Arithmetic L-functions and their Sato-Tate distributions

Andrew V. Sutherland

Massachusetts Institute of Technology

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A simple thing we don't know

Let $X/\mathbb{Q}$ be a nice (smooth, projective, geometrically integral) curve of genus g. For each good prime p the trace of Frobenius

$$a_p := p + 1 - \#X(\mathbb{F}_p)$$

satisfies $|a_p| \le 2g\sqrt{p}$, by the Weil bounds, and $x_p := a_p/\sqrt{p} \in [-2g, 2g]$. In particular $g \ge |x_p|/2$ for all primes p.

[Katz12]: Is the lower bound on g ever sharp?

For $g = 1$ this follows from the Sato–Tate conjecture (now a theorem). The question remains open for all $g > 1$.

For $g = 2$ we know $|x_p| \ge 2/3$ for a positive density of p [Taylor18]. For $g > 2$ we know essentially nothing...

The L-function of a curve

Let $X/\mathbb{Q}$ be a nice curve of genus g. The L-function of X is given by

$$L(X,s) = L(\mathrm{Jac}(X),s) := \sum_{n\geq 1} a_n n^{-s} := \prod_p L_p(p^{-s})^{-1}.$$

For primes p of good reduction for X we have the zeta function

$$Z(X_p;s) := \exp\left(\sum_{r\geq 1} \#X(\mathbb{F}_{p^r})\frac{T^r}{r}\right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

and the L-polynomial $L_p \in \mathbb{Z}[T]$ in the numerator satisfies

$$L_p(T) = T^{2g}\chi_p(1/T) = 1 - a_pT + \cdots + p^gT^{2g},$$

where $\chi_p(T)$ is the charpoly of the Frobenius endomorphism of $\mathrm{Jac}(X_p)$.

The Selberg class with polynomial Euler factors

The Selberg class S^{poly} consists of Dirichlet series $L(s) = \sum_{n \geq 1} a_n n^{-s}$:

1. $L(s)$ has an analytic continuation that is holomorphic at $s \neq 1$;
2. For some $\gamma(s) = Q^s \prod_{i=1}^r \Gamma(\lambda_i s + \mu_i)$ and ε, the completed L-function $\Lambda(s) := \gamma(s) L(s)$ satisfies the functional equation

$$\Lambda(s) = \varepsilon \overline{\Lambda(1 - \bar{s})},$$

   where $Q > 0$, $\lambda_i > 0$, $\mathrm{Re}(\mu_i) \geq 0$, $|\varepsilon| = 1$. Define $\deg L := 2 \sum_i^r \lambda_i$.
3. $a_1 = 1$ and $a_n = O(n^{\epsilon})$ for all $\epsilon > 0$; the Ramanujan bound.
4. $L(s) = \prod_p L_p(p^{-s})^{-1}$ for some $L_p \in \mathbb{Z}[T]$ with $\deg L_p \leq \deg L$; in other words $L(s)$ has an Euler product.

The Dirichlet series $L_{\mathrm{an}}(s, X) := L(X, s + \frac{1}{2})$ satisfies (3) and (4), and conjecturally lies in S^{poly}; for $g = 1$ this is known via modularity.

Strong multiplicity one

Theorem (Kaczorowski-Perelli 2001)

If $A(s) = \sum_{n\geq 1} a_n n^{-s}$ *and* $B(s) = \sum_{n\geq 1} b_n n^{-s}$ *lie in* S^{poly} *and* $a_p = b_p$ *for all but finitely many primes* p*, then* $A(s) = B(s)$*.*

Corollary

If $L_{\mathrm{an}}(s, X)$ *lies in* S^{poly} *then it is determined by (any choice of) all but finitely many coefficients* a_p*.*

Henceforth we assume that $L_{\mathrm{an}}(s, X) \in S^{\mathrm{poly}}$.

Let $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^s\Gamma(s)$ and define $\Lambda(X, s) := \Gamma_{\mathbb{C}}(s)^g L(X, s)$. Then

$$\Lambda(X, s) = \varepsilon N^{1-s}\Lambda(X, 2 - s).$$

where the root number $\varepsilon = \pm 1$ and the analytic conductor $N \in \mathbb{Z}_{\geq 1}$ are determined by the a_p (one can take these as definitions).

Testing the functional equation

Let $G(x)$ be the inverse Mellin transform of $\Gamma_{\mathbb{C}}(s)^g = \int_0^\infty G(x)x^{s-1}dx$, and define

$$S(x) := \frac{1}{x}\sum a_n G(n/x),$$

so that $\Lambda(X,s) = \int_0^\infty S(x)x^{-s}dx$, and for all $x > 0$ we have

$$S(x) = \varepsilon S(N/x).$$

The function $G(x)$ decays rapidly, and for sufficiently large c_0 we have

$$S(x) \approx S_0(x) := \frac{1}{x}\sum_{n \le c_0 x} a_n G(n/x),$$

with an explicit bound on the error $|S(x) - S_0(x)|$.

Effective strong multiplicity one

Fix a finite set of small primes $\mathcal{S}$ (e.g. $\mathcal{S} = \{2\}$) and an integer M that we know is a multiple of the conductor N (e.g. $M = \Delta(X)$).

There is a finite set of possibilities for $\varepsilon = \pm 1$, $N|M$, and the Euler factors $L_p \in \mathbb{Z}[T]$ for $p \in \mathcal{S}$ (the coefficients of $L_p(T)$ are bounded).

Suppose we can compute a_n for $n \le c_1\sqrt{M}$ whenever $p \nmid n$ for $p \in \mathcal{S}$.

We now compute $\delta(x) := |S_0(x) - \varepsilon S_0(N/x)|$ with $x = c_1\sqrt{N})$ for every possible choice of ε, N, and $L_p(T)$ for $p \in \mathcal{S}$. If all but one choice makes $\delta(x)$ larger than our explicit error bound, we know the correct choice.

For a suitable choice of c_1 this is guaranteed to happen.[1] One can explicitly determine a set of $O(N^\epsilon)$ candidate values of c_1, one of which is guaranteed to work; in practice the first one usually works.

[1] Subject to our assumptions; if it does not happen then we have found an explicit counterexample to the Hasse-Weil conjecture.

Conductor bounds

The formula of Brumer and Kramer gives explicit bounds on the p-adic valuation of the algebraic conductor N of $\mathrm{Jac}(X)$:

$$v_p(N) \leq 2g + pd + (p-1)\lambda_p(d),$$

where $d = \lfloor \frac{2g}{p-1} \rfloor$ and $\lambda_p(d) = \sum i d_i p^i$, with $d = \sum d_i p^i$ with $0 \leq d_i < p$.

g	$p = 2$	$p = 3$	$p = 5$	$p = 7$	$p > 7$
1	8	5	2	2	2
2	20	10	9	4	4
3	28	21	11	13	6

For $g \leq 2$ these bounds are tight (see `www.lmfdb.org` for examples).

For hyperelliptic curves N divides $\Delta(X)$; for a suitable definition of $\Delta(X)$ one expects this to hold in general.

Arithmetic L-functions

A more precise description of the properties S^{poly} is intended to capture is given by the axioms for analytic L-functions; see [FPRS 2019].

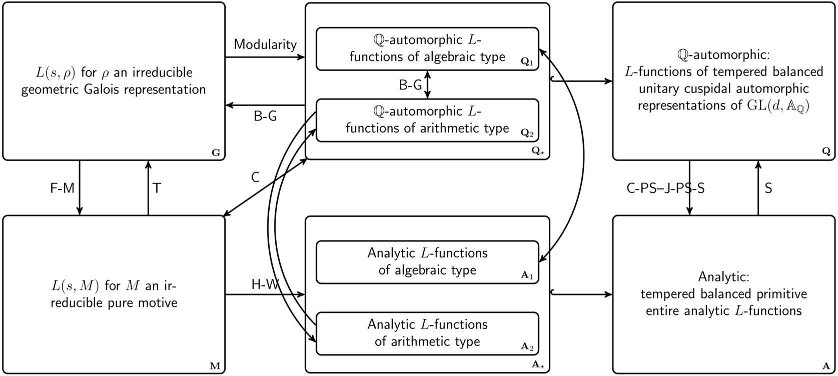
Among these one can distinguish those of arithmetic type.
These are analytic L-functions $L(s) = \sum a_n n^{-s}$ for which there exists $w_{ar} \in \mathbb{Z}$ and a number field K such that $a_n n^{w_{ar}/2} \in \mathcal{O}_K$ for all n.

The smallest F and w_{ar} are the field of coefficients and arithmetic weight of $L(s)$. For curves over number fields we always have $F = \mathbb{Q}$ (whether X is defined over $\mathbb{Q}$ or not), so $L(X)$ is a rational L-function, and the arithmetic weight $w_{ar} = 1$ agrees with the motivic weight.

More generally, one expects that the L-function of any pure motive of weight w should have $w_{ar} = w$, and moreover, that every arithmetic L-function should come from a motive.

Example: $L(s) = 1 + 16 \cdot 19^{-s} - 10 \cdot 25^{-s} + 16 \cdot 43^{-s} + 2 \cdot 49^{-s} - \cdots$

Conjectured relationships between sets of L-functions



F–M	Fontaine–Mazur	T	Taylor
B–G	Buzzard–Gee	C	Clozel
C–PS	Codgell–Piatetski-Shapiro	H–W	Hasse–Weil
J-PS-S	Jacquet–Piatetski-Shapiro–Shalika	S	Selberg

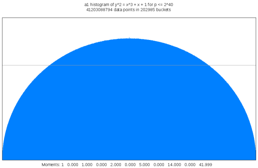
*Figure taken from page 21 of *Analytic L-functions: Definitions theorems and connections*, by D.W. Farmer, A. Pitale, N.C. Ryan, and R. Schmidt, arXiV:1711.10375.

Sato–tate distributions of rational L-functions

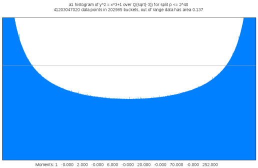
Given an arithmetic L-function $L(s)$ we can study the distribution of its (analytically normalized) coefficients, or equivalently, the distribution of its normalized Euler factors.

If we assume $L(s)$ is motivic (we do), we can associate a Sato-Tate group to $L(s)$; take the Sato-Tate group of a corresponding motive.

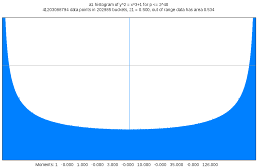
For rational L-functions of degree 2 and weight 1 there are three possible Sato-Tate distributions:



$\mathrm{SU}(2)$



$\mathrm{U}(1)$

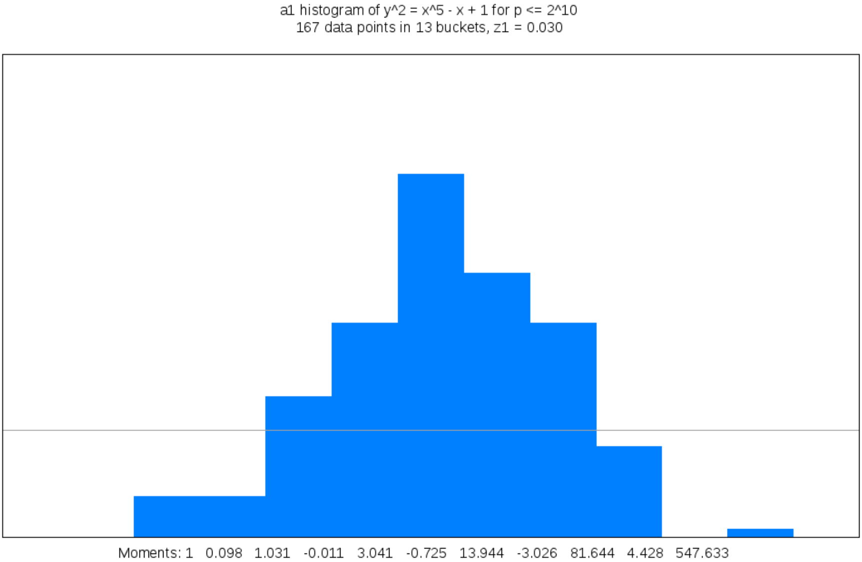


$N(\mathrm{U}(1))$

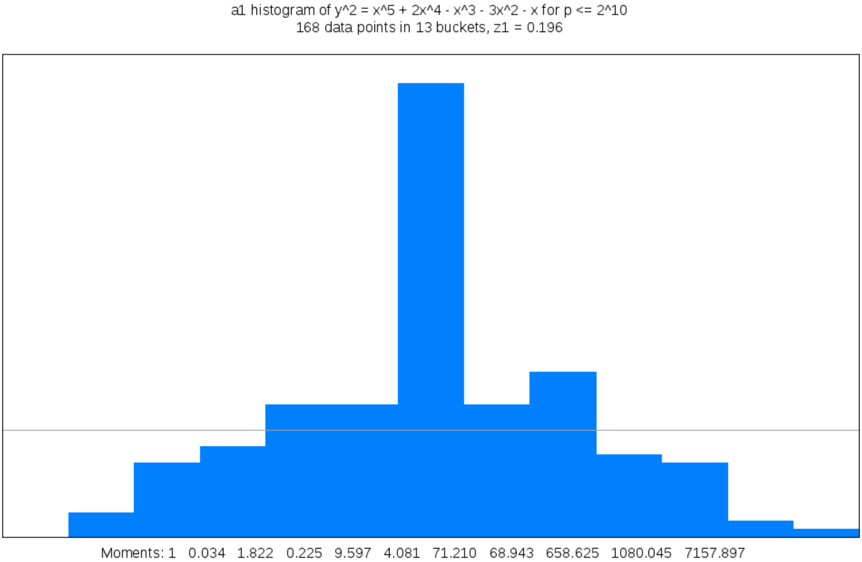
Some rational L-functions of weight w and degree d

w	d	L-function
0	1	$L(\chi,s)$ for a Dirichlet character with $\chi^2=1$, including $\zeta(s)$
	2	$L(f,s)$ for weight 1 CMFs with $\mathbb{Q}(f)=\mathbb{Q}$
	n	$\zeta_K(s)$ with $[K:\mathbb{Q}]=n$
		$L(\rho,s)$ for Artin representation with $\dim\rho=n$ and $\mathrm{tr}(\rho)$ rational
1	2	$L(f,s)$ for weight 2 CMFs with $\mathbb{Q}(f)=\mathbb{Q}$
		$L(E,s)$ for elliptic curves $E/\mathbb{Q}$
	4	$L(f,s)$ for parallel weight 2 HMFs with $\mathbb{Q}(f)=\mathbb{Q}$
		$L(E,s)$ for elliptic curves E/K with $[K:\mathbb{Q}]=2$
		$L(X,s)$ for genus 2 curves $X/\mathbb{Q}$
2	2	$L(f,s)$ for weight 3 CMFs with $\mathbb{Q}(f)=\mathbb{Q}$
	3	$L(\mathrm{Sym}^2(E),s)$ for elliptic curves $E/\mathbb{Q}$
		$L(H,s)$ for hypergeometric motives H with Hodge vector $[1,1,1]$
3	2	$L(f,s)$ for weight 4 CMFs with $\mathbb{Q}(f)=\mathbb{Q}$
	4	$L(\mathrm{Sym}^3(E),s)$ for elliptic curves $E/\mathbb{Q}$
		$L(H,s)$ for hypergeometric motives H with Hodge vector $[1,1,1,1]$

Sato-Tate group $G\subseteq O(d)$ if w is even, $G\subseteq \mathrm{USp}(d)$ if w is odd; $wd\equiv 0 \bmod 2$.



click histogram to animate (requires adobe reader)



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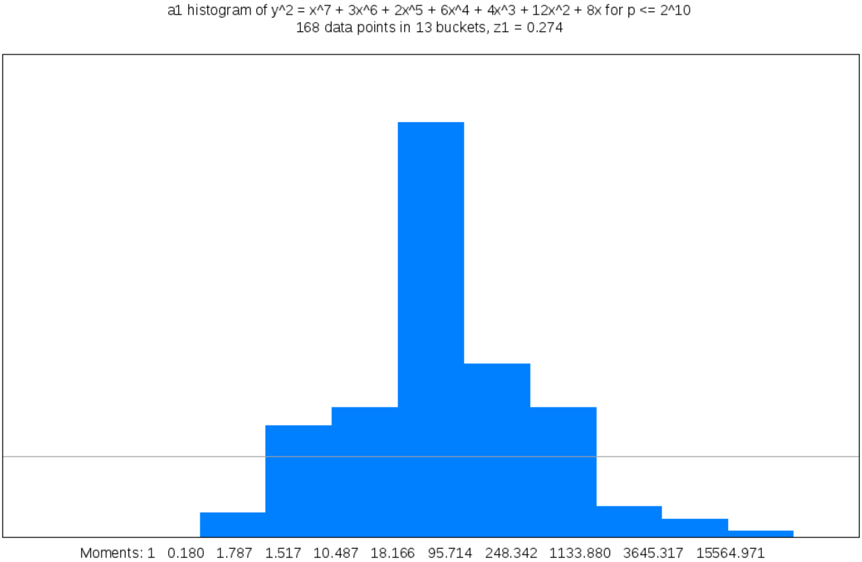
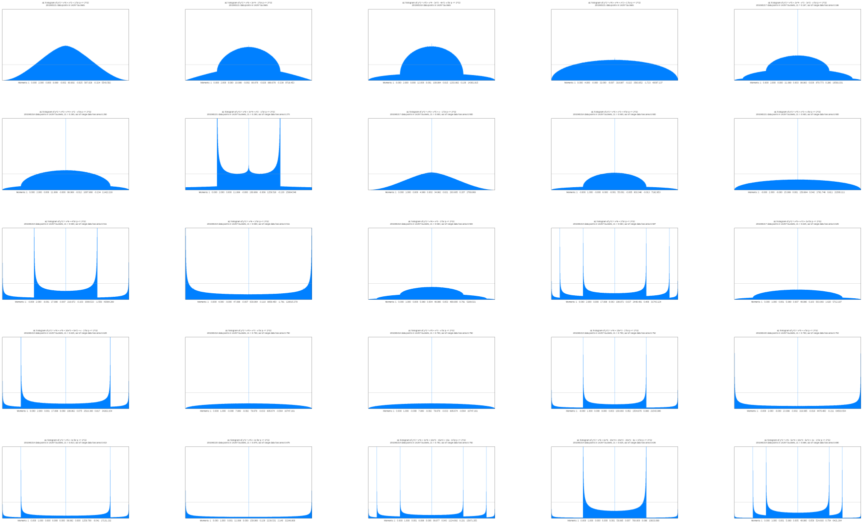
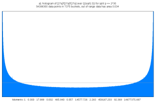


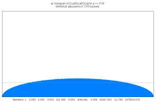
click histogram to animate (requires adobe reader)

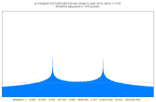
Exceptional distributions for abelian surfaces over $\mathbb{Q}$

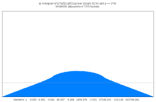


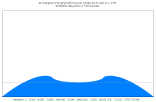
Connected Sato-Tate groups of abelian threefolds:

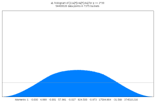

$\mathrm{U}(1)_3$

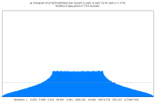

$\mathrm{SU}(2)_3$

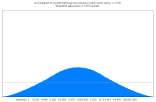

$\mathrm{U}(1) \times \mathrm{U}(1)_2$

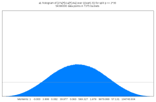

$\mathrm{U}(1) \times \mathrm{SU}(2)_2$

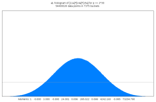

$\mathrm{SU}(2) \times \mathrm{U}(1)_2$

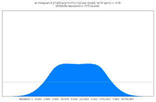

$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$

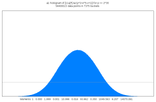

$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$

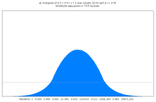

$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$

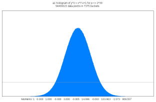

$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(1)$


$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$


$\mathrm{U}(1) \times \mathrm{USp}(4)$


$\mathrm{SU}(2) \times \mathrm{USp}(4)$


$\mathrm{U}(3)$


$\mathrm{USp}(6)$

Algorithms to compute L-functions

Given $X/\mathbb{Q}$ of genus g, we want to compute $L_p(T)$ for all good $p \le B$.

	complexity per prime (ignoring factors of $O(\log\log p)$)		
algorithm	$g=1$	$g=2$	$g=3$
point enumeration	$p\log p$	$p^2\log p$	$p^3(\log p)^2$
group computation	$p^{1/4}\log p$	$p^{3/4}\log p$	$p(\log p)^2$
p-adic cohomology	$p^{1/2}(\log p)^2$	$p^{1/2}(\log p)^2$	$p^{1/2}(\log p)^2$
CRT (Schoof-Pila)	$(\log p)^5$	$(\log p)^8$	$(\log p)^{14*}$
average poly-time	$(\log p)^4$	$(\log p)^4$	$(\log p)^4$

For $L(X,s) = \sum a_n n^{-s}$, we only need a_{p^2} for $p^2 \le B$, and a_{p^3} for $p^3 \le B$. We can compute all of these in $O(B)$ time using any $O(p)$ method.

Bottom line: It all comes down to computing a_p's.

*For hyperelliptic curves [Abelard18].

Arithmetic schemes

Let X be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, an arithmetic scheme.
The Hasse–Weil zeta function (or arithmetic zeta function) of X is

$$\zeta_X(s) := \prod_{x\in X}(1-N(x)^{-s})^{-1} = \prod \zeta_{X_p}(s) = \prod Z_{X_p}(p^{-s}),$$

where the product is over closed points x, the norm $N(x) := \#\kappa(x)$ is the cardinality of the residue field $\kappa(x)$, and $X_p := X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ is the reduction of X modulo p. The local zeta function $Z_{X_p}(T)$ is

$$Z_{X_p}(T) := \exp\left(\sum_{r\geq 1}^{\infty} \#X_p(\mathbb{F}_{p^r})\frac{T^r}{r}\right) \in 1 + T\mathbb{Z}[[T]],$$

which is known to lie in $\mathbb{Q}(T)$ (by work of Dwork and Grothendieck).

For $X_p(\mathbb{F}_{p^r}) := \operatorname{Hom}_{\mathbb{F}_p}(\operatorname{Spec}(\mathbb{F}_{p^r}), X)$ we then have

$$\#X_p(\mathbb{F}_{p^r}) = \sum_{e|r} e\#\{x \in X : \kappa(x) \simeq \mathbb{F}_{p^e}\}.$$

Arithmetic zeta functions and L-functions

Let $X/\mathbb{Q}$ be a nice curve with integral model $\mathcal{X}$, which we can view as an arithmetic scheme. What is the relationship between $L_X(s)$ and $\zeta_{\mathcal{X}}(s)$?

We have $Z_{X_p}(T) = Z_{\mathcal{X}_p}(T)$ at all good primes p of $\mathcal{X}$, in which case the L-polynomials $L_{X_p}(T)$ and $L_{\mathcal{X}_p}(T)$ in their numerators will agree.

From our multiplicity one perspective, this is all we need; the local zeta functions $Z_{\mathcal{X}_p}(T)$ at good primes determine $L_X(s)$ (for any choice of $\mathcal{X}$).

In general L-polynomials $L_{X_p}(T)$ in $L_X(s) = \prod_p L_{X_p}(p^{-s})$ may differ from the numerator of the local zeta functions $Z_{\mathcal{X}_p}(T)$ at bad primes.

For example, if X is 49a1 and $\mathcal{X}$ is the arithmetic scheme given by its minimal Weierstrass equation $y^2z + xyz = x^3 - x^2z - 2xz^2 - z^3$, then

$$L_{\mathcal{X}_7}(T) = -7T^2 + 1 \neq 1 = L_{X_7}(T).$$

On the other hand, when X is 11a1 we actually have $L_X(s) = \zeta_{\mathcal{X}}(s)$.

Harvey's results for arithmetic schemes

Theorem (Harvey 2015)

Let X be an arithmetic scheme.

1. *There is a deterministic algorithm that, given a prime p, outputs $Z_{X_p}(T)$ in $p(\log p)^{1+o(1)}$ time using $O(\log p)$ space.*
2. *There is a deterministic algorithm that, given a prime p, outputs $Z_{X_p}(T)$ in $\sqrt{p}\,(\log p)^{2+o(1)}$ time using $O(\sqrt{p}\log p)$ space.*
3. *There is a deterministic algorithm that, given an integer N, outputs $Z_{X_p}(T)$ for $p \le N$ in $N(\log N)^{3+o(1)}$ time using $O(N\log^2 N)$ space.*

In these complexity bounds, X is fixed, only p or N are part of the input (the arithmetic scheme X is effectively "hardwired" into the algorithm).

If one constrains X and fixes its representation, the dependence on X can be made explicit; for plane curves one obtains $g^{14}N(\log N)^{3+o(1)}$.

These are not just existence statements; Harvey gives explicit algorithms.

Practical average polynomial-time algorithms

To date all practical implementations compute $L_p(T) \bmod p$ by computing Hasse–Witt (Cartier–Manin) matrices $A_p \in \mathbb{F}_p^{g\times g}$ for $p \le B$.

We have $a_p \equiv \operatorname{tr}(A_p) \bmod p$, which determines $a_p \in \mathbb{Z}$ for $p > 16g^2$. (for $p \le 16g^2$ one can simply count point naïvely).

Fast implementations are currently available in the following cases:

- Hyperelliptic curves over $\mathbb{Q}$ [HS14, HS16].
- Geometrically hyperelliptic genus 3 curves over $\mathbb{Q}$ [HMS16].
- Smooth plane quartics over $\mathbb{Q}$ [CHS20].
- Superelliptic curves over $\mathbb{Q}$ [S20].

A toy implementation of Harvey's algorithm for smooth plane curves of arbitrary genus is available, but much still remains to be done...

Average polynomial-time in genus 1

Let $X: y^2 = f(x)$ with $\deg f = 3, 4$ and $f(0) \neq 0$, and let f_k^n be the coefficient of x^k in f^n. Then $a_p \equiv f_{p-1}^{(p-1)/2} \bmod p$ for all good p.

The relations $f^{n+1} = f \cdot f^n$ and $(f^{n+1})' = (n+1)f' \cdot f^n$ yield the identity

$$k f_0 f_k^n = \sum_{1 \leq i \leq d} (i(n+1) - k) f_i f_{k-i}^n,$$

for all $k, n \geq 0$. Suppose for simplicity $\deg f = 3$, and define

$$v_k^n := [f_{k-2}^n, f_{k-1}^n, f_k^n], \qquad M_k^n := \begin{bmatrix} 0 & 0 & (3n+3-k)f_3 \\ kf_0 & 0 & (2n+2-k)f_2 \\ 0 & kf_0 & (n+1-k)f_1 \end{bmatrix},$$

so that we have the recurrence $v_k^n = \frac{1}{kf_0} v_{k-1}^n M_k^n$.

Average polynomial-time in genus 1

We then have
$$v_k^n = \frac{1}{(f_0)^k k!} v_0^n M_1^n \cdots M_k^n.$$
We want to compute $a_p \equiv f_{2n}^n \bmod p$ with $n := (p-1)/2$.
This is just the last entry of the vector v_{2n}^n reduced modulo $p = 2n+1$.

Observe that $2(n+1) \equiv 1 \bmod p$, so $2M_k^n \equiv M_k \bmod p$, where
$$M_k := \begin{bmatrix} 0 & 0 & (3-2k)f_3 \\ kf_0 & 0 & (2-2k)f_2 \\ 0 & kf_0 & (1-2k)f_1 \end{bmatrix}$$
is an integer matrix whose entries do not depend on $p = 2n+1$, and
$$v_{2n}^n \equiv -\left(\frac{f_0}{p}\right) V_0 M_1 \cdots M_{p-1} \bmod p \qquad \text{(where } V_0 = [0,0,1]\text{)}.$$

Accumulating remainder tree

Given matrices $M_0, \ldots, M_{n-1}$ and moduli $m_1, \ldots, m_n$, to compute

$$\begin{aligned} M_0 \bmod m_1 \\ M_0 M_1 \bmod m_2 \\ M_0 M_1 M_2 \bmod m_3 \\ M_0 M_1 M_2 M_3 \bmod m_4 \\ \cdots \\ M_0 M_1 \cdots M_{n-2} M_{n-1} \bmod m_n \end{aligned}$$

multiply adjacent pairs and recursively compute

$$\begin{aligned} (M_0 M_1) \bmod m_2 m_3 \\ (M_0 M_1)(M_2 M_3) \bmod m_4 m_5 \\ \cdots \\ (M_0 M_1) \cdots (M_{n-2} M_{n-1}) \bmod m_n \end{aligned}$$

and adjust the results as required (for better results, use a forest).

Complexity analysis

Assume $\log|f_i| = O(\log B)$. The recursion has depth $O(\log B)$ and in each recursive step we multiply and reduce 3×3 matrices with integer entries whose total bitsize is $O(B \log B)$.

We can do all the multiplications/reductions at any given level of the recursion in time $O(\mathsf{M}(B \log B)) = B(\log B)^{2+o(1)}$.

Total complexity is $B(\log B)^{3+o(1)}$, or $(\log p)^{4+o(1)}$ per prime $p \leq B$.

For a single prime p we can give an $O(p^{1/2}(\log p)^{1+o(1)})$ algorithm using the same matrices.

This is a silly way to compute a_p in genus 1, but it is in practice the fastest than method known for $g > 2$ and $p \leq B$ (for any reasonable value of B).

Open problem: Given a polynomial-time algorithm that takes as input a defining equation for a nice curve $X/\mathbb{F}_p$ and outputs $\#X(\mathbb{F}_p)$.

Efficiently handling a single prime

Simply computing $V_0 M_1 \cdots M_{p-1}$ modulo p is surprisingly quick (faster than semi-naïve point-counting); it takes $p(\log p)^{1+o(1)}$ time.

But we can do better.

Viewing $M_k \bmod p$ as $M \in \mathbb{F}_p[k]^{3\times 3}$, we compute

$$A(k) := M(k)M(k+1)\cdots M(k+r-1) \in \mathbb{F}_p[k]^{3\times 3}$$

with $r \approx \sqrt{p}$ and then instantiate $A(k)$ at roughly r points to get

$$M_1 M_2 \cdots M_{p-1} \equiv_p A(1)A(r+1)A(2r+1)\cdots A(p-r).$$

Using standard product tree and multipoint evaluation techniques this takes $O(\mathsf{M}(p^{1/2})\log p) = p^{1/2}(\log p)^{2+o(1)}$ time.

Bostan-Gaudry-Schost: $p^{1/2}(\log p)^{1+o(1)}$ time [BGS07].

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