1. Locales

1.1. Morphisms between locales of subobjects. Suppose that \( f: X \to Y \) is a morphism of simplicial sets. Let \( \mathcal{X} = \text{Sub}(X) \) (respectively \( \mathcal{Y} = \text{Sub}(Y) \)) denote the locale of subobjects of \( X \) (resp. \( Y \)). The map \( f \) induces an adjunction

\[
\begin{array}{ccc}
\mathcal{Y} & \xRightarrow{f_*} & \mathcal{X} \\
\text{adjunction} & & \\
\end{array}
\]

where \( \mathcal{F}(X') = \bigvee \{ Y' \subset Y | f^{-1}(Y') \subset X' \} \). Note that \( f^{-1} \) preserves finite meets, so \( \mathcal{F}: \mathcal{X} \to \mathcal{Y} \) denotes a morphism of locales. This morphism further induces an adjunction between Grothendieck topoi

\[
\begin{array}{ccc}
\text{Shv}(\mathcal{Y}) & \xrightarrow{\mathcal{F}^*} & \text{Shv}(\mathcal{X}) \\
\text{adjunction} & & \end{array}
\]

defined as follows for sheaves \( A \in \text{Shv}(\mathcal{X}) \) and \( B \in \text{Shv}(\mathcal{Y}) \). For any \( U \in \text{Sub}(X) \) we take

\[
(1) \quad \mathcal{F}^* B(U) := B(f(U)),
\]

where \( f(U) \in \text{Sub}(Y) \) is the image of \( U \) in \( Y \). For any \( V \in \text{Sub}(Y) \) we take

\[
(2) \quad \mathcal{F}_* A(V) := A(f^{-1}(V)),
\]

where \( f^{-1}(V) \) is the preimage of \( V \) in \( X \).

1.2. Morphisms between locales. We introduced the above ideas using a map of simplicial sets \( f: X \to Y \), but in actuality the definitions of \( \mathcal{F}^* \) and \( \mathcal{F}_* \) work whenever \( \mathcal{F} \) is a open morphism of locales, as we shall soon see. To get to that point, however, it is best to start all over, by taking \( \mathcal{X} \) and \( \mathcal{Y} \) to be arbitrary locales and considering an arbitrary adjunction

\[
\begin{array}{ccc}
\mathcal{Y} & \xRightarrow{f_*} & \mathcal{X} \\
\text{adjunction} & & \\
\end{array}
\]

as a morphism \( \mathcal{F} = f_* \) of locales. The left adjoint \( f^* \) induces an adjunction of Grothendieck topoi denoted

\[
\begin{array}{ccc}
\text{Shv}(\mathcal{Y}) & \xrightarrow{\mathcal{F}^*} & \text{Shv}(\mathcal{X}) \\
\text{adjunction} & & \end{array}
\]

by \( \mathcal{F}_* \), which is given by the formula

\[
(3) \quad \mathcal{F}_*(A)(V) := A(f^*V).
\]

The left adjoint \( \mathcal{F}^* \) is easy to understand on representable functors, but in general is just understandable as a colimit.

We say that \( \mathcal{F} \) is an open morphism if \( f^* \) has a left adjoint \( f_!: \mathcal{X} \to \mathcal{Y} \) satisfying the “Frobenius identity”

\[
f_!(x \land f^*y) = f_!(x) \land y
\]

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for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \). In this case, we can do a better job of understanding \( \mathcal{F}^* \). Namely, for a sheaf \( B \in \text{Shv}(\mathcal{Y}) \) and object \( U \in \mathcal{X} \), we define
\[
\mathcal{F}^* B(U) := B(f_!(U))
\]
and still
To bring it together, a morphism \( f: X \to Y \) of simplicial sets always induces an open morphism of locales. Thus, Equations (1) and (2) becomes Equations (3) and (4).

2. Elementary localic databases

Let \( \text{Loc} \) denote the category of locales (see [??]). Recall that a morphism between locales is an adjunction \((f^*, f_*)\) for which \( f^* \) preserves finite meets; such a morphism is denoted by the right adjoint part.

**Definition 2.0.1.** A type specification consists of a pair \( D = (D, \Gamma) \), where \( D \) is a category and \( \Gamma \in \text{Pre}(D) \) is a presheaf on \( D \). A morphism of type specifications, denoted
\[
(F, F^\sharp): D \to D',
\]
consists of a functor \( F: D' \to D \) and a map of presheaves \( F^\sharp: F^*(\Gamma) \to \Gamma' \) on \( D' \).

**Example 2.0.2.** Let \( D = (\text{Fin} \downarrow \{m, a\}) \) and \( D' = (\text{Fin} \downarrow \{F\}) \) be categories. Given \( \sigma: C \to \{m, a\} \), let \( \Gamma(\sigma) = \text{Hom}(C, \mathbb{R}) \), and similarly given \( \sigma': C' \to \{F\} \), let \( \Gamma'(\sigma') = \text{Hom}(C', \mathbb{R}) \). There is a morphism of type specifications
\[
(\mu, \mu^\sharp): (D, \Gamma) \to (D', \Gamma')
\]
given as follows.

First off, the functor \( \mu: D' \to D \) sends \( \sigma': C' \to \{F\} \), to \( \mu(\sigma') := C' \times \{m, a\} \to \{m, a\} \). This induces a functor
\[
\mu^* : \text{Pre}(D) \to \text{Pre}(D'); \quad X(-) \mapsto X(\mu(-)).
\]

Now, let \( \mu^\sharp : \mu^* \Gamma \to \Gamma' \) denote the map of presheaves on \( D' \) defined as follows for \( \sigma: C' \to \{F\} \). We calculate that \( \mu^* \Gamma(\sigma) = \Gamma(\mu(\sigma)) = (\mathbb{R} \times \mathbb{R})^{C'} \) and that \( \Gamma'(C') = \mathbb{R}^{C'} \). Now define
\[
\mu^\sharp : (\mathbb{R} \times \mathbb{R})^{C'} \to \mathbb{R}^{C'}
\]
by composing with the usual multiplication map \( \mathbb{R} \times \mathbb{R} \to \mathbb{R} \).

This change of type specifications allows one to turn any “mass/acceleration” table into a “Force” table. We write it as \( F = ma \).

More generally, suppose that \( x_1, \ldots, x_m \) are the variables of \( D \), taking values in \( \Gamma(x_1), \ldots, \Gamma(x_m) \) and \( X_1, \ldots, X_n \) are the “variables” of \( D' \) taking values in \( \Gamma(X_1), \ldots, \Gamma(X_n) \), then we can define \( X_i \) in terms of any set of \( x_1, \ldots, x_m \) using any formula. Writing down the variables involved in these formulas constitutes a map
\[
D' = (\text{Fin} \downarrow \{X_1, \ldots, X_n\}) \longrightarrow D = (\text{Fin} \downarrow \{x_1, \ldots, x_m\}).
\]

**Definition 2.0.3.** Let \( D = (D, \Gamma) \) denote a type specification. A schema on \( D \) consists of a pair \((L, \sigma)\), where \( L \) is a locale and \( \sigma: L \to \text{Pre}(D) \) preserves colimits and finite limits. A morphism of schemas is denoted
\[
\mathcal{F} = (f^*, f^\sharp): (L, \sigma) \to (L', \sigma'),
\]
where \( f^*: L \xrightarrow{\cong} L' \) is a morphism of locales, and \( f^\sharp: \sigma \circ f^* \to \sigma' \) is a natural transformation of functors \( L' \to \text{Pre}(D) \).
Definition 2.0.4. Let $\mathcal{D} = (D, \Gamma)$ denote a type specification, and let $X = (L, \sigma)$ denote a schema over $\mathcal{D}$. The universal sheaf on $X$, denoted $\mathcal{U}_X: L^{\text{op}} \to \text{Sets}$, is the functor which assigns

$$\ell \mapsto \text{Hom}_{\text{Pre}(D)}(\sigma(\ell), \Gamma) \quad \text{for } \ell \in L.$$ 

It is clearly contravariant and takes joins in $L$ to limits in $\text{Sets}$.

Let $X' = (L', \sigma')$. Given a morphism $F = (f, f^\#): X \to X'$ of schemas, there is an induced morphism $U_F: \mathcal{U}_{X'} \to F^\ast(\mathcal{U}_X)$ given as follows for $\ell' \in L'$:

$$U_{L'}(\ell') = \text{Hom}_{\text{Pre}(D)}(\sigma'\ell', \Gamma) \xrightarrow{f^\#} \text{Hom}_{\text{Pre}(D)}(\sigma f^\ast \ell', \Gamma) = F^\ast(\mathcal{U}_{L})(\ell').$$

Example 2.0.5. Let $\mathcal{D} = (D, \Gamma)$ and $\mathcal{D}' = (D', \Gamma')$ denote type specifications, and let $(F, F^\sharp): \mathcal{D} \to \mathcal{D}'$ denote a morphism of type specifications. Recall that $F: \mathcal{D}' \to \mathcal{D}$ is a functor and $F^\sharp: F^\ast \Gamma \to \Gamma'$ is a morphism of presheaves on $\mathcal{D}'$.

Let $X = (L, \sigma)$ denote a schema over $\mathcal{D}$, where $\sigma: L \to \text{Pre}(D)$. Given $\ell \in L$, the value of the universal sheaf $\mathcal{U}_X$ at $\ell$ is

$$\mathcal{U}_X(\ell) = \text{Hom}_{\text{Pre}(D)}(\sigma(\ell), \Gamma).$$

If we apply the left adjoint $F^\ast$ to both sides and then compose with $F^\sharp$ we obtain a map

$$\mathcal{U}_X(\ell) = \text{Hom}(\sigma(\ell), \Gamma) \to \text{Hom}(F^\ast \sigma(\ell), F^\ast \Gamma)$$

$$\to \text{Hom}(F^\ast \sigma(\ell), \Gamma')$$

This may be seen as how databases on $\mathcal{D}$ become databases on $\mathcal{D}'$.

Definition 2.0.6. Let $\mathcal{D} = (D, \Gamma)$ denote a type specification. A database of type $\mathcal{D}$ is a sequence $(X, K_X, \tau_X)$, where $X = (L, \sigma)$ is a schema, $K_X \in \text{Shv}(L)$ is a sheaf of sets on $L$ and $\tau_X: K_X \to \mathcal{U}_X$ is a morphism of sheaves over the universal sheaf $\mathcal{U}_X$.

A morphism of databases consists of a pair

$$(F, F^\sharp): (X, K_X, \tau_X) \to (Y, K_Y, \tau_Y),$$

where $F: X \to X'$ is a morphism of schema and $F^\sharp: K_Y \to F_\ast K_X$ is a morphism of sheaves on $Y$, such that the diagram

$$\begin{array}{ccc}
\mathcal{K}_Y & \xrightarrow{\tau_Y} & \mathcal{U}_Y \\
F^\sharp \downarrow & & \downarrow F_* \\
F_* K_X & \xrightarrow{\mathcal{F}_* \tau_X} & \mathcal{F}_* \mathcal{U}_X
\end{array}$$

commutes.

3. Theory

There may be issues with the variance of locales. That is, we may be naming by left adjoints instead of right adjoints. This calls a lot of the following into question. For now, we are just making a first pass through the space.
Definition 3.0.7. Suppose given a diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\sigma_L} & M \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sigma_M} & \{\ast\}
\end{array}
\]

where \( L \) and \( M \) are locales and \( C \) is a category. Define \( \Gamma^L_M \in \text{Shv}(L) \) to be the sheaf

\[
\Gamma^L_M(x) := \{ f: \downarrow x \to M | \sigma_M \circ f = \sigma_L \circ i_x \}.
\]

For locales \( L, M \) (in absence of \( C, \sigma_L, \sigma_M \)) define \( \Gamma^L_M \in \text{Shv}(L) \) by \( \Gamma^L_M(x) := \{ f: \downarrow x \to M \} \). Clearly, this is the same thing as one gets by taking \( C = \{\ast\} \) to be the terminal category and \( \sigma_L, \sigma_M \) to be the unique functors.

Given a map \( \pi: M \to L \), let \( \Gamma_\pi \) be as defined by the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\pi} & M \\
\downarrow & & \downarrow \\
\{\ast\} & \xrightarrow{} & \text{Pre}(D)
\end{array}
\]

Definition 3.0.8. Suppose given a diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\sigma_L} & M \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sigma_M} & \{\ast\}
\end{array}
\]

where \( L \) and \( M \) are locales and \( C \) is a category. Define \( \mathcal{U}^L_M \in \text{Shv}(L) \) to be the sheaf whose value on \( x \in L \) is given by

\[
\mathcal{U}^L_M(x) := \{ (f, f^\sharp): f: \downarrow x \to M, f^\sharp: \sigma_L(x) \to \sigma_M \circ f(x) \}.
\]

Let \( D \) be a locale and \( \Gamma \) a presheaf on \( D \). Given a functor \( \sigma: L \to \text{Pre}(D) \), where \( L \) is a locale, we have a diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\sigma} & \{\ast\} \\
\downarrow & & \downarrow \\
\text{Pre}(D) & \xrightarrow{} & \Gamma
\end{array}
\]

so we can define \( \mathcal{U}^L_\Gamma \in \text{Shv}(L) \) as above. Tracing through definitions, this sheaf is simply given on \( x \in L \) by

\[
\mathcal{U}^L_\Gamma(x) = \text{Hom}_{\text{Pre}(D)}(\sigma(x), \Gamma),
\]

as expected.

Definition 3.0.9. Let \( C \) denote a category. A schema on \( C \) is a left exact left adjoint \( \Gamma: M \to C \), where \( M \) is a locale.

Definition 3.0.10. Let \( \Gamma: M \to C \) denote a schema on \( C \). A database on \( \Gamma \) consists of a sequence \((\sigma, K, \tau)\), where \( \sigma: L \to C \) is a schema, \( K \in \text{Shv}(L) \) is a sheaf on \( L \), and \( \tau: K \to \mathcal{U}^L_\Gamma \) is a morphism of sheaves.

Let \( \sigma_L: L \to C \) denote a schema and let \( K_L \in \text{Shv}(L) \) denote a sheaf. This data constitutes a schema \((\sigma_L \times K_L): L \to (C \times \text{Sets}^{\text{op}})\).
4. Next steps...

- I’d like to study locales given by \( \text{Sub}(S) \) for a set \( S \). There should be an adjunction \( \text{Sets} \xrightarrow{\text{Sub}} \text{Loc} \) which I can use. Locales of the form \( \text{Sub}(S) \) may turn out to have a particularly easy “grammar” in that the elements of \( S \) serve well as “variables.” There should be some kind of more general theory of variables, but I think it’s easiest for \( \text{Sub}(S) \).

- A computer program is data interpreted as instructions. This should be phraseable in our language. But what is instruction?