1. Notation

Let $F: A \to C$ and $G: B \to C$ be functors. We denote by $(F \downarrow G)$ the category whose objects are pairs $(a, b, f)$ where $a \in \text{Ob}(A), b \in \text{Ob}(B)$ and $f: F(a) \to G(b)$ is a morphism in $C$, and whose morphism sets are given by

$$\text{Hom}_{(F \downarrow G)}((a, b, f), (a', b', f')) = \{(x: a \to a', y: b \to b') | yf = f'x\}.$$ 

If $F: A \to \text{Cat}$ is a functor and $B$ is a category, we sometimes write $(A \downarrow \{B\})$ to denote $(F \downarrow \text{incl}_B)$, where $\text{incl}_B: \{\ast\} \to \text{Cat}$ is the category $B$.

If $C = \text{Cat}$ we introduce a new category $(F \llarrow G)$ whose objects are the same as those in $(F \downarrow G)$ but for which morphism sets are given by

$$\text{Hom}_{(F \llarrow G)}((a, b, f), (a', b', f')) = \{(x, y, \alpha) | x: a \to a', y: b \to b', \alpha: yf \to f'x\}.$$

We write $(F \uparrow G)$ to denote the category with the direction of the natural transformation reversed: same objects, but morphism sets given by

$$\text{Hom}_{(F \uparrow G)}((a, b, f), (a', b', f')) = \{(x, y, \alpha) | x: a \to a', y: b \to b', \alpha: f'x \to yf\}.$$ 

Again, we may write $(F \downarrow \{B\})$ to denote $(F \downarrow \text{incl}_B)$. Explicitly, the objects in $(F \downarrow \{B\})$ are functors $F(a) \to B$, where $a \in A$, and the morphisms $(F(a) \to B) \to (F(a') \to B)$ are pairs $(x, \alpha)$ giving a natural transformation diagram

$$\begin{array}{ccc}
F(a) & \xrightarrow{x} & F(a') \\
\downarrow f & \alpha \Downarrow & \downarrow f' \\
B & & B
\end{array}$$

Given $F: A \to \text{Cat}$, let $\bar{F}: A \to \text{Cat}$ denote the functor $a \mapsto F(a)^{\text{op}}$. Clearly, $(F \downarrow \{B\})$ is isomorphic to $(\bar{F} \downarrow \{B^{\text{op}}\})$. The category $(\bar{F} \downarrow \{B^{\text{op}}\})$ can be identified with the category whose objects are functors $F(a) \to B$, where $a \in A$, and the morphisms $(F(a) \to B) \to (F(a') \to B)$ are pairs $(x, \alpha)$ giving a natural transformation diagram

$$\begin{array}{ccc}
F(a) & \xrightarrow{x} & F(a') \\
\downarrow f & \alpha \Leftarrow & \downarrow f' \\
B & & B
\end{array}$$

Clearly, this is the category $(F \uparrow \{B\}) = (\bar{F} \downarrow \{B^{\text{op}}\})$.

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2. Grothendieck construction

Let \( T : S \to \text{Cat} \) be a functor. The Grothendieck construction for \( T \) is given by
\[
\text{Gr}(T) := (\{*\} \Downarrow T).
\]
Explicitly, \( \text{Gr}(T) \) is the category with objects \((S, s)\) where \( S \in \text{Ob}(S) \) and \( s \in T(S) \). The morphism sets in \( \text{Gr}(T) \) are given by
\[
\text{Hom}((S, s), (S', s')) = \{ (y : S \to S', \alpha : T(y)(s) \to s') \}.
\]
There is a natural projection \( \text{Gr}(T) \to S \) given by \((S, s) \mapsto S\). It is a split fibration; that is, for any \( y : S \to S' \) and \( s \in S \), there is a canonical choice of map \((S, s) \to (S', T(y)(s))\) lying over \( y \).

**Lemma 2.0.1.** Let \( \sigma : C \to S \) and \( T : S \to \text{Cat} \) be functors. The diagram
\[
\begin{array}{ccc}
\text{Gr}(T\sigma) & \longrightarrow & \text{Gr}(T) \\
\downarrow & & \downarrow \\
C & \longrightarrow & S
\end{array}
\]
is a pullback square.

Recall that a pullback square as in Lemma 2.0.1 defines a functor from the sections of \( \text{Gr}(T) \to S \) to the sections of \( \text{Gr}(T\sigma) \to C \).

**Lemma 2.0.2.** Suppose given two functors \( f, g : S \to \text{Cat} \); let \( \text{Nat}(f, g) \) denote the set of natural transformations from \( f \) to \( g \). There is a natural isomorphism
\[
\text{Hom}_S(\text{Gr}(f), \text{Gr}(g)) \cong \text{Nat}(f, g).
\]

**Definition 2.0.3.** There is a functor \( \Gamma : (\text{Cat} \downarrow \{\text{Cat}\}) \to \text{Cat} \) defined as follows. On objects, \( \Gamma(f : C \to \text{Cat}) \) is defined to be the global sections of the functor \( \pi : \text{Gr}(f) \to C \), i.e. the category whose objects are functors \( s : C \to \text{Gr}(f) \) with \( \pi \circ s = \text{id}_C \), and whose morphisms are natural transformations \( s \to s' \).

To define \( \Gamma \) on morphisms, suppose we are given a natural transformation diagram
\[
\begin{array}{ccc}
C & \xrightarrow{x} & D \\
\downarrow^f & \swarrow & \searrow^g \\
\text{Cat} & \leftarrow & \text{Cat}
\end{array}
\]
By Lemma 2.0.1 there is a canonical functor \( \Gamma(g) \to \Gamma(gx) \). By Lemma 2.0.2 there is a canonical functor \( \Gamma(gx) \to \Gamma(f) \) induced by \( \alpha \). The composition gives the desired functor.

**Lemma 2.0.4.** Suppose that \( \sigma_i : C_i \to \text{Cat} \), for \( i = 1, 2, 3 \), are three functors and suppose that \( f : C_2 \to C_1 \) and \( g : C_2 \to C_3 \) are functors too. There is an induced functor \( C_4 := C_1 \amalg_{C_2} C_3 \) \( \to \text{Cat} \). Taking Grothendieck constructions yeilds the
diagram

All of the vertical squares are pullback squares by Lemma 2.0.1.

The natural map

\[ \Gamma(\sigma_4) \rightarrow \Gamma(\sigma_1) \times_{\Gamma(\sigma_2)} \Gamma(\sigma_3) \]

is an isomorphism.

3. Monad

**Lemma 3.0.5.** Suppose that \( C \) is a category and \( T \) is a monad on \( C \). If \( C \in C \) is a \( T \)-algebra, then \( T \) lifts to a monad on the category \( (C \downarrow C) \).

**Proof.** Given \( f : X \rightarrow C \) we define \( T(f) \) to be the composition \( T(X) \rightarrow T(C) \rightarrow C \).

It satisfies the axioms for a monad.

\[ \square \]

**Remark 3.0.6.** The material below emphasizes the role of \( \text{Cat} \), but everything said (Lemma 3.0.7 and 3.0.8) also works for \( \text{Sets} \). In particular, replace the category \( \text{FC} \) with the category \( \text{Fin} \) of finite sets, and replace the category \( \text{Cat} \) with the category \( \text{Sets} \).

**Lemma 3.0.7.** Let \( \text{FC} \) denote the category of finite categories. Consider the endofunctor \( d := (\text{FC} \downarrow (\cdot)) : \text{Cat} \rightarrow \text{Cat} \). There are natural transformations \( \eta : \text{id}_{\text{Cat}} \rightarrow d \) and \( \mu : d^2 \rightarrow d \), which gives \( d \) the structure of a monad on \( \text{Cat} \).

**Proof.** For \( C \in \text{Cat} \), define \( \eta_C : C \rightarrow dC \) to be the functor given on objects by \( c \mapsto \{\ast\} \xrightarrow{\sigma} C \), and given on morphisms by

\[
(f : c \rightarrow c') \mapsto \begin{pmatrix}
\{\ast\} \\
\{\ast\}
\end{pmatrix} \\
\begin{array}{c}
\text{id}_C \\
\sigma
\end{array}
\]

This is clearly a natural transformation.

Defining \( \mu \) is more difficult. Let \( C \in \text{Cat} \) be a category; we will define \( \mu_C \).

Let \( X \) be a finite category and let \( \sigma : X \rightarrow (\text{FC} \downarrow C) \) be an object in \( dC \). We need to establish some notation for \( \sigma \). On objects \( x \in X \), denote \( \sigma(x) \) by

\[ \sigma_x : S_x \rightarrow C, \]
where $S_x$ is a finite category. On morphisms $f : x \to y$ in $X$, denote $\sigma(f)$ by

\[
(\sigma_f, \sigma_f^\sharp) : \begin{array}{c}
S_x \\
\downarrow \sigma_x \\
\downarrow \sigma_f \\
\downarrow \sigma_y \\
C
\end{array}
\xrightarrow{\sigma_f} S_y.
\]

Define $\mu(X, \sigma) \in \text{Cat}$ as the category with objects

\[\text{Ob}(\mu(X, \sigma)) := \{(x, s_x) | x \in X, s_x \in S_x\}\]

and morphisms

\[\text{Hom}_{\mu(X, \sigma)}((x, s_x), (y, s_y)) = \{(f, f^\sharp) | f : x \to y, (f^\sharp : \sigma_f(s_x) \to s_y) \in S_y\}.\]

Note that since $X$ and each $S_x$ is finite, so is $\mu(X, \sigma)$.

There is a canonical functor $\mu(X, \sigma) \to C$ given on objects by $(x, s_x) \mapsto \sigma(s_x)$, and given on morphisms by sending $(f, f^\sharp) : (x, s_x) \to (y, s_y)$ to the composite

\[\sigma_x(s_x) \xrightarrow{\sigma_f^\sharp} \sigma_y \sigma_f(s_x) \xrightarrow{\sigma_y(f^\sharp)} \sigma_y(s_y).\]

Thus $\mu(X, \sigma)$ is indeed in $(\text{FC} \Downarrow C)$.

So far, we have only established $\mu$ on objects. On morphisms

\[
\begin{array}{c}
X \\
\downarrow \sigma
\end{array}
\xrightarrow{m} \begin{array}{c}
X' \\
\downarrow \sigma'
\end{array}
\]

define $\mu(m, m^\sharp) : \mu(X, \sigma) \to \mu(X', \sigma')$ as the functor given as follows. Send the object $(x, s_x) \in \mu(X, \sigma)$ to $(m(x), m^\sharp_x(s_x)) \in \mu(X', \sigma')$. Send $(f, f^\sharp) : (x, s_x) \to (y, s_y)$ to

\[(m(x), (m^\sharp_y \circ f^\sharp : \sigma'_m(m^\sharp_x(s_x)) \to m^\sharp_y(s_y)))) : \mu(X, \sigma) \to \mu(X', \sigma').\]

There is an obvious natural transformation

\[
\begin{array}{c}
\mu(X, \sigma) \\
\downarrow
\end{array}
\xrightarrow{=} \begin{array}{c}
\mu(X', \sigma') \\
\downarrow
\end{array}
\]

because $m^\sharp : \sigma' \circ m \to \sigma$ was assumed natural.
We draw the following diagram, in case it is useful to the reader:

We forgo showing that this really is a monad.

Lemma 3.0.8. The global sections functor $\Gamma: (\text{FC} \Downarrow \text{Cat}) \to \text{Cat}$ gives $\text{Cat}$ the structure of a $d$-algebra.

Therefore, one can consider $d$ as a monad on $(\text{Cat} \Downarrow \{\text{Cat}\})$.

Proof. The second assertion follows from the first by Lemma 3.0.5.