1. INTRODUCTION

We will attempt to find a smooth version of the classical (discrete) game of chess. Defined rigorously below, a smooth version of a game is in some sense reminiscent of the probability distributions discussed in quantum mechanics, in the sense that a piece is not necessarily concentrated in one place, but distributed throughout the board. For example, if a player believes that there is a 30% chance that move $A$ is correct and a 70% chance that move $B$ is correct, then she can move in accordance with that probability distribution, using the move $.3A + .7B$.

Let us briefly describe smooth chess, so the reader has something to keep in mind. Play in smooth chess begins with the board set up as in classical chess. Play alternates between players, with each player on his turn required to move a total of 1 unit of material (distributed in any way he chooses). The only exception occurs if the player cannot legally make a full move, for example if he has only $7\text{ Kings}$ left, in which case he must move the maximum amount possible.

The guiding principles of smooth chess are as follows. Each square “holds” one full unit of material. Fractional pieces may slide through squares, provided there is “enough room” to do so. A fractional piece may land on a square provided that doing so would not result in there being in excess of 1 unit of material of its own color on the square. In other words, he cannot overflow the square with his own material. Capturing occurs when a player moves to a square and in so doing overflows the square in such a way that, by removing his opponent’s material, the quantity of material on the square can be brought down to exactly 1. In this case, the opposing player chooses the distribution of his material which is captured.
One of the goals in inventing a smooth version of chess was to provide chess with a topology such that similar positions result in similar outcomes. We will describe the topology on smooth chess in section 4. Given a board position \( P \), let \( \text{Opt}(P) \in [-1,1] \) denote the amount that white wins by, assuming optimal play starting at \( P \). One of the most interesting questions one may ask about smooth chess is whether \( \text{Opt} \) is a continuous function from the space of board positions to the interval \([−1,1]\). In fact, the game as we present it in section 3.7 is the result of a long process of refinements, whose aim was to find a smooth version of chess which is faithful to the classical version and which has this continuity property. As in the recent phenomenon of “categorification” in mathematics, the process of “smoothification” presents one with many options, and it can be unclear which is the best, and in what senses that can even be evaluated. Thus we can neither say that our choices are the best aesthetically, nor can we prove that the function \( \text{Opt} \) is continuous for smooth chess. Our aim is merely to offer one suggestion for how this kind of thing could work.

This paper is organized as follows. In Section 2 we write down the ways we understand chess as a board game, in such a way that the smoothing process makes the most sense. In Section 3.7 we describe the rules of smooth chess.

I wish to apologize to combinatorial game theorists – my presentation does not at all follow Conway’s. This is not because I have gone through the trouble to check that Conway’s conception will not work – it’s because I do not have the proper training or, frankly, the urge to think about it without a knowledgeable collaborator. However, there is another reason as well, and that is that it seemed to me that the precise ways in which smooth chess really is a generalization of chess is made most apparent using the presentation I give. Regardless, as this paper is not currently being published but simply disseminated, I hope that any person who wishes to consider the basic idea of smooth chess, either as is or using accepted game-theoretic ideas, will be interested in collaborating on such a project.

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2. Chess as a board game

**Definition 2.1.** A setup \((B, S_A, S_P, M)\) consists of a (directed) graph \(B\), called the board, two disjoint subsets \(S_A, S_P \subseteq B_0\) of its underlying set, such that \(S_A \amalg S_P = B_0\), and a set \(M\) of material. We refer to elements of the underlying set \(B_0\) of \(B\) as squares. Thus, the squares come in two varieties, called the set of administrative squares, \(S_A\), and the set of play squares, \(S_P\). We refer to elements \(m \in M\) as materia (in the discrete world, these can be thought of as pieces).

A position in \((B, S_A, S_P, M)\) consists of a function \(p: B_0 \times M \to \{0,1\}\), which assigns to every square and every materium, a “true/false” value, corresponding to whether that materium is represented on that square. Let \(P = \{0,1\}^{B_0 \times M}\) be the set of positions. We can also write \(P = \mathcal{P}(B_0 \times M)\), the power set of \(B_0 \times M\).

Given a position \(p\), a piece in \(p\) is an element \(x = (b, m) \in B_0 \times M\) such that \(p(b, m) = 1\). Of course, a position \(p: B_0 \times M \to \{0,1\}\) is the characteristic function for some subset of \(B_0 \times M\), which we might also write as \(p\). Then a piece is just an element \(x \in p\).
Example 2.2. Let us discuss the setup for a game called baby chess. Let \( B'' \) be the complete graph with vertices \( \{(i,j) | 1 \leq i,j \leq 8\} \). Let \( B' \) be the graph that looks just like \( B'' \), except with
- a terminal vertex called \textit{captured} (i.e. there is a single edge \( b \rightarrow \text{captured} \) for every \( b \in B_0 \) and no other edges to or from \textit{captured}), and
- a vertex called \textit{storage} that has a single edge to every vertex \( (i,1) \in B'' \), a single edge to every vertex \( (i,8) \in B'' \), and no other edges to or from it.

Let \textit{turn} be the graph with two vertices and an edge in each direction. We will call the two vertices of \textit{turn} “white’s” and “black’s.” Let \( B = B' \sqcup \text{turn} \).

Here, the play squares \( S_P = |B''| \) consists of the set of squares in the \( 8 \times 8 \) subboard; the rest of the squares (\textit{captured}, etc.) are administrative squares.

The material for baby chess consists of the set
\[
M = \left( \{w,b\} \times \{K,Q,R,B,N,P\} \right) \sqcup \{\text{turn}\},
\]
consisting of a white and black King, Queen, Rook, Bishop, Knight, and Pawn, as well as a \textit{turn} indicator.

Chess has a classical “initial position” \( p_0 \in P \), consisting, for each color, of two rooks, two knights, two bishops, a queen, a king, and eight pawns. For example \( p_0((1,1),wR) = 1 \) and \( p_0((1,8),wR) = 1 \) and \( p_0(x,wR) = 0 \) for all \( x \notin \{(1,1),(1,8)\} \). For our setup, we need to include a few more pieces in \( p_0 \): a \textit{turn} marker on the vertex “whites”, and eight queens, rooks, bishops, and knights of each color in storage.

Unfortunately, the setup for chess is different than the setup for baby chess, whatever that is. In baby chess, one can see that there is no room in the set of positions for one to discuss the history of the game. In other words, the position at a certain stage in the game cannot reveal whether the king has been moved (which governs castling in chess), or whether a pawn has moved “up two” (which governs en passant in chess).

While the material is the same as for baby chess, the setup for chess must include a bigger “board”. Let \( B'' \) be the graph with vertices \( \{(i,j,n) | 1 \leq i,j \leq 8, n \in \mathbb{N}\} \), and with an edge \( (i_1,j_1,n_1) \rightarrow (i_2,j_2,n_2) \) if and only if \( n_2 = n_1 + 1 \).

Now, there is enough information in a chess “position” to allow one to discern what the legal subsequent moves will be. We will never again discuss baby chess, but basically it is chess without castling or \textit{en passant}.

One might wonder why chose the chess board to include a complete graph on the squares \( 8 \times 8 \) rather than, say, an edge \( (i_1,j_1) \rightarrow (i_2,j_2) \) if and only if the two squares are adjacent in the sense that \( |i_1 - i_2| + |j_1 - j_2| = 1 \). In this paper, one might call this graph the “king on an empty board” graph. Since various pieces can move various ways, and since those ways are limited by the board position at the time, we decided it was best to let the board itself allow all possible moves, and then limit them for game play in a different way.

**Definition 2.3.** Suppose that \((B,S_A,S_P,M)\) is a setup, \( p \subset B_0 \times M \) a position, and \( x = (b,m) \in p \) a piece. The set of legal submoves for \( x \in p \) is a subset \( \text{sm}(x,p) \subset B/b \) of the set of squares adjacent to (or more precisely, squares with an edge coming from) \( b \). Let \( \text{sm}(p) = \bigcup_x \text{sm}(x,p) \) denote the set of all submoves for a position \( p \).
Each submove \( f \in \text{sm}(p) \) can be written as
\[
f = (m_f, b_f, t_f),
\]
where \( m_f \) is the materium, \( b_f \) is the initial square, and \( t_f \) is the terminal square of the submove. A submove \((m, b, t)\) is said to be administrative if either \( b \) or \( t \) is an element of \( S_A \), the set of administrative squares. A submove that is not administrative is called chosen.

Given a board position \( p \), a play on \( p \) is a finite sequence \(((x_1, f_1), \ldots, (x_n, f_n))\) such that each \( x_i \in p \) is a piece, each \( f_i \in \text{sm}(x_i, p_i) \) is a legal submove, and exactly one of the submoves \( f_i = (m_i, b_i, t_i) \) is chosen. A set of legal moves from \( p \) is a subset \( \text{m}(p) \) of the set of plays on \( p \).

In chess, the set of legal moves for a piece \( x \in p \) varies from piece to piece, and (historical) position to (historical) position. For example, a rook on an empty board can legally move along rows or columns. However, on a non-empty board the rook’s movement is restricted by pieces of its own color and pieces of the opposite color. As another example, if there is a pawn on the penultimate rank and nothing on the last rank, it can move to the eighth rank, and from there to the captured square, and one can move, say, a Queen from the storage square to the eighth rank. Finally, the player moves the turn indicator. Note that these four legal submoves are part of the same move. The first is chosen and the rest are administrative.

The promotion of a pawn is one example of a legal move. Capturing provides another: if a black pawn is next to a white king, the king can move to the pawn’s square and subsequently the pawn will be moved to the captured square. The reason for this is simply that we believe that smooth chess works better under this convention.

The only thing we still must discuss is how games end and how we score them.

**Definition 2.4.** A set of ending positions for a setup \((B, S_A, S_P, M)\) is a subset \( E \subseteq P = \mathcal{P}(B_0 \times M) \), such that for any \( p \in E \) and any \( x \in p \), one has \( M(x, p) = \emptyset \).

Fix any set \( V \), called the set of win values. A scoring function for \((B, S_A, S_P, M, V)\) is a function \( s: E \rightarrow V \).

A board game \( G = ((B, S_A, S_P, M), \text{sm}, m, E, V, s) \) consists of a setup \((B, S_A, S_P, M)\), a set of legal submoves \( \text{sm}(x, p) \) for every piece \( x \) in every position \( p \), a set of legal moves \( m(p) \) for each \( p \), a set \( E \) of ending positions for \((B, S_A, S_P, M)\), a set \( V \) of win values, and a scoring function \( s: E \rightarrow V \).

A game of \( G \) is a finite sequence \((m_1, m_2, \ldots, m_n) \in m^n \) of legal moves.

For chess, the set \( E \) of ending positions is the set of checkmates together with the set of stalemates. (One could also connect a “resignation” square and a “draw offer” square to the turn subboard, and then accordingly modify \( E \), but this is going overboard.) The set of win values is \( \{-1, 0, 1\} \) corresponding to black win, draw, and white win. The scoring function is obvious; for example if white checkmates black, then the score is 1.

**Example 2.5.** Note that board games, as we have defined them, can include multiplayer games as well as solitary games. In fact, different players can even take on
different roles; this information is simply subsumed in the set \( m \) of legal moves, which will be dependent on the placement of the turn indicator. Thus, for example, the classical games of backgammon, Monopoly, and Go can each be made into a board game in our sense. One might wonder about how to deal with dice. The answer is to do two things. First, consider the dice roll as another player and include it in the turn indicator before each “real player’s” turn. Second, include some new administrative squares for where the dice lands, and encode the relationship between the dice and the play in the set \( m \) of legal moves.

3. Smooth Chess as a smoothing of chess

We will set up the definition of a smooth board game. Let \([0, 1] \subset \mathbb{R}\) denote the closed unit interval.

**Definition 3.1.** A smooth setup is just a setup \((B, S_A, S_P, M)\). However a smooth position in \((B, S_A, S_P, M)\) consists of a function \( B_0 \times M \rightarrow [0, 1] \), which assigns to every square and materium a value corresponding to “how much of” that materium exists on that square. We take the set of smooth positions to be \( P = [0, 1]^{B_0 \times M} \).

Given a (smooth) position \( p : B_0 \times M \rightarrow [0, 1] \), a fractional piece in \( p \) is an element \( x = (b, m, v) \in B \times M \times (0, 1] \) such that \( v \leq p(b, m) \). By abuse of notation, we write \( x \in p \) when \( x \) is a fractional piece in a given position. The value of \( x = (b, m, v) \) is \( v \). If \( v = 1 \) we call \( x \) a full piece.

In smooth chess, our setup is exactly the same as for chess; i.e. same board, same administrative and play squares, and same material. The initial position for smooth chess is also the same as for chess; for example there are eight full white pawns on the second rank. The difference is that we will allow fractional movement of pieces.

Legal submoves and plays are defined for smooth board games analogously to the way they are for board games, except that pieces are replaced with fractional pieces, and submoves are replaced by sums of submoves whose total value is 1.

**Definition 3.2.** Suppose that \((B, S_A, S_P, M)\) is a setup, \( p : B_0 \times M \rightarrow [0, 1] \) a smooth position, and \( x = (b, m, v) \in p \) a fractional piece. The set of legal submoves for \( x \in p \) is a subset \( \text{sm}(x, p) \subset B_0/b \) of the set of squares adjacent to (or more precisely, squares with an edge coming from) \( b \). Let \( \text{sm}(p) = \bigcup_x \text{sm}(x, p) \) denote the set of all submoves for a position \( p \).

Each submove \( f \in \text{sm}(p) \) can be written as

\[
    f = (m_f, b_f, v_f, t_f),
\]

where \( m_f \) is the materium, \( b_f \) is the initial square, \( 0 < v_f \leq 1 \) is the value, and \( t_f \) is the terminal square of the submove. A submove \((m, b, v, t)\) is said to be administrative if either \( b \) or \( t \) is an element of \( S_A \), the set of administrative squares. A submove that is not administrative is called chosen. The value of a submove \((m, b, v, t)\) is \( v \). Note that, since the value of a move is non-zero, we can write

\[
    (m, b, v_1, t) = \frac{v_1}{v_2} (m, b, v_2, t)
\]

for two submoves which only differ in value.

We then require that \( \text{sm}(p) \) be closed under multiplication by elements of \([0, 1] \in \mathbb{R}\). Note that we will be able to restrict which sequences of submoves are legal when
we choose $m$, so this “closure” restriction in no way actually limits the class of smooth board games.

Given a board position $p$, a *play on $p$* is a finite sequence $((x_1, f_1), \ldots, (x_n, f_n))$ such that

- each $x_i \in p$ is a fractional piece, and
- each $f_i \in \text{sm}(x_i, p_i)$ is a legal submove.

The *value* of a play is the sum of the values of the chosen submoves in the play. A *set of legal moves from $p$* is a subset $m(p)$ of the set of plays on $p$ which have value 1. If this set is empty, then $m(p)$ is a subset of the set of plays on $p$ which have the maximum attainable value.

In smooth chess, we allow fractional pieces to move on an empty board just like regular pieces do. For example a half rook on an empty board can move along rows or columns. In chess, two white pieces can never share a square, and in smooth chess two full pieces cannot share a square either. However, fractional pieces can.

The way we think of smooth chess is as though every play square can hold at most 1 unit of material. Thus .6 white rooks and .3 white queens can be on a $(1, 1)$. If .1 white kings wanted to join, they could, but .2 could not. Capture occurs when a square is overflowed, in the sense that the total value on that square is greater than 1. In this case, the capturing player leaves all the pieces on the overflowed square, and the opponent moves any distribution of his own material to the captured square before he chooses his next move. This convention was chosen by the authors because we believe it leads to a better game, but of course the alternate convention still leads to a smooth board game. Note that even though he chooses this move, it is considered “administrative” because material is being sent to an administrative square.

Another strange convention involves castling. Typically, we think of castling as a process in which two pieces move between play squares, but this is illegal under our rules. So we define castling to be choosing to move the King either two to the right or two to the left, then (administratively) moving the rook to the captured square, and finally (administratively) moving a rook from the storage square to its appropriate position. We ask the reader for leeway for our error above in which we only put eight rooks in storage; it would have been cumbersome to explain the reasoning for a ninth rook at that point.

The definitions of a set $E$ of ending positions, a set $V$ of win values, and a scoring function $s : E \rightarrow V$ for smooth board games are defined exactly as they are for board games.

**Definition 3.3.** A *smooth board game* $G = ((B, S_A, S_P, M), \text{sm}, m, E, V, s)$ consists of a setup $(B, S_A, S_P, M)$, a set of legal submoves $\text{sm}(x, p)$ for every fractional piece $x$ in every position $p$, a set of legal moves $m(p)$ for each $p$, a set $E$ of ending positions, a set $V$ of win values, and a scoring function $s : E \rightarrow V$.

**Definition 3.4.** For either a board game or a smooth board game $G$, a *game in progress of $G$* is a finite sequence $(f_1, f_2, \ldots, f_n) \in m^n$ of legal moves, and a *finished game of $G$* is a game in progress of $G$ such that the terminal position $t_n$ of $f_n = (m_n, b_n, v_n, t_n)$ is an element of the set $E$ of ending positions.

There is a function taking games in progress to positions, which we denote $(f_1, \ldots, f_n) \mapsto p(f_1, \ldots, f_n)$. 
Now that we have defined smooth board games, we need to define the smoothing of a board game.

**Definition 3.5.** Let \( G = ((B, S_A, S_P, M), \text{sm}, \text{m}, E, V, s) \) be a board game. A *smoothing of* \( G \) *is a smooth board game* \( G_s = ((B, S_A, S_P, M), \text{sm}_s, \text{m}_s, E_s, V_s, s_s) \) *that satisfies the following axioms.*

1. The two setups are the same.
2. Every game in progress of \( G \) is a game in progress of \( G_s \), and every finished game in \( G \) is finished in \( G_s \).
3. Given a game in progress \( f = (f_1, \ldots, f_n) \) of \( G \), the set of legal submoves \( \text{sm}_s(p(f)) \) for \( G_s \) is the closure of the set of legal submoves \( \text{sm}(p(f)) \) for \( G \), under multiplication by elements of \([0, 1] \in \mathbb{R}\).
4. Given a game in progress \( f = (f_1, \ldots, f_n) \) of \( G \), the set \( \text{m}(p(f_1, \ldots, f_n)) \) of legal moves in \( G \) is equal to the subset
   \[ \{f = (m, b, v, t) \in \text{m}_s(p(f_1, \ldots, f_n)) | v = 1\} \]
   of legal moves in \( G_s \) which consist of only one chosen submove (of value 1).
5. The set \( V_s \) of win values for \( G_s \) is the closure of \( V \) under multiplication by elements of \([0, 1] \in \mathbb{R}\).
6. For any finished game \( f = (f_1, \ldots, f_n) \) of \( G \), the scores \( s(f) = s_s(f) \in V \) are equal.

**Remark 3.6.** Note that there may be many different smoothings of the same board game \( G \). The above axioms dictate that the only differences between these smoothings can only exist when there are non-full pieces on the board, which of course occurs much of the time in a smooth game! Aesthetically, one hopes that a given smoothing of \( G \) will be somehow faithful to the spirit of \( G \). Unfortunately, this seems difficult to quantify.

3.7. **Rules of smooth chess.** There may be many smoothings of the same game. We now describe our particular incarnation of smooth chess.

As mentioned before, and as dictated by the fact that smooth chess is a smoothing of chess, the setup \((B, S_A, S_P, M)\) for smooth chess is the same as for chess, and the set of win values \( V \) is \([-1, 1]\), the closure of \([-1, 0, 1]\) under multiplication by \([0, 1] \in \mathbb{R}\).

### 4. Analysis of Smooth Chess

#### 5. Examples

#### 6. Theorems

#### 7. Open Questions
8. Old stuff, probably trash

Just to provide a frame of reference, we begin by giving a quick review of the rules of chess.


Notation 8.2. Let $T$ be the set of types $\{K, Q, R, B, N, P\}$, whose elements are called the King, Queen, Rook, Bishop, Knight, and Pawn, respectively. Let $C$ denote the set of colors $\{w, b\}$, whose elements are white and black. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ denote the set of squares. A board position is a function $P: C \times T \times S \to \{0, 1\}$ such that for all $s \in S$, one has

$$\sum_{c \in C, t \in T} P(c, t, s) \in \{0, 1\}.$$ 

The position function $P$ has many transposes, which are useful for describing positions. For example, $P_{(1,6)}: C \times T \to \{0, 1\}$ is the function which describes what material is on square $(1,6)$, and $P_{wQ}: S \to \{0, 1\}$ is the function which expresses which square the white Queen is on. Note that not all board positions are “legal chess positions” for example one could have a pawn on the first rank; however this will not concern us.

A piece is a sequence $(P, c, t, s)$, where $P$ is a board position and $(c, t, s) \in C \times T \times S$, such that $P(c, t, s) = 1$.

It is difficult to find a notation for presenting a position that is typographically economical. Instead of describing our notation, it is easier to simply give an example. We use the fact that each square can have only one piece on it, and therefore write $P_{(1,6)} = wQ$ or $P_{wQ} = (1,6)$ instead of $P(w, Q, (1,6)) = 1$. In this case we say “in position $P$, there is a white Queen on square $(1,6)$.” We may also refer to the square $(1,6)$ as A6. Confusion arising from the fact that the second column and the bishop are both denoted with the same letter should not arise, because the column, B, is written in Roman font, whereas the Bishop, B, is written in math mode.

Play begins with the board set up in the well-known way, which we call the opening position. A piece that is positioned as it was in the opening position is also said to be in its opening position. Each piece has a well-known set of ways in which it can move. These come in a few varieties, which we briefly describe below. A more detailed account of the opening position and the legal moves can be found in any beginning book on chess or at www.wikipedia.org.

Sliding pieces: Bishops, Rooks, and Queens are called sliding pieces, and each has an allowable set of directions in which it can slide: Bishops slide along diagonals, Rooks slide along rows and columns, and Queens can do either. Each move of a sliding piece $(c, t)$ is broken up into two parts: the intermediate squares and the terminal squares of the move. The intermediate squares must be unoccupied. The terminal square may not be occupied by a piece of the same color $c$, but it may be occupied by a piece of the opposing color. In this case, the sliding piece $(c, t)$ replaces the opposing piece, the opposing piece is removed from the board, and $(c, t)$ is said to capture the opposing piece.
**KNights:** KNights jump in a well-known 2-1 pattern. There are no intermediate squares in a kNight move, however there is a terminal square. The terminal square may not be occupied by a piece of the same color, but it may be occupied by a piece of the opposing color, in which case the kNight captures the opposing piece.

**Pawns:** Pawns usually move forward one square. They may not do so if the square in front of them is occupied by a piece of either color. There are four exceptions to this description of pawn movement. First, pawns may move diagonally forward to a square if that square is occupied by a piece of the opposing color, in which case the pawn captures the opposing piece. Second, a pawn in its opening position may move forward two squares, provided neither the intermediate nor the terminal square is occupied. Third, there is en passant, which roughly says that if a pawn $P$ moves forward 2 squares and an opposing pawn could have captured it had it only moved forward one square, the opposing pawn may “pretend” that $P$ only moved forward one square and capture it accordingly. Finally, if a pawn moves forward one square and lands on the first or last rank of the board, then it must be replaced by either a kNight, Bishop, Rook, or Queen, as decided by the player who moved the pawn.

**Kings:** Kings may move one square in any direction, provided that the square to which it moves is not occupied by a piece of the same color; if it is occupied by a piece of opposing color then the King captures the opposing piece. A king is said to be in check if, were it the opponent’s move, it could be captured. If a player’s King is not in check but the player’s only legal move is to move his King into check, then he may refuse to do so and instead claim a stalemate. If a player captures the opponent’s King, she wins the game.

**Castling:** If a King and a Rook have not been moved since the opening and if there is no material between them, then they may “castle” in a well-known way, provided that, in the initial position, the King is not in check. However, castling is similar to the 2-square pawn move in that there is an en passant rule. That is, an opposing piece may “pretend” that the King moved onto each intermediate square in succession, and may capture the King accordingly.

**Remark 8.3.** Note that this presentation of the rules of chess differs from the classical one in terms of how it deals with check, checkmate, and stalemate. Namely, we allow the King to be captured and do not have checkmate as such. However, the difference is insubstantial in that there it is equivalent to the classical version of chess. That is, there is an obvious isomorphism between the strategies for the two games (assuming players are permitted to resign in both versions).

8.4. **Rules of Smooth Chess.** Fix a gamely subset $Q \subset [0, 1]$ of allowable quantities. From here forward, we refer to the classical game of chess as 1-chess.

**Notation 8.5.** Let $T = \{K, Q, R, B, N, P\}$, $C = \{w, b\}$, and $S = \{1, 2, 3, 4, 5, 6, 7, 8\}^2$, as above. A board position is a function $P : C \times T \times S \to Q$ to the set of allowable quantities, such that for all $s \in S$, one has

$$|P_s| = \sum_{c \in C, t \in T} P(c, t, s) \in Q.$$
This quantity is called the amount of material on square \( s \). The position function \( P \) has many transposes, which are useful for describing positions. For example, \( P_{(1,6)} : C \times T \to [0,1] \) is the function which describes what material is on square \((1,6)\), and \( P_{wQ} : S \to [0,1] \) is the function which expresses which square the white Queen is on.

A fractional piece is a sequence \( p = (P,c,t,s,x) \), where \( P \) is a board position, \((c,t,s)\) is an element of \( C \times T \times S \), and \( x \in Q \) is an allowable quantity such that \( 0 < x \leq P(c,t,s) \). The number \( x \) is called the value of \( p \).

We denote subfunctions of the position function in a manner that is similar to that for 1-chess. For example, we write \( P_{(1,6)} = .2wQ + .6bR \) if \( P(w,Q,(1,6)) = .2, P(b,R,(1,6)) = .6, \) and if for all other pieces \( p \in C \times T, P(p,(1,6)) = 0 \). In this case we say “in position \( P \), there is .2 white Queens and .6 black Rooks on square \((1,6).\)” Alternately, we write \( P_{wQ} = .2(1,6) + .5(2,2) \) if the white Queen is so distributed. In this case, .3 of the white Queen has been captured, since it is not on the board.

Play begins with the board set up as in 1-chess. Play alternates between players, with each player on his turn required to move a total of 1 unit of material (distributed in any way). The only exception is if the player cannot legally make a full move, in which case he must move the maximum amount possible.

The guiding principles of smooth chess are as follows. Each square “holds” one full unit of material. Fractional pieces may slide through squares, provided there is “enough room” to do so. A fractional piece may land on a square provided that doing so would not result in there being in excess of 1 unit of material of its own color on the square. In other words, he cannot overflow the square with his own material. Capturing occurs when a player moves to a square and in so doing overflows the square in such a way that, by removing his opponent’s material, the quantity of material on the square can be brought down to 1. In this case, the opposing player chooses the distribution of his material which is captured.

We give a little more detail below.

**Sliding pieces:**