THE BOX MONADS

DAVID I. SPIVAK

1. Notation

Let \{\ast\} denote the terminal category (which has one arrow). Let \textbf{Cat} denote the category of small categories (morphisms are functors). Given a category \(A \in \textbf{Cat}\), let \(\text{Ob}(A)\) denote the set of objects in \(A\).

Let \(F: A \to C\) and \(G: B \to C\) be functors. We denote by \((F \downarrow_C G)\) the category whose objects are pairs \((a, b, f)\) where \(a \in \text{Ob}(A), b \in \text{Ob}(b)\) and \(f: F(a) \to G(b)\) is a morphism in \(C\). These are pictured as \(Fa \downarrow \downarrow \downarrow Ga\).

The morphism sets are given by

\[
\text{Hom}_{(F \downarrow_C G)}((a, b, f), (a', b', f')) = \{(x, y) | x: a \to a', y: b \to b', Gy \circ f = f' \circ Fx\}.
\]

In pictures, a morphism \((a, b, f) \to (a', b', f')\) in \((F \downarrow_C G)\) is written

\[
\begin{array}{ccc}
Fa & \xrightarrow{Fx} & Fa' \\
\downarrow f & & \downarrow f' \\
Gb & \xrightarrow{Gy} & Gb'.
\end{array}
\]

Suppose now that \(C\) is a 2-category; we will use the morphism-terminology of \(C = \textbf{Cat}\) because that is the main example we ultimately want to consider. In other words, we will call morphisms in \(C\) “functors” and 2-arrows in \(C\) “natural transformations.” We still consider \(A\) and \(B\) as 1-categories.

Example 1.1. Let \(C = \textbf{Cat}\). Suppose that \(F: A \to \textbf{Cat}\) is a functor and \(B\) is a category, we sometimes write \((A \downarrow_{\textbf{Cat}} \{B\})\) to denote \((F \downarrow_{\textbf{Cat}} i_B)\), where \(i_B: \{\ast\} \to \textbf{Cat}\) is the category \(B\). In other words, \((A \downarrow_{\textbf{Cat}} \{B\})\) is the category of \(A\)-shaped diagrams in \(B\).

We can add a layer of complexity to \((F \downarrow_C G)\), by adding the possibility that the morphisms are not commutative diagrams in \(C\), but instead involve a natural transformation (i.e. a 2-arrow in \(C\)). To that end, we define a category \((F \Downarrow_C G)\) whose objects are the same as those in \((F \downarrow_C G)\), pictured in [1], but for which morphisms sets are given by

\[
\text{Hom}_{(F \Downarrow_C G)}((a, b, f), (a', b', f')) = \\
\{(x, y, \alpha) | x: a \to a', y: b \to b', \alpha: Gy \circ f \to f' \circ Fx\}.
\]

This project was supported in part by the Office of Naval Research.
In pictures, a morphism \((a, b, f) \rightarrow (a', b', f')\) in \((F \downarrow_C G)\) is written
\[
\begin{array}{ccc}
Fa & \xrightarrow{Fx} & Fa' \\
\downarrow f & \Rightarrow & \downarrow f' \\
Gb & \xrightarrow{Gy} & Gb'.
\end{array}
\]

We write \((F \uparrow_C G)\) to denote a similar category but with some reversal of 1-arrows. Its objects are the same as those in \((F \downarrow_C G)\) and \((F \downarrow_C G)\); they are sequences \((a, b, f)\), as seen in (1). The morphism sets are given by
\[
\text{Hom}_{(F \uparrow_C G)}((a, b, f), (a', b', f')) = \{(x, y, \alpha) | x: a' \rightarrow a, y: b' \rightarrow b, \alpha: f \circ Fx \rightarrow Gy \circ f'\}.
\]

In pictures, a morphism \((a, b, f) \rightarrow (a', b', f')\) in \((F \uparrow_C G)\) is written
\[
\begin{array}{ccc}
Fa & \xleftarrow{Fx} & Fa' \\
\downarrow f & \Rightarrow & \downarrow f' \\
Gb & \xleftarrow{Gy} & Gb'.
\end{array}
\]

For those who prefer, we can simply write \((F \uparrow_C G) = (G^\text{op} \downarrow_{C^\text{op}} F^\text{op})\), where \(F^\text{op}: A^\text{op} \rightarrow C^\text{op}\) and \(G^\text{op}: B^\text{op} \rightarrow C^\text{op}\) are the opposites of \(F\) and \(G\).

**Example 1.2.** Suppose that \(C = \text{Cat}\) and that \(B\) is a category, considered as a functor \(\{\ast\} \xrightarrow{B} C\). We may write \((F \downarrow \{B\})\) to denote \((F \downarrow i_B)\). Explicitly, the objects in \((F \downarrow \{B\})\) are functors \(F(a) \xrightarrow{\alpha} B\), where \(a \in A\), and the morphisms \((Fa \xrightarrow{\alpha} B) \longrightarrow (Fa' \xrightarrow{\alpha'} B)\) are pairs \((x, \alpha)\), where \(x: a \rightarrow a'\) and \(\alpha: f \rightarrow f' \circ Fx\), as seen in the natural transformation diagram
\[
\begin{array}{ccc}
Fa & \xrightarrow{Fx} & Fa' \\
\downarrow f & \Rightarrow & \downarrow f' \\
B & \xleftarrow{\alpha} & B.
\end{array}
\]

On the other hand, while the objects in \((F \uparrow \{B\})\) are the same as those in \((F \downarrow \{B\})\), a morphism \((Fa \xleftarrow{\alpha} B) \longrightarrow (Fa' \xrightarrow{\alpha'} B)\) has a reversal in the top 1-arrow. It is a pair \((x, \alpha)\) where \(x: a' \rightarrow a\) and \(\alpha: f \circ Fx \rightarrow f'\), as seen in the natural transformation diagram
\[
\begin{array}{ccc}
Fa & \xleftarrow{Fx} & Fa' \\
\downarrow f & \Rightarrow & \downarrow f' \\
B & \xleftarrow{\alpha} & B.
\end{array}
\]

2. **The Box monads**

In this section we will be working in the base category \(\text{Cat}\), so when we write \((X \downarrow Y)\) we mean \((X \downarrow \text{Cat}, Y)\). In particular, let \(\text{FC}\) denote the category of finite categories and consider the full inclusion \(\text{FC} \rightarrow \text{Cat}\). We will be considering, for
any category \( C \in \text{Cat} \), the categories \((\text{FC} \downarrow \{C\})\) and \((\text{FC} \uparrow \{C\})\). These are particular cases of the categories considered in Example 1.2.

**Theorem 2.1.** Let \( \text{FC} \) denote the category of finite categories. Consider the endofunctor \( d := (\text{FC} \downarrow \{-\}) : \text{Cat} \to \text{Cat} \). There are natural transformations \( \eta : \text{id}_{\text{Cat}} \to d \) and \( \mu : d^2 \to d \) that give \( d \) the structure of a monad on \( \text{Cat} \).

**Proof.** For \( C \in \text{Cat} \), define \( \eta_C : C \to dC \) to be the functor given on objects by \( c \mapsto \{\ast\} \) and \( \mu : d^2 \to d \) that give \( d \) the structure of a monad on \( \text{Cat} \).

Let \( X \) be a finite category and let \( \sigma : X \to (\text{FC} \downarrow \{C\}) \) be an object in \( d^2 C \). We need to establish some notation for \( \sigma \). On objects \( x \in X \), denote \( \sigma_x \) by \( \sigma_x : S_x \to C \), where \( S_x \) is a finite category. On morphisms \( f : x \to y \) in \( X \), denote \( \sigma(f) \) by

\[
\begin{array}{ccc}
S_x & \xrightarrow{\sigma_f} & S_y \\
\sigma_x & \searrow & \sigma_y \\
& C & \swarrow \\
& & \sigma_y(f)
\end{array}
\]

Define \( \mu(X, \sigma) \in \text{Cat} \) as the category with objects

\[
\text{Ob}(\mu(X, \sigma)) := \{ (x, s_x) | x \in X, s_x \in S_x \}
\]

and morphisms

\[
\text{Hom}_{\mu(X, \sigma)}((x, s_x), (y, s_y)) = \{ (f, f') : (x, s_x) \to (y, s_y) \in S_y \}.
\]

Note that since \( X \) and each \( S_x \) is finite, so is \( \mu(X, \sigma) \).

There is a canonical functor \( \mu(X, \sigma) \to C \) given on objects by \( (x, s_x) \mapsto \sigma_x(s_x) \) and given on morphisms by sending \( (f, f') : (x, s_x) \to (y, s_y) \) to the composite

\[
\sigma_x(s_x) \xrightarrow{\sigma_y f} \sigma_y \sigma_f(s_x) \xrightarrow{\sigma_y f'} \sigma_y(s_y).
\]

Thus \( \mu(X, \sigma) \) is indeed in \((\text{FC} \downarrow \{C\})\).
So far, we have only established $\mu$ on objects. On morphisms

$$\begin{array}{ccc}
X & \xrightarrow{m} & X' \\
\downarrow^{\sigma} & m^t \Downarrow & \downarrow^{\sigma'} \\
(\mathbf{FC} \downarrow \{C\}) & & \\
\end{array}$$

define $\mu(m, m^t): \mu(X, \sigma) \to \mu(X', \sigma')$ as the functor given as follows. Send the object $(x, s_x) \in \mu(X, \sigma)$ to $(m(x), m^t_x(s_x)) \in \mu(X', \sigma')$. Send $(f, f^t): (x, s_x) \to (y, s_y)$ to

$$(m(x), (m^t_y \circ f^t): \sigma'_{m,f}(m^t_x(s_x)) \to m^t_y(s_y))): \mu(X, \sigma) \to \mu(X', \sigma').$$

There is an obvious natural transformation

$$\begin{array}{ccc}
\mu(X, \sigma) & \xrightarrow{=} & \mu(X', \sigma') \\
\downarrow & & \downarrow \\
C & & C \\
\end{array}$$

because $m^t: \sigma' \circ m \to \sigma$ was assumed natural.

We draw the following diagram, in case it is useful to the reader:

$$\begin{array}{ccc}
S_x & \xrightarrow{\sigma_f} & S_y \\
\downarrow^{m^t_x} & m^t_y \Downarrow & \downarrow^{\sigma_y} \\
S'_m x & \xrightarrow{\sigma'_{m,x}} & S'_m y \\
\downarrow^{\sigma'_{m,f} \uparrow} & \downarrow^{\sigma'_{m,y}} & \downarrow^{\sigma'_{m,y}} \\
C & & C \\
\end{array}$$

We forgo showing that this really is a monad.

\[\square\]

**Corollary 2.2.** Let $\mathbf{FC}$ denote the category of finite categories. Consider the endofunctor $e := (\mathbf{FC} \uparrow \{-\}): \mathbf{Cat} \to \mathbf{Cat}$. There are natural transformations $\eta: \text{id}_\mathbf{Cat} \to e$ and $\mu: e^2 \to e$, which gives $e$ the structure of a monad on $\mathbf{Cat}$.

**Proof.** Recall the endofunctor $d: \mathbf{Cat} \to \mathbf{Cat}$ given by $d = (\mathbf{FC} \uparrow \{-\})$ as in Theorem 2.1. For any category $C$ there is an isomorphism (natural in $C$)

$$(\mathbf{FC} \downarrow \{C\}) \cong (\mathbf{FC} \uparrow \{C^{\text{op}}\})^{\text{op}}$$

given by sending an object $\sigma: S \to C$ to the object $\sigma^{\text{op}}: S^{\text{op}} \to C^{\text{op}}$ and by sending a morphism $(\sigma_f, \sigma_f^t): (\sigma_x: S_x \to C) \to (\sigma_y: S_y \to C)$, pictured as

$$\begin{array}{ccc}
S_x & \xrightarrow{\sigma_f} & S_y \\
\downarrow^{\sigma_x} & \sigma_x^t \Downarrow & \downarrow^{\sigma_y} \\
C & & C \\
\end{array}$$
to the morphism pictured as

\[
\begin{array}{ccc}
S^\text{op}_y & \xrightarrow{\sigma_f} & S^\text{op}_z \\
\downarrow \sigma^\text{op}_y & & \downarrow \sigma^\text{op}_z \\
C^\text{op} & \xleftarrow{\sigma^\text{op}_f} & S^\text{op}_y \\
\end{array}
\]

Now we can simply apply the maps \(\mu\) and \(\eta\) from Theorem 2.1. For \(\mu\) we have

\[
(\text{FC} \uparrow \{\text{FC} \uparrow \{C\}\}) \cong (\text{FC} \downarrow \{\text{FC} \downarrow \{C^\text{op}\}\})^{\text{op}}
\]

\[
\cong (\text{FC} \downarrow \{\text{FC} \downarrow \{C^\text{op}\}\})^{\text{op}}
\]

\[
\xrightarrow{\mu} (\text{FC} \downarrow \{C^\text{op}\})^{\text{op}} \cong (\text{FC} \uparrow \{C\})
\]

and it is natural in \(C\). Writing \(\text{op}: \text{Cat} \rightarrow \text{Cat}\) to denote the involution, we also have

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{(\text{FC}\uparrow(-))} & \text{Cat} \\
\downarrow \text{id}_{\text{Cat}} & & \downarrow \text{id}_{\text{Cat}} \\
\text{Cat} & \xrightarrow{(\text{FC}\downarrow(-))} & \text{Cat} \\
\end{array}
\]

so the natural transformation \(\eta: \text{id}_{\text{Cat}} \rightarrow (\text{FC} \downarrow \{-\})\) extends to a natural transformation \(\text{id}_{\text{Cat}} \rightarrow (\text{FC} \uparrow \{-\})\).

All the monad diagrams commute for \(e\) because they do for \(d\).

\[\square\]

In the next section we will define the “global sections functor”

\[\Gamma: (\text{FC} \downarrow \{\text{Cat}\}) \rightarrow \text{Cat},\]

which is related to the Grothendieck construction. For now, we just record a proposition as to the purpose of this material.

**Proposition 2.3.** The global sections functor \(\Gamma: (\text{FC} \downarrow \{\text{Cat}\}) \rightarrow \text{Cat}\) gives \(\text{Cat}\) the structure of a \(d\)-algebra.

**Proof.***

\[\square\]

### 3. Grothendieck Construction

Let \(f: S \rightarrow \text{Cat}\) be a functor. The Grothendieck construction for \(f\) is given by

\[\text{Gr}(f) := \{(s) \downarrow f\}\]

Explicitly, \(\text{Gr}(f)\) is the category with objects \((S, s)\) where \(S \in \text{Ob}(S)\) and \(s \in f(S)\). The morphism sets in \(\text{Gr}(f)\) are given by

\[\text{Hom}((S, s), (S', s')) = \{(y: S \rightarrow S', \alpha: f(y)(s) \rightarrow s')\}\]

There is a natural projection \(\pi_f: \text{Gr}(f) \rightarrow S\) given by \((S, s) \mapsto S\). It is a split fibration; that is, for any \(y: S \rightarrow S'\) and \(s \in S\), there is a canonical choice of map \((S, s) \rightarrow (S', f(y)(s))\) lying over \(y\).
Lemma 3.1. Let $f: S \to \textbf{Cat}$ be a diagram of categories. A functor $\sigma: C \to S$ induces a diagram

\[
\begin{array}{ccc}
Gr(f) & \to & Gr(f) \\
\downarrow^{\pi_{f\sigma}} & & \downarrow^{\pi_f} \\
C & \searrow & S,
\end{array}
\]

which is a pullback square.

Proof. *** \hfill \Box

Definition 3.2. Let $\pi: T \to S$ be a functor. A section of $\pi$ is a functor $g: S \to T$ such that $\pi g = \text{id}_S$. A morphism of sections of $\pi$ is a natural transformation over $S$.

Recall that a pullback square as in Lemma 3.1 defines a functor

$g \mapsto (\text{id}_C \times_{\text{id}_S} (g \circ \sigma))$

from the sections of $\pi_f: Gr(f) \to S$ to the sections of $\pi_{f\sigma}: Gr(f\sigma) \to C$.

Lemma 3.3. Suppose given two functors $f, g: S \to \textbf{Cat}$; let $\text{Nat}(f, g)$ denote the set of natural transformations from $f$ to $g$. There is a natural isomorphism

$\text{Hom}_S(Gr(f), Gr(g)) \cong \text{Nat}(f, g)$.

Proof. *** \hfill \Box

Definition 3.4. There is a functor $\Gamma: (\textbf{Cat} \uparrow \{\textbf{Cat}\}) \to \textbf{Cat}$ defined as follows. On objects, $\Gamma(f: C \to \textbf{Cat})$ is defined to be the category of sections of the functor $\pi: Gr(f) \to C$. To define $\Gamma$ on morphisms, suppose we are given a natural transformation diagram

\[
\begin{array}{ccc}
C & \xleftarrow{f} & D \\
\downarrow^{\alpha} & \searrow & \downarrow^{g} \\
\text{Cat} & \xleftarrow{\pi} & \text{Cat}
\end{array}
\]

By Lemma 3.1 there is a canonical functor $\Gamma(g) \to \Gamma(gx)$. By Lemma 3.3 there is a canonical functor $\Gamma(gx) \to \Gamma(f)$ induced by $\alpha$. The composition gives the desired functor.

Lemma 3.5. Suppose that $\sigma_i: C_i \to \textbf{Cat}$, for $i = 1, 2, 3$, are three functors and suppose that $f: C_2 \to C_1$ and $g: C_2 \to C_3$ are functors too. There is an induced functor $C_4 := C_1 \amalg_{C_2} C_3 \overset{\sigma_4}{\to} \textbf{Cat}$. Taking Grothendieck constructions yeilds the
All of the vertical squares are pullback squares by Lemma 3.1.

The natural map

\[ \Gamma(\sigma_4) \longrightarrow \Gamma(\sigma_1) \times_{\Gamma(\sigma_2)} \Gamma(\sigma_3) \]

is an isomorphism.