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Calculus III and Stokes' Theorem

Let A be a real affine space of dimension n and $C^\infty(A)$ the ring of real valued functions on A. All the tangent spaces of A are isomorphic as real vector spaces and we write TA for a representative of this isomorphism class. Let $\bigwedge^\bullet(TA)^\vee$ denote the exterior algebra of the vector space dual of TA. Then the tensor product

$$C^\infty(A) \otimes_{\mathbb{R}} \bigwedge^\bullet (TA)^\vee$$

is a differential graded algebra, which we call the de Rham complex of A and denote $\Omega^\bullet(A)$. We refer the ith graded piece $\Omega^i(A), (0 \leq i \leq n)$, of the de Rham complex as the module of i-forms of A.

Example 0.1. Suppose $A = \mathbb{R}^3$. Then $\Omega^\bullet(A)$ is the DGA

$$C^\infty\mathbb{R}^3 \xrightarrow{d} C^\infty\mathbb{R}^3\langle dx, dy, dz\rangle \xrightarrow{d} C^\infty\mathbb{R}^3\langle dx\wedge dy, dy\wedge dz, dz\wedge dx\rangle \xrightarrow{d} C^\infty\mathbb{R}^3\langle dx\wedge dy\wedge dz\rangle.$$

The chain complex axiom that $d^2 = 0$ corresponds to the calculus fact that Curl(∇f)=0 and Div(Curl$F$) $= 0$ for a function f and a vector field F. Since $\Omega^\bullet(A)$ is the de Rham complex, a student of differential geometry knows that $H^1(\Omega^\bullet(A)) = 0$ if A is simply connected; this is theorem 10.3.6 in Stewart. Such a student also knows that $H^0(A) = \mathbb{R}^i$ where i is the number of connected components of A. This corresponds to the Calculus fact that the gradient of a function f is zero if and only if f is locally constant; if A has i connected components, then the $\mathbb{R}$-module of locally constant functions on A is clearly $\mathbb{R}^i$.

The above discussion holds for the category of affine spaces, but to delve more deeply into the connection between differential geometry and Calculus III, we choose a Euclidian metric g and an orientation on TA.

The metric $g : TA \times TA \to \mathbb{R}$ induces a map $g^\sharp : TA \to (TA)^\vee$ (where $(TA)^\vee$ is the vector-space dual of TA and called the cotangent space of A) via the adjunction $\mathrm{Hom}_{\mathbb{R}}(TA \otimes_{\mathbb{R}} TA, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(TA, \mathrm{Hom}_{\mathbb{R}}(TA, \mathbb{R}))$. The fact that the g is positive definite (by virtue of being a metric) implies that $g^\sharp$ is an isomorphism. Thus we may identify TA with $(TA)^\vee$ given our choice of g. For simplicity, we write g to denote $g^\sharp$, which will cause no confusion since we will never again need to refer to the function $TA \times TA \to \mathbb{R}$.

Consider the $C^\infty A$-module isomorphism $h_i(A) : \Omega^i(A) \to \Omega^{n-i}(A)$ which sends the basis vector $b \in \Omega^i$ to the unique basis vector $c \in \Omega^{n-i}$ such that $b \wedge c$ is the basis vector of unit length and positive orientation in $\Omega^n(A)$.

Example 0.2. In the case $A = \mathbb{R}^3$, the De Rham complex $\Omega^\bullet A$ is isomorphic to the DGA

$$C^\infty\mathbb{R}^3 \xrightarrow{\nabla} C^\infty\mathbb{R}^3\langle i,j,k\rangle \xrightarrow{\text{Curl}} C^\infty\mathbb{R}^3\langle i,j,k\rangle \xrightarrow{\text{Div}} C^\infty\mathbb{R}^3,$$

which we denote $\text{Calc}^\bullet(A)$. The differentials in $\text{Calc}^\bullet(A)$ are given by $\nabla = gd$, $\text{Curl} = gh_2dg^{-1}$, and $\text{Div} = h_3dg^{-1}h_2^{-1}$. The reader should check these maps, as they are easier to understand than it may appear at first sight.

It is clear that the first differential in $\text{Calc}^\bullet(A)$ relies only on the metric g, whereas the last two rely on the orientation (in order to form the h_i.) In Calculus III, we have chosen a metric and orientation of $\mathbb{R}^3$ without expressing it, and we therefore simply identify Ω^2 with Ω^1, and Ω^3 with Ω^0. Hence, there is no way to tell in which graded piece a given scalar field or vector field lives.

We end this example with a calculation of Curl. A 1-form in $\text{Calc}^\bullet(\mathbb{R}^3)$ is of the form $\alpha = f_1i + f_2j + f_3k$, and we have

$$g^{-1}\alpha = f_1dx + f_2dy + f_3dz \in \Omega^1 A.$$

Applying the differential, we get

$$\begin{aligned} gh_2(dg^{-1}\alpha) &= gh_2\Big(\frac{\partial f_1}{\partial y}dy\wedge dx + \frac{\partial f_1}{\partial z}dz\wedge dx + \\ &\qquad \frac{\partial f_2}{\partial x}dx\wedge dy + \frac{\partial f_2}{\partial z}dz\wedge dy + \\ &\qquad \frac{\partial f_3}{\partial x}dx\wedge dz + \frac{\partial f_3}{\partial y}dy\wedge dz\Big) \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)k. \end{aligned}$$

Finally we get to the statement of Stokes' theorem. Consider first the "smooth singular" chain complex $C_\bullet M$ for a manifold M: the chain groups are

$$C_iM = \mathbb{R}\langle \text{Hom}_{sm}(\Delta^i, M)\rangle,$$

i.e. C_iM is the free $\mathbb{R}$-module generated by the smooth functions from the manifold with corners Δ^i to M. The inclusion $\partial\Delta^i \to \Delta^i$ induces a map of free modules

$$\mathbb{R}\langle\partial^*\rangle : C_iM \to C_{i-1}M,$$

which we denote simply by ∂.

Let $C^\bullet M$ denote the dual cochain complex with cochain groups

$$C^iM = \text{Hom}_{\mathbb{R}}(C_iM, \mathbb{R})$$

and differential operator $d' : C^{i-1}M \to C^iM$ given by $(d\phi)(a) = \phi(\partial a)$, for $\phi \in C^{i-1}M$ and $a \in C_iM$. We denote this differential operator by d' so as not to confuse it with the differential in the De Rham complex.

There is a map of cochain groups $\int : \Omega^\bullet A \to C^\bullet A$; i.e. a form $\alpha \in \Omega^i A$ induces a cochain $a \mapsto \int_a \alpha \in \mathbb{R}$ for $a \in C_iA$. Stokes' theorem simply says that these maps induce a map of chain complexes

$$\Omega^\bullet A \to C^\bullet A.$$

That is, for any $\alpha \in \Omega^i A$ and any $a \in C_{i+1}A$, Stokes' theorem states that the proposed equation

$$\int_a d\alpha = \left(\int d\alpha\right)(a) =^? d'\left(\int \alpha\right)(a) = \left(\int \alpha\right)(\partial a) = \int_{\partial a} \alpha$$

holds.