map of fuzzy sets, an element cannot be sent to an element of lower strength.

Lemma 1.1. Suppose that $S$ and $T$ are fuzzy sets. If $f : S \to T$ is a morphism of fuzzy sets, then for all $a, b \in (0, 1]$, if $x \in S(a)$ then $f(x) \in T(b)$ for some $b \geq a$.

Proof. Since $x \in S_{\geq a}$, we have by definition that $f(x) \in T_{\geq a}$, so $x \in T(b)$ for some $b \geq a$.

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Lemma 1.2. The forgetful functor \( \text{Fuz} \to \text{Shv}(I) \) is fully faithful and has a left adjoint \( \rho \). Thus Fuz is closed under taking colimits.

Proof. Given a sheaf \( S : I^{\text{op}} \to \text{Sets} \) and \( a \in (0, 1] \), let \( (mS)^{\geq a} = S^{\geq a} / \sim \), where for \( x, x' \in S^{\geq a} \), we set \( x \sim x' \) if there exists \( b \leq a \) such that \( \rho_{a,b}(x) = \rho_{a,b}(x') \). Clearly, \( mS \) is a fuzzy set, and one checks that \( m \) is left adjoint to the forgetful functor.

To compute the colimit of a diagram in Fuz, one applies the forgetful functor, takes the colimit in Shv(I), and applies the left adjoint.

\[ \square \]

Let \( \Delta \) denote the simplicial indexing category, and denote its objects by \([n]\) for \( n \in \mathbb{N} \).

Definition 1.3. A fuzzy simplicial set is a functor \( \Delta^{\text{op}} \to \text{Fuz} \). A morphism of fuzzy simplicial sets is a natural transformation of functors. The category of fuzzy simplicial sets is denoted \( \text{sFuz} \).

A fuzzy simplicial set is a simplicial set in which every simplex has a strength. A simplex has strength at most the minimum of its faces. All degeneracies of a simplex have the same strength as the simplex.

A fuzzy simplicial set \( X : \Delta^{\text{op}} \to \text{Fuz} \) can be rewritten as a sheaf \( X : (\Delta \times I)^{\text{op}} \to \text{Sets} \), where \( \Delta \) has the trivial Grothendieck topology and \( \Delta \times I \) has the product Grothendieck topology. We write \( X^{\leq a} \) to denote the set \( X([n], [0, a)) \).

For \( n \in \mathbb{N} \) and \( i \in I \), let \( \Delta^n_i \in \text{sFuz} \) denote the functor represented by \((n, i)\). If \( i = [0, a] \) we may also write \( \Delta^n_a \) to denote \( \Delta^n_{(0,a)} \). Note that a map \( f : [n] \to [m] \) induces a unique map \( F : \Delta^n_a \to \Delta^m_b \) if and only if \( a \leq b \); otherwise there can be no such \( F \).

Any fuzzy simplicial set \( X \) can be canonically written as the colimit of its diagram of simplices:

\[
\text{colim}_{\Delta^{\geq a} - X} \Delta^n_{\leq a} \xrightarrow{\sim} X
\]

2. Uber-metric spaces

We define a category of uber-metric spaces, which are metric spaces except with the possibility of \( d(x, y) = \infty \) or \( d(x, y) = 0 \) for \( x \neq y \).

Definition 2.1. An uber-metric space is a pair \((X, d)\), where \( X \) is a set and \( d : X \times X \to [0, \infty) \), such that for all \( x, y, z \in X \),

(1) \( d(x, x) = 0 \),

(2) \( d(x, y) = d(y, x) \), and

(3) \( d(x, z) \leq d(x, y) + d(y, z) \).

Here we consider \( x \leq \infty \) and \( x + \infty = \infty + x = \infty \) for all \( x \in [0, \infty] \). We call \( d \) an uber-metric or just a metric on \( X \).

A morphism of uber-metric spaces, denoted \( f : (X, d_X) \to (Y, d_Y) \) is a function \( f : X \to Y \) such that \( d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) \) for all \( x_1, x_2 \in X \). Such functions are also called non-expansive.

These objects and morphisms define a category called the category of uber-metric spaces and denoted UM.

Lemma 2.2. The category UM is closed under colimits.
Proof. We must show that $\text{UM}$ has an initial object, arbitrary coproducts, and coequalizers. The set $\emptyset$ is the initial object in $\text{UM}$.

Let $A$ be a set and for all $a \in A$, let $(X_a, d_a)$ denote a metric space. Let $X_A$ denote the set $\bigsqcup_{a \in A} X_a$; and let $d_A$ denote the metric such that for all $y, y' \in X_A$, if there exists $a \in A$ such that $y, y' \in X_a$ then $d_A(y, y') = d_a(y, y')$, but if instead $y$ and $y'$ are in separate components then $d_A(y, y') = \infty$. One checks that $(X_A, d_A)$ is an uber-metric space and that it satisfies the universal property for a coproduct.

Finally, suppose that

$$A \xrightarrow{f} X \xrightarrow{g} Y$$

is a coequalizer diagram of sets. Write $x \sim x'$ if there exists $a \in A$ with $x = f(a), y = g(a)$; then $Y = X/\sim$ is the set of equivalence classes. If $y = m(x)$. If $d_X$ is a metric on $X$, we define a metric (\cite{Wiki}) $d_Y$ on $Y$ by

$$d_Y([x], [x']) = \inf(d_X(p_1, q_1) + d_X(p_2, q_2) + \cdots + d_X(p_n, q_n)),$$

where the infimum is taken over all pairs of sequences $(p_1, \ldots, p_n), (q_1, \ldots, q_n)$ of elements of $X$, such that $p_1 \sim x, q_1 \sim x'$, and $p_{i+1} \sim q_i$ for all $1 \leq i \leq n - 1$. Again, one checks that $(Y, d_Y)$ is an uber-metric space which satisfies the universal property of a coequalizer.

\[ \square \]

3. Metric realization

In order to define a metric realization functor $Re: \mathbf{sFuz} \rightarrow \text{UM}$, we first define it on the representable sheaves in $\mathbf{sFuz}$ and then extend to the whole category using colimits (i.e. using a left Kan extension).

Recall the usual metric on Euclidean space $\mathbb{R}^m$ and let $\mathbb{R}^m_{\geq 0}$ denote the $m$-tuples all of whose entries are non-negative. Recall also that objects of $I$ are of the form $[0, a)$ for $0 < a \leq 1$. For an object $([n], [0, a]) \in \mathbb{N} \times I$, define $Re(\Delta^n_{\leq a})$, as a set, to be

$$\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 + x_1 + \cdots + x_n = 1 - a\}$$

We take as our metric on $Re(\Delta^n_{\leq a})$ to be that induced by its inclusion as a subspace of $\mathbb{R}^{n+1}$.

A morphism $([n], [0, a]) \rightarrow ([m], [0, b])$ exists if $a \leq b$, and in that case consists of a morphism $\sigma: [n] \rightarrow [m]$. We define $Re(\sigma, a \leq b): Re(\Delta^n_{\leq a}) \rightarrow Re(\Delta^m_{\leq b})$ to be the map

$$(x_0, x_1, \ldots, x_n) \mapsto \frac{1 - b}{1 - a} \left( \sum_{i_0 \in \sigma^{-1}(0)} x_{i_0}, \sum_{i_1 \in \sigma^{-1}(1)} x_{i_1}, \ldots, \sum_{i_m \in \sigma^{-1}(m)} x_{i_m} \right).$$

Note that this map is non-expansive because $1 - b \leq 1 - a$.

We are ready to define $Re$ on a general $X$ as

$$Re(X) := \operatorname{colim}_{\Delta^n_{\leq a} \rightarrow X} Re(\Delta^n_{\leq a}).$$

This functor preserves colimits, so it has a right adjoint, which we denote $\text{Sing}: \text{UM} \rightarrow \mathbf{sFuz}$. It is given on $Y \in \text{UM}$ by

$$\text{Sing}(Y)_{\leq a} = \text{Hom}_{\text{UM}}(Re(\Delta^n_{\leq a}), Y).$$
REFERENCES

[Ish] Isbell. Category of metric spaces.