1. Notation

Let \( \{\ast\} \) denote the terminal category (which has one arrow). Let \( \textbf{Cat} \) denote the category of small categories (morphisms are functors). Given a category \( A \in \textbf{Cat} \), let \( \text{Ob}(A) \) denote the set of objects in \( A \).

Let \( F: A \to C \) and \( G: B \to C \) be functors. We denote by \( (F \downarrow C \; G) \) the category whose objects are pairs \( (a, b, f) \) where \( a \in \text{Ob}(A), b \in \text{Ob}(b) \) and \( f: F(a) \to G(b) \) is a morphism in \( C \). These are pictured as \(Fa \downarrow \downarrow \downarrow Gb\).

(1)

The morphism sets are given by

\[
\text{Hom}_{(F \downarrow C \; G)}((a, b, f), (a', b', f')) = \{(x, y) | x: a \to a', y: b \to b', Gy \circ f = f' \circ Fx\}.
\]

In pictures, a morphism \((a, b, f) \to (a', b', f')\) in \( (F \downarrow C \; G) \) is written

\[
\begin{array}{ccc}
& & Fa' \\
& f & \downarrow \\
F & \downarrow f & Gb \\
& & Gb'
\end{array}
\]

Suppose now that \( C \) is a 2-category; we will use the morphism-terminology of \( C = \textbf{Cat} \) because that is the main example we ultimately want to consider. In other words, we will call morphisms in \( C \) “functors” and 2-arrows in \( C \) “natural transformations.” We still consider \( A \) and \( B \) as 1-categories.

Example 1.1. Let \( C = \textbf{Cat} \). Suppose that \( F: A \to \textbf{Cat} \) is a functor and \( B \) is a category, we sometimes write \((A \downarrow_{\textbf{Cat}} \{B\})\) to denote \((F \downarrow_{\textbf{Cat}} i_B)\), where \( i_B: \{\ast\} \to \textbf{Cat} \) is the category \( B \). In other words, \((A \downarrow_{\textbf{Cat}} \{B\})\) is the category of \( A \)-shaped diagrams in \( B \).

We can add a layer of complexity to \((F \downarrow C \; G)\), by adding the possibility that the morphisms are not commutative diagrams in \( C \), but instead involve a natural transformation (i.e. a 2-arrow in \( C \)). To that end, we define a category \((F \updownarrow_{C} \; G)\) whose objects are the same as those in \((F \downarrow_{C} \; G)\), pictured in [1], but for which morphisms sets are given by

\[
\text{Hom}_{(F \updownarrow C \; G)}((a, b, f), (a', b', f')) = \{(x, y, \alpha) | x: a \to a', y: b \to b', \alpha: Gy \circ f = f' \circ Fx\}.
\]

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In pictures, a morphism \((a, b, f) \to (a', b', f')\) in \((F \downarrow_C G)\) is written

\[
\begin{align*}
Fa & \xrightarrow{Fx} Fa' \\
\downarrow f & \xrightarrow{\alpha} \downarrow f' \\
Gb & \xrightarrow{Gy} Gb'.
\end{align*}
\]

We write \((F \uparrow_C G)\) to denote a similar category but with some reversal of 1-arrows. Its objects are the same as those in \((F \downarrow_C G)\) and \((F \downarrow C G)\); they are sequences \((a, b, f)\), as seen in (1). The morphism sets are given by

\[
\text{Hom}_{(F \uparrow_C G)}((a, b, f), (a', b', f')) = \{(x, y, \alpha) | x: a' \to a, y: b' \to b, \alpha: f \circ Fx \to Gy \circ f'\}.
\]

In pictures, a morphism \((a, b, f) \to (a', b', f')\) in \((F \uparrow_C G)\) is written

\[
\begin{align*}
Fa & \xleftarrow{Fx} Fa' \\
\downarrow f & \xleftarrow{\alpha} \downarrow f' \\
Gb & \xleftarrow{Gy} Gb'.
\end{align*}
\]

For those who prefer, we can simply write \((F \uparrow_C G)\) = \((G \downarrow_C F)\), where \(F^\text{op}: A^\text{op} \to C^\text{op}\) and \(G^\text{op}: B^\text{op} \to C^\text{op}\) are the opposites of \(F\) and \(G\).

**Example 1.2.** Suppose that \(C = \text{Cat}\) and that \(B\) is a category, considered as a functor \(\{\ast\} \xrightarrow{B} C\). We may write \((F \downarrow \{B\})\) to denote \((F \downarrow i_B)\). Explicitly, the objects in \((F \downarrow \{B\})\) are functors \(F(a) \xrightarrow{f} B\), where \(a \in A\), and the morphisms \((Fa \xrightarrow{f} B) \to (Fa' \xrightarrow{f'} B)\) are pairs \((x, \alpha)\), where \(x: a \to a'\) and \(\alpha: f \to f' \circ Fx\), as seen in the natural transformation diagram

\[
\begin{align*}
Fa & \xrightarrow{Fx} Fa' \\
\downarrow f & \xrightarrow{\alpha} \downarrow f' \\
B & \xrightarrow{B} B
\end{align*}
\]

On the other hand, while the objects in \((F \uparrow \{B\})\) are the same as those in \((F \downarrow \{B\})\), a morphism \((Fa \xrightarrow{f} B) \to (Fa' \xrightarrow{f'} B)\) has a reversal in the top 1-arrow. It is a pair \((x, \alpha)\) where \(x: a' \to a\) and \(\alpha: f \to f' \circ Fx\), as seen in the natural transformation diagram

\[
\begin{align*}
Fa & \xleftarrow{Fx} Fa' \\
\downarrow f & \xleftarrow{\alpha} \downarrow f' \\
B & \xleftarrow{B} B
\end{align*}
\]

2. **The Box monads**

In this section we will be working in the base category \(\text{Cat}\), so when we write \((X \downarrow Y)\) we mean \((X \downarrow_{\text{Cat}} Y)\). In particular, let \(\text{FC}\) denote the category of finite categories and consider the full inclusion \(\text{FC} \to \text{Cat}\). We will be considering, for
any category \( C \in \textbf{Cat} \), the categories \((\textbf{FC} \downarrow \{C\})\) and \((\textbf{FC} \uparrow \{C\})\). These are particular cases of the categories considered in Example 1.2.

**Theorem 2.1.** Let \( \textbf{FC} \) denote the category of finite categories. Consider the endofunctor \( d := (\textbf{FC} \downarrow \{-\}) : \textbf{Cat} \to \textbf{Cat} \). There are natural transformations \( \eta : \text{id}_{\textbf{Cat}} \to d \) and \( \mu : d^2 \to d \) that give \( d \) the structure of a monad on \( \textbf{Cat} \).

**Proof.** For \( C \in \textbf{Cat} \), define \( \eta_C : C \to dC \) to be the functor given on objects by \( \{\ast\}^\to C \to \{\ast\} \), and given on morphisms by

\[
\begin{array}{ccc}
(f : c \to c') & \mapsto & \{\ast\} \\
\downarrow & f & \downarrow \\
C & \Rightarrow & c'
\end{array}
\]

This is clearly a natural transformation.

Defining \( \mu \) is more difficult. Let \( C \in \textbf{Cat} \) be a category; we will define \( \mu_C : d^2 C \to dC \).

Let \( X \) be a finite category and let \( \sigma : X \to (\textbf{FC} \downarrow \{C\}) \) be an object in \( d^2 C \).

We need to establish some notation for \( \sigma \). On objects \( x \in X \), denote \( \sigma(x) \) by \( \sigma_x : S_x \to C \), where \( S_x \) is a finite category. On morphisms \( f : x \to y \) in \( X \), denote \( \sigma(f) \) by

\[
\begin{array}{ccc}
\sigma_x & \Rightarrow & \sigma_y \\
\sigma_f & \Rightarrow & \sigma_y
\end{array}
\]

Define \( \mu(X, \sigma) \in \textbf{Cat} \) as the category with objects

\[
\text{Ob}(\mu(X, \sigma)) := \{(x, s_x) | x \in X, s_x \in S_x\}
\]

and morphisms

\[
\text{Hom}_{\mu(X, \sigma)}((x, s_x), (y, s_y)) = \{(f, f^\sharp) | f : x \to y, (f^\sharp : \sigma_f(s_x) \to s_y) \in S_y\}.
\]

Note that since \( X \) and each \( S_x \) is finite, so is \( \mu(X, \sigma) \).

There is a canonical functor \( \mu(X, \sigma) \to C \) given on objects by \( (x, s_x) \mapsto \sigma_x(s_x) \) and given on morphisms by sending \( (f, f^\sharp) : (x, s_x) \to (y, s_y) \) to the composite

\[
\sigma_x(s_x) \xrightarrow{\sigma_f} \sigma_y \sigma_f(s_x) \xrightarrow{\sigma_y(f^\sharp)} \sigma_y(s_y).
\]

Thus \( \mu(X, \sigma) \) is indeed in \( (\textbf{FC} \downarrow \{C\}) \).
So far, we have only established $\mu$ on objects. On morphisms

$\xymatrix{ X \ar[r]^m \ar[dr]_{m^t} & X' \ar[d]^{\sigma'} \ar[l]_{\sigma} \\
 & (FC \downarrow \{C\}) &}$

define $\mu(m, m^t): \mu(X, \sigma) \to \mu(X', \sigma')$ as the functor given as follows. Send the object $(x, s_x) \in \mu(X, \sigma)$ to $(m(x), m^t(s_x)) \in \mu(X', \sigma')$. Send $(f, f^t): (x, s_x) \to (y, s_y)$ to

$(m(x), (m^t_y \circ f^t: \sigma'_{m, f}(m^t_x(s_x)) \to m^t_y(s_y))): \mu(X, \sigma) \to \mu(X', \sigma')$.

There is an obvious natural transformation

$\xymatrix{ \mu(X, \sigma) \ar[r] \ar[dr] & \mu(X', \sigma') \\
 & C &}$

because $m^t: \sigma' \circ m \to \sigma$ was assumed natural.

We draw the following diagram, in case it is useful to the reader:

$\xymatrix{ S_x \ar[r]^{\sigma_f} \ar[dr]_{m^t_x} & S_y \ar[d]^{\sigma_y} \ar[l]^{\sigma^t_x} \\
 & C & S'_y \ar[l]_{\sigma_{m, y}} \ar[u]_{m^t_y} & S'_m \ar[l]_{\sigma^t_{m, y}} \ar[u]_{\sigma_x} \ar[r]_{\sigma'_{m, y}} & S_x \ar[u]_{\sigma_x} \ar[l]_{\sigma_f} }$

We forgo showing that this really is a monad.

\[\square\]

**Corollary 2.2.** Let $FC$ denote the category of finite categories. Consider the endofunctor $e := (FC \uparrow \{-\}): \textbf{Cat} \to \textbf{Cat}$. There are natural transformations $\eta: \text{id}_{\textbf{Cat}} \to e$ and $\mu: e^2 \to e$, which gives $e$ the structure of a monad on $\textbf{Cat}$.

**Proof.** Recall the endofunctor $d: \textbf{Cat} \to \textbf{Cat}$ given by $d = (FC \uparrow \{-\})$ as in Theorem 2.1. For any category $C$ there is a isomorphism (natural in $C$)

$(FC \downarrow \{C\}) \cong (FC \uparrow \{C^{\text{op}}\})^{\text{op}}$

given by sending an object $\sigma: S \to C$ to the object $\sigma^{\text{op}}: S^{\text{op}} \to C^{\text{op}}$ and by sending a morphism $(\sigma_f, \sigma^t_f): (\sigma_x: S_x \to C) \to (\sigma_y: S_y \to C)$, pictured as

$\xymatrix{ S_x \ar[r]^{\sigma_f} \ar[dr]_{\sigma_x} & S_y \ar[d]^{\sigma_y} \ar[l]_{\sigma_x} \\
 & C & S'_y \ar[l]_{\sigma_{m, y}} \ar[u]_{m^t_y} & S'_m \ar[l]_{\sigma^t_{m, y}} \ar[u]_{\sigma_x} \ar[r]_{\sigma'_{m, y}} & S_x \ar[u]_{\sigma_x} \ar[l]_{\sigma_f} }$
to the morphism pictured as

\[
\begin{array}{c}
S_y^\text{op} \\
\downarrow \sigma_f \\
σ_{\gamma}^\text{op} \\
\downarrow \sigma_{\delta} \\
C^\text{op} \\
\end{array}
\]

Now we can simply apply the maps \(μ\) and \(η\) from Theorem 2.1. For \(μ\) we have

\[
(\text{FC} \uparrow \{\text{FC} \uparrow \{C\}\}) \cong (\text{FC} \downarrow \{\text{FC} \downarrow \{C^{\text{op}}\}\})^{\text{op}}
\]

\[
\cong (\text{FC} \downarrow \{\text{FC} \downarrow \{C^{\text{op}}\}\})^{\text{op}}
\]

\[
\cong (\text{FC} \uparrow \{C^{\text{op}}\})^{\text{op}} \cong (\text{FC} \uparrow \{C\})
\]

and it is natural in \(C\). Writing \(op: \text{Cat} \rightarrow \text{Cat}\) to denote the involution, we also have

\[
\begin{array}{c}
\text{Cat} \\
\downarrow \text{id}_{\text{Cat}} \\
\text{Cat} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Cat} \\
\downarrow \text{id}_{\text{Cat}} \\
\text{Cat} \\
\end{array}
\]

so the natural transformation \(η: \text{id}_{\text{Cat}} \rightarrow (\text{FC} \downarrow \{-\})\) extends to a natural transformation \(\text{id}_{\text{Cat}} \rightarrow (\text{FC} \uparrow \{-\})\).

All the monad diagrams commute for \(e\) because they do for \(d\).

In the next section we will define the “global sections functor”

\[
Γ: (\text{FC} \downarrow \{\text{Cat}\}) \rightarrow \text{Cat},
\]

which is related to the Grothendieck construction. For now, we just record a proposition as to the purpose of this material.

**Proposition 2.3.** The global sections functor \(Γ: (\text{FC} \downarrow \{\text{Cat}\}) \rightarrow \text{Cat}\) gives \(\text{Cat}\) the structure of a \(d\)-algebra.

**Proof.***

\[
□
\]

3. **Grothendieck construction**

Let \(f: S \rightarrow \text{Cat}\) be a functor and let \(\{\ast\}\) denote the terminal category. The Grothendieck construction for \(f\) is given by

\[
Gr(f) := (\{\ast\} \downarrow f).
\]

Explicitly, \(Gr(f)\) is the category with objects \((S, s)\) where \(S \in \text{Ob}(S)\) and \(s \in f(S)\). The morphism sets in \(Gr(f)\) are given by

\[
\text{Hom}((S, s), (S', s')) = \{(y: S \rightarrow S', α: f(y)(s) \rightarrow s')\}.
\]

There is a natural projection \(π_f: Gr(f) \rightarrow S\) given by \((S, s) \mapsto S\). It is a split fibration; that is, for any \(y: S \rightarrow S'\) and \(s \in S\), there is a canonical choice of map \((S, s) \rightarrow (S', f(y)(s))\) lying over \(y\).
Lemma 3.1. Let \( f: S \to \text{Cat} \) be a diagram of categories. A functor \( \sigma: C \to S \) induces a diagram

\[
\begin{array}{ccc}
Gr(f) & \xrightarrow{\pi_f} & Gr(f) \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sigma} & S,
\end{array}
\]

which is a pullback square.

Proof. ***

\[ \square \]

Definition 3.2. Let \( \pi: T \to S \) be a functor. A section of \( \pi \) is a functor \( g: S \to T \) such that \( \pi g = \text{id}_S \). A morphism of sections of \( \pi \) is a natural transformation over \( S \).

Recall that a pullback square as in Lemma 3.1 defines a functor

\[ g \mapsto (\text{id}_C \times \text{id}_S (g \circ \sigma)) \]

from the sections of \( \pi_f: Gr(f) \to S \) to the sections of \( \pi_{f\sigma}: Gr(f\sigma) \to C \).

Lemma 3.3. Suppose given two functors \( f, g: S \to \text{Cat} \); let \( \text{Nat}(f, g) \) denote the set of natural transformations from \( f \) to \( g \). There is a natural isomorphism

\[ \text{Hom}_S(Gr(f), Gr(g)) \cong \text{Nat}(f, g). \]

Proof. ***

\[ \square \]

Definition 3.4. There is a functor \( \Gamma: (\text{Cat} \uparrow \{\text{Cat}\}) \to \text{Cat} \) defined as follows. On objects, \( \Gamma(f: C \to \text{Cat}) \) is defined to be the category of sections of the functor \( \pi: Gr(f) \to C \). To define \( \Gamma \) on morphisms, suppose we are given a natural transformation diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & D \\
f \downarrow & & \downarrow \\
\text{Cat} & \xrightarrow{g} & 
\end{array}
\]

By Lemma 3.1 there is a canonical functor \( \Gamma(g) \to \Gamma(gx) \). By Lemma 3.3 there is a canonical functor \( \Gamma(gx) \to \Gamma(f) \) induced by \( \alpha \). The composition gives the desired functor.

Lemma 3.5. Suppose that \( \sigma_i: C_i \to \text{Cat} \), for \( i = 1, 2, 3 \), are three functors and suppose that \( f: C_2 \to C_1 \) and \( g: C_2 \to C_3 \) are functors too. There is an induced functor \( C_4 := C_1 \sqcup C_2 \to \text{Cat} \). Taking Grothendieck constructions yields the
All of the vertical squares are pullback squares by Lemma 3.1.

The natural map

$$\Gamma(\sigma_4) \longrightarrow \Gamma(\sigma_1) \times_{\Gamma(\sigma_2)} \Gamma(\sigma_3)$$

is an isomorphism.