Theorem 0.0.1. There is a fully faithful functor from the category of small multi-categories to the category of database multi-categories.

Proof. Let $\mathcal{M}$ be a small multi-category. For any object in $M \in \text{Ob}(\mathcal{M})$, let $H(M)$ denote the set

$$H_M := \prod_{n \geq 1} \text{Hom}_\mathcal{M}(X_1, \ldots, X_n; M).$$

Let $\mathbf{DT}_\mathcal{M} = \text{Ob}(\mathcal{M})$, let $U_M = (\coprod_{D \in \mathbf{DT}_\mathcal{M}} H_D)$, and let $\pi_\mathcal{M}: U_M \to \mathbf{DT}_\mathcal{M}$ denote the obvious map.

For each morphism $f: (X_1, \ldots, X_n) \to M$ in $\mathcal{M}$, let $S_f$ denote the operational table whose schema is $\sigma_f := (X_1, \ldots, X_n; M)$ and whose row set is

$$R_f := \{(f_1, \ldots, f_n, g) \in H_{X_1} \times \cdots \times H_{X_n} \times H_M \mid g = f \circ (f_1, \ldots, f_n)\}.$$ 

There is an obvious map $\delta_f: R \to \Gamma(\sigma_f) = H_{X_1} \times \cdots \times H_{X_n} \times H_M$. That is, we set $S_f = (\sigma_f, \delta_f)$. Let $S_\mathcal{M} = \{S_f \mid f \in \mathcal{M}\}$.

In the following paragraph we will show that the pair $(\pi_\mathcal{M}, S_\mathcal{M})$ is a database multi-category. That is, it contains the identities, and is closed under operational table composition. That it contains the identities is obvious: for any $M \in \text{Ob}(\mathcal{M})$ we find $S_{\text{id}_M}$ in $S_\mathcal{M}$.

Let $(S_{f_1}, \ldots, S_{f_n}; S_f)$ denote a composable sequence of tables in $S_\mathcal{M}$; denote the inputs of $S_{f_i}$ by $\sigma_i$, the inputs of $S_f$ by $\sigma: (C_1, \ldots, C_n) \to \mathbf{DT}$, and the output of $S_f$ by $M$. It is obvious that $(f_1, \ldots, f_n; f)$ is a composable sequence of arrows in $\mathcal{M}$. The operational table composite of $(S_{f_1}, \ldots, S_{f_n}; S_f)$ has as input type $\sigma_1 \Pi \cdots \Pi \sigma_n$ and has as output type $M$; it is obtained as the join of the sequence. It is easy to see that this join is precisely $S_{f_0(f_1, \ldots, f_n)}$.

We have now constructed our functor on objects. Suppose given a morphism of multi-categories $F: \mathcal{M} \to \mathcal{N}$. It clearly induces a map $\pi(F): \pi_\mathcal{M} \to \pi_\mathcal{N}$. Let $T_\mathcal{N}$ denote the counterpart to $S_\mathcal{M}$. It is clear that $F$ also induces a function $S_\mathcal{M} \to T_\mathcal{N}$ by sending each $S_f = (\quad$ to

\[ \square \]