THE CATEGORY OF AUTOMATA

The following definition, restricted to objects, is [Sipser, Definition 1.5].

**Definition 0.1.** An automaton consists of a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$ where $Q$ and $\Sigma$ are sets, $q_0 \in Q$ is an element of $Q$ and $F \subset Q$ is a subset of $Q$, and where $\delta: \Sigma \times Q \to Q$ is a function. The elements of $Q$ are called states and the elements of $\Sigma$ are called letters.

Suppose that $A' = (Q', \Sigma', \delta', q'_0, F')$ is another automaton. A morphism of automata, denoted $(f, g): A \to A'$, consists of a function $f: Q \to Q'$ and a function $g: \Sigma \to \Sigma'$ such that $f(q_0) = q'_0$, $f(F) \subset F'$ and the diagram

$$
\begin{align*}
Q \times \Sigma & \xrightarrow{\delta} Q \\
Q' \times \Sigma' & \xrightarrow{\delta'} Q'
\end{align*}
$$

commutes. We denote the category of automata by Automata.

**Definition 0.2.** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton. We call the monoid $\Sigma^*$ of sequences in $\Sigma$ the action monoid for $A$, and note that $\delta$ gives $Q$ the structure of a right $\Sigma^*$-set. For $a \in \Sigma^*$ and $q \in Q$, we write $q \cdot a$ to denote the result of acting $a$ on $q$.

The language of $A$, denoted $L(A)$ is the subset of elements $a \in \Sigma^*$ such that $q_0 \cdot a \subset F$. Note that a morphism $(f, g): A \to A'$ induces a morphism of sets $L(A) \to L(A')$, so that $L: \text{Automata} \to \text{Sets}$ is a functor.

I’m not sure what the point of the following definition is, but it is natural to define.

**Definition 0.3.** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton. An element $a \in \Sigma^*$ is said to preserve acceptance if, for all $q \in F$ one has $q \cdot a \in F$. Note that the set of letters that preserve acceptance is a submonoid of $\Sigma^*$.

**References**