The Sato-Tate conjecture for abelian varieties

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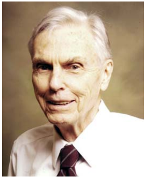
Mikio Sato



John Tate

Joint work with F. Fité, K.S. Kedlaya, and V. Rotger, and also with D. Harvey.

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, which we can write in the form

$$y^2 = x^3 + ax + b.$$

Let p be a prime of good reduction for E.
The number of $\mathbb{F}_p$-points on the reduction of E modulo p is

$$\#\overline{E}(\mathbb{F}_p) = p + 1 - t_p.$$

The trace of Frobenius t_p is an integer in the interval $[-2\sqrt{p}, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p = -t_p/\sqrt{p} \in [-2, 2]$, as p varies over primes of good reduction.

Example: $y^2 = x^3 + x + 1$

p	t_p	x_p	p	t_p	x_p	p	t_p	x_p
3	0	**0.000000**	71	13	**−1.542816**	157	−13	**1.037513**
5	−3	**1.341641**	73	2	**−0.234082**	163	−25	**1.958151**
7	3	**−1.133893**	79	−6	**0.675053**	167	24	**−1.857176**
11	−2	**0.603023**	83	−6	**0.658586**	173	2	**−0.152057**
13	−4	**1.109400**	89	−10	**1.059998**	179	0	**0.000000**
17	0	**0.000000**	97	1	**−0.101535**	181	−8	**0.594635**
19	−1	**0.229416**	101	−3	**0.298511**	191	−25	**1.808937**
23	−4	**0.834058**	103	17	**−1.675060**	193	−7	**0.503871**
29	−6	**1.114172**	107	3	**−0.290021**	197	−24	**1.709929**
37	−10	**1.643990**	109	−13	**1.245174**	199	−18	**1.275986**
41	7	**−1.093216**	113	−11	**1.034793**	211	−11	**0.757271**
43	10	**−1.524986**	127	2	**−0.177471**	223	−20	**1.339299**
47	−12	**1.750380**	131	4	**−0.349482**	227	0	**0.000000**
53	−4	**0.549442**	137	12	**−1.025229**	229	−2	**0.132164**
59	−3	**0.390567**	139	14	**−1.187465**	233	−3	**0.196537**
61	12	**−1.536443**	149	14	**−1.146925**	239	−22	**1.423062**
67	12	**−1.466033**	151	−2	**0.162758**	241	22	**−1.417145**

`http://math.mit.edu/~drew`

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (This is also known for E/k, where k is a totally real number field).

[Taylor et al.]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[classical]

Sato-Tate groups in dimension 1

The *Sato-Tate group* of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ derived from the ℓ-adic Galois representation attached to E.

The refined Sato-Tate conjecture implies that the normalized trace distribution of E converges to the trace distribution of G under the Haar measure (the unique translation-invariant measure).

G	G/G^0	Example curve	k	$\mathrm{E}[a_1^0], \mathrm{E}[a_1^2], \mathrm{E}[a_1^4] \ldots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$

In dimension 1 there are three possible Sato-Tate groups, two of which arise for elliptic curves defined over $\mathbb{Q}$.

Zeta functions and L-polynomials

For a smooth projective curve $C/\mathbb{Q}$ of genus g and a prime p define

$$Z(\overline{C}/\mathbb{F}_p;T) := \exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right),$$

where $N_k = \#\overline{C}(\mathbb{F}_{p^k})$. This is a rational function of the form

$$Z(\overline{C}/\mathbb{F}_p;T) = \frac{L_p(T)}{(1-T)(1-pT)},$$

where $L_p(T)$ is an integer polynomial of degree $2g$.

For $g=1$ we have $L_p(t) = pT^2 + c_1T + 1$, and for $g=2$,

$$L_p(T) = p^2T^4 + c_1pT^3 + c_2T^2 + c_1T + 1.$$

Normalized L-polynomials

The normalized polynomial

$$\bar{L}_p(T) := L_p(T/\sqrt{p}) = \sum_{i=0}^{2g} a_i T^i \in \mathbb{R}[T]$$

is monic, symmetric ($a_i = a_{2g-i}$), and unitary (roots on the unit circle). The coefficients a_i necessarily satisfy $|a_i| \le \binom{2g}{i}$.

We now consider the limiting distribution of $a_1, a_2, \ldots, a_g$ over all primes $p \le N$ of good reduction, as $N \to \infty$.

In this talk we will focus primarily on genus $g = 2$.

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L-polynomials of Abelian varieties

Let A be an abelian variety of dimension $g \geq 1$ over a number field k.

Let $\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ be the Galois representation arising from the action of G_k on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n].$$

For each prime $\mathfrak{p}$ of good reduction for A, let $q = \|\mathfrak{p}\|$ and define

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T),$$
$$\bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{q}) = \sum a_i T^i.$$

In the case that A is the Jacobian of a genus g curve C, this agrees with our earlier definition in terms of the zeta function of C.

The Sato-Tate problem for an abelian variety

For each prime $\mathfrak{p}$ of k where A has good reduction, the polynomial $\bar{L}_{\mathfrak{p}} \in \mathbb{R}[T]$ is monic, symmetric, unitary, and of degree $2g$.

Every such polynomial arises as the characteristic polynomial of a conjugacy class in the unitary symplectic group $\mathrm{USp}(2g)$.

Each probability measure on $\mathrm{USp}(2g)$ determines a distribution of conjugacy classes (hence a distribution of characteristic polynomials).

The *Sato-Tate problem*, in its simplest form, is to find a measure for which these classes are equidistributed. Conjecturally, such a measure arises as the Haar measure of a compact subgroup of $\mathrm{USp}(2g)$.

The Sato-Tate group ST_A

Let $\rho_\ell\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ be the ℓ-adic Galois representation arising from the action of G_k on $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$.

Let G_k^1 be the kernel of the cyclotomic character $\chi_\ell\colon G_k \to \mathbb{Q}_\ell^\times$.
Let $G_\ell^{1,\mathrm{Zar}}$ be the Zariski closure of $\rho_\ell(G_k^1)$ in $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$.
Choose an embedding $\iota\colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and let $G^1 = G_\ell^{1,\mathrm{Zar}} \otimes_\iota \mathbb{C}$.

Definition [Serre]

$\mathrm{ST}_A \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G^1 \subseteq \mathrm{Sp}_{2g}(\mathbb{C})$.
For each prime $\mathfrak{p}$ of good reduction for A, let $s(\mathfrak{p})$ denote the conjugacy class of $\|\mathfrak{p}\|^{-1/2}\rho_\ell(\mathrm{Frob}_\mathfrak{p}) \in G^1$ in ST_A.

Conjecturally, ST_A does not depend on ℓ; this is known for $g \leq 3$.
In any case, the characteristic polynomial of $s(\mathfrak{p})$ is always $\bar{L}_\mathfrak{p}(T)$.

Equidistribution

Let μ_{ST_A} denote the image of the Haar measure on $\mathrm{Conj}(\mathrm{ST}_A)$ (which does not depend on the choice of ℓ or the embedding ι).

Conjecture [Refined Sato-Tate]

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to μ_{ST_A}.

In particular, the distribution of $\bar{L}_{\mathfrak{p}}(T)$ matches the distribution of characteristic polynomials of random matrices in ST_A.

We can test this numerically by comparing statistics of the coefficients $a_1, \ldots, a_g$ of $\bar{L}_{\mathfrak{p}}(T)$ over $\|\mathfrak{p}\| \leq N$ to the predictions given by μ_{ST_A}.

The Sato-Tate axioms (weight 1)

A subgroup G of $\mathrm{USp}(2g)$ satisfies the *Sato-Tate axioms* if:

1. G is closed.
2. (Hodge circles) There is a subgroup H that is the image of a homomorphism $\theta\colon \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u and u^{-1} with multiplicity g, and H can be chosen so that its conjugates generate a dense subset of G^0.
3. (Rationality) For each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

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3. (Rationality) For each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$.

For any fixed g, the set of subgroups of $\mathrm{USp}(2g)$ that satisfy the Sato-Tate axioms is **finite** up to conjugacy.

Theorem

For $g \leq 3$, the group ST_A satisfies the Sato-Tate axioms.

Conjecturally, this holds for all g.

Sato-Tate groups in genus 2

Theorem 1 [FKRS 2012]

Up to conjugacy, 55 subgroups of $\mathrm{USp}(4)$ satisfy the Sato-Tate axioms:

$$
\begin{aligned}
\mathrm{U}(1):&\quad C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O,\\
&\quad J(C_1), J(C_2), J(C_3), J(C_4), J(C_6),\\
&\quad J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
&\quad C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2):&\quad E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
\mathrm{U}(1)\times\mathrm{U}(1):&\quad F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}\\
\mathrm{U}(1)\times\mathrm{SU}(2):&\quad \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
\mathrm{SU}(2)\times\mathrm{SU}(2):&\quad \mathrm{SU}(2)\times\mathrm{SU}(2),\ N(\mathrm{SU}(2)\times\mathrm{SU}(2))\\
\mathrm{USp}(4):&\quad \mathrm{USp}(4)
\end{aligned}
$$

Sato-Tate groups in dimension 2

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& J(D_2), J(D_3), J(D_4), J(D_6), J(T), J(O),\\
& C_{2,1}, C_{4,1}, C_{6,1}, D_{2,1}, D_{3,2}, D_{4,1}, D_{4,2}, D_{6,1}, D_{6,2}, O_1\\
\mathrm{SU}(2):\quad & E_1, E_2, E_3, E_4, E_6, J(E_1), J(E_2), J(E_3), J(E_4), J(E_6)\\
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\mathrm{U}(1)\times\mathrm{SU}(2):\quad & \mathrm{U}(1)\times\mathrm{SU}(2),\ N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
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\mathrm{USp}(4):\quad & \mathrm{USp}(4)
\end{aligned}$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

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\mathrm{U}(1)\times\mathrm{U}(1):&\quad F, F_a, F_c, F_{a,b}, F_{ab}, F_{ac}, F_{ab,c}, F_{a,b,c}\\
\mathrm{U}(1)\times\mathrm{SU}(2):&\quad \mathrm{U}(1)\times\mathrm{SU}(2), N(\mathrm{U}(1)\times\mathrm{SU}(2))\\
\mathrm{SU}(2)\times\mathrm{SU}(2):&\quad \mathrm{SU}(2)\times\mathrm{SU}(2), N(\mathrm{SU}(2)\times\mathrm{SU}(2))\\
\mathrm{USp}(4):&\quad \mathrm{USp}(4)
\end{aligned}
$$

Of these, exactly 52 arise as ST_A for an abelian surface A (34 over $\mathbb{Q}$).

Note that our theorem says nothing about equidistribution, which is currently known only in special cases [FS 2012, Johansson 2013].

Sato-Tate groups in dimension 2 with $G^0 = \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
1	1	C_1	C_1	0	0, 0, 0, 0, 0	8, 96, 1280, 17920	4, 18, 88, 454
1	2	C_2	C_2	1	0, 0, 0, 0, 0	4, 48, 640, 8960	2, 10, 44, 230
1	3	C_3	C_3	0	0, 0, 0, 0, 0	4, 36, 440, 6020	2, 8, 34, 164
1	4	C_4	C_4	1	0, 0, 0, 0, 0	4, 36, 400, 5040	2, 8, 32, 150
1	6	C_6	C_6	1	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
1	4	D_2	D_2	3	0, 0, 0, 0, 0	2, 24, 320, 4480	1, 6, 22, 118
1	6	D_3	D_3	3	0, 0, 0, 0, 0	2, 18, 220, 3010	1, 5, 17, 85
1	8	D_4	D_4	5	0, 0, 0, 0, 0	2, 18, 200, 2520	1, 5, 16, 78
1	12	D_6	D_6	7	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
1	2	$J(C_1)$	C_2	1	1, 0, 0, 0, 0	4, 48, 640, 8960	1, 11, 40, 235
1	4	$J(C_2)$	D_2	3	1, 0, 0, 0, 1	2, 24, 320, 4480	1, 7, 22, 123
1	6	$J(C_3)$	C_6	3	1, 0, 0, 2, 0	2, 18, 220, 3010	1, 5, 16, 85
1	8	$J(C_4)$	$\mathrm{C}_4 \times \mathrm{C}_2$	5	1, 0, 2, 0, 1	2, 18, 200, 2520	1, 5, 16, 79
1	12	$J(C_6)$	$\mathrm{C}_6 \times \mathrm{C}_2$	7	1, 2, 0, 2, 1	2, 18, 200, 2450	1, 5, 16, 77
1	8	$J(D_2)$	$\mathrm{D}_2 \times \mathrm{C}_2$	7	1, 0, 0, 0, 3	1, 12, 160, 2240	1, 5, 13, 67
1	12	$J(D_3)$	D_6	9	1, 0, 0, 2, 3	1, 9, 110, 1505	1, 4, 10, 48
1	16	$J(D_4)$	$\mathrm{D}_4 \times \mathrm{C}_2$	13	1, 0, 2, 0, 5	1, 9, 100, 1260	1, 4, 10, 45
1	24	$J(D_6)$	$\mathrm{D}_6 \times \mathrm{C}_2$	19	1, 2, 0, 2, 7	1, 9, 100, 1225	1, 4, 10, 44
1	2	$C_{2,1}$	C_2	1	0, 0, 0, 0, 1	4, 48, 640, 8960	3, 11, 48, 235
1	4	$C_{4,1}$	C_4	3	0, 0, 2, 0, 0	2, 24, 320, 4480	1, 5, 22, 115
1	6	$C_{6,1}$	C_6	3	0, 2, 0, 0, 1	2, 18, 220, 3010	1, 5, 18, 85
1	4	$D_{2,1}$	D_2	3	0, 0, 0, 0, 2	2, 24, 320, 4480	2, 7, 26, 123
1	8	$D_{4,1}$	D_4	7	0, 0, 2, 0, 2	1, 12, 160, 2240	1, 4, 13, 63
1	12	$D_{6,1}$	D_6	9	0, 2, 0, 0, 4	1, 9, 110, 1505	1, 4, 11, 48
1	6	$D_{3,2}$	D_3	3	0, 0, 0, 0, 3	2, 18, 220, 3010	2, 6, 21, 90
1	8	$D_{4,2}$	D_4	5	0, 0, 0, 0, 4	2, 18, 200, 2520	2, 6, 20, 83
1	12	$D_{6,2}$	D_6	7	0, 0, 0, 0, 6	2, 18, 200, 2450	2, 6, 20, 82
1	12	T	A_4	3	0, 0, 0, 0, 0	2, 12, 120, 1540	1, 4, 12, 52
1	24	O	S_4	9	0, 0, 0, 0, 0	2, 12, 100, 1050	1, 4, 11, 45
1	24	O_1	S_4	15	0, 0, 6, 0, 6	1, 6, 60, 770	1, 3, 8, 30
1	24	$J(T)$	$\mathrm{A}_4 \times \mathrm{C}_2$	15	1, 0, 0, 8, 3	1, 6, 60, 770	1, 3, 7, 29
1	48	$J(O)$	$\mathrm{S}_4 \times \mathrm{C}_2$	33	1, 0, 6, 8, 9	1, 6, 50, 525	1, 3, 7, 26

Sato-Tate groups in dimension 2 with $G^0 \neq \mathrm{U}(1)$.

d	c	G	G/G^0	z_1	z_2	$M[a_1^2]$	$M[a_2]$
3	1	E_1	C_1	0	0, 0, 0, 0, 0	4, 32, 320, 3584	3, 10, 37, 150
3	2	E_2	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	1, 6, 17, 78
3	3	E_3	C_3	0	0, 0, 0, 0, 0	2, 12, 110, 1204	1, 4, 13, 52
3	4	E_4	C_4	1	0, 0, 0, 0, 0	2, 12, 100, 1008	1, 4, 11, 46
3	6	E_6	C_6	1	0, 0, 0, 0, 0	2, 12, 100, 980	1, 4, 11, 44
3	2	$J(E_1)$	C_2	1	0, 0, 0, 0, 0	2, 16, 160, 1792	2, 6, 20, 78
3	4	$J(E_2)$	D_2	3	0, 0, 0, 0, 0	1, 8, 80, 896	1, 4, 10, 42
3	6	$J(E_3)$	D_3	3	0, 0, 0, 0, 0	1, 6, 55, 602	1, 3, 8, 29
3	8	$J(E_4)$	D_4	5	0, 0, 0, 0, 0	1, 6, 50, 504	1, 3, 7, 26
3	12	$J(E_6)$	D_6	7	0, 0, 0, 0, 0	1, 6, 50, 490	1, 3, 7, 25
2	1	F	C_1	0	0, 0, 0, 0, 0	4, 36, 400, 4900	2, 8, 32, 148
2	2	F_a	C_2	0	0, 0, 0, 0, 1	3, 21, 210, 2485	2, 6, 20, 82
2	2	F_c	C_2	1	0, 0, 0, 0, 0	2, 18, 200, 2450	1, 5, 16, 77
2	2	F_{ab}	C_2	1	0, 0, 0, 0, 1	2, 18, 200, 2450	2, 6, 20, 82
2	4	F_{ac}	C_4	3	0, 0, 2, 0, 1	1, 9, 100, 1225	1, 3, 10, 41
2	4	$F_{a,b}$	D_2	1	0, 0, 0, 0, 3	2, 12, 110, 1260	2, 5, 14, 49
2	4	$F_{ab,c}$	D_2	3	0, 0, 0, 0, 1	1, 9, 100, 1225	1, 4, 10, 44
2	8	$F_{a,b,c}$	D_4	5	0, 0, 2, 0, 3	1, 6, 55, 630	1, 3, 7, 26
4	1	G_4	C_1	0	0, 0, 0, 0, 0	3, 20, 175, 1764	2, 6, 20, 76
4	2	$N(G_4)$	C_2	0	0, 0, 0, 0, 1	2, 11, 90, 889	2, 5, 14, 46
6	1	G_6	C_1	0	0, 0, 0, 0, 0	2, 10, 70, 588	2, 5, 14, 44
6	2	$N(G_6)$	C_2	1	0, 0, 0, 0, 0	1, 5, 35, 294	1, 3, 7, 23
10	1	$\mathrm{USp}(4)$	C_1	0	0, 0, 0, 0, 0	1, 3, 14, 84	1, 2, 4, 10

Galois types

Let A be an abelian surface defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{\mathbb{Q}}})$.
The group $\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

The *Galois type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$.

An isomorphism $[G, E] \simeq [G', E']$ is an isomorphism $G \simeq G'$ of groups and an equivariant isomorphism $E \simeq E'$ of $\mathbb{R}$-algebras.

One may have $G \simeq G'$ and $E \simeq E'$ but $[G, E] \not\simeq [G', E']$.

Galois types and Sato-Tate groups in dimension 2

Theorem 2 [FKRS 2012]

Up to conjugacy, the Sato-Tate group G of an abelian surface A is uniquely determined by its Galois type, and vice versa.

We also have $G/G^0 \simeq \mathrm{Gal}(K/k)$, and G^0 is uniquely determined by the isomorphism class of $\mathrm{End}(A_K)_{\mathbb{R}}$, and vice versa:

$$\begin{array}{rlcrl} \mathrm{U}(1) & \mathrm{M}_2(\mathbb{C}) & \qquad & \mathrm{U}(1)\times\mathrm{SU}(2) & \mathbb{C}\times\mathbb{R} \\ \mathrm{SU}(2) & \mathrm{M}_2(\mathbb{R}) & & \mathrm{SU}(2)\times\mathrm{SU}(2) & \mathbb{R}\times\mathbb{R} \\ \mathrm{U}(1)\times\mathrm{U}(1) & \mathbb{C}\times\mathbb{C} & & \mathrm{USp}(4) & \mathbb{R} \end{array}$$

There are 52 distinct Galois types of abelian surfaces.

The proof uses the *algebraic Sato-Tate group* of Banaszak and Kedlaya, which, for $g \leq 3$, uniquely determines ST_A.

Exhibiting Sato-Tate groups of abelian surfaces

Remarkably, the 34 Sato-Tate groups that can arise over $\mathbb{Q}$ can all be realized as the Sato-Tate group of the Jacobian of a hyperelliptic curve.

The remaining 18 groups all arise as subgroups of these 34.

These subgroups can be obtained by extending the field of definition appropriately (in fact, one can realize all 52 groups using just 9 curves).

Genus 2 curves realizing Sato-Tate groups with $G^0 = \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
C_1	$x^6 + 1$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3})$
C_2	$x^5 - x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2})$
C_3	$x^6 + 4$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
C_4	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a); a^4 + 17a^2 + 68 = 0$
C_6	$x^6 + 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
D_2	$x^5 + 9x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
D_3	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
D_4	$x^5 + 3x$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
D_6	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3}, a); a^3 + 3a - 2 = 0$
T	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 - 7a + 7 = b^4 + 4b^2 + 8b + 8 = 0$
O	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-11}, a, b);$
			$a^3 - 4a + 4 = b^4 + 22b + 22 = 0$
$J(C_1)$	$x^5 - x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_2)$	$x^5 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(C_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(C_4)$	$x^6 + x^5 - 5x^4 - 5x^2 - x + 1$	$\mathbb{Q}$	see entry for C_4
$J(C_6)$	$x^6 - 15x^4 - 20x^3 + 6x + 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{3}, a); a^3 + 3a^2 - 1 = 0$
$J(D_2)$	$x^5 + 9x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$
$J(D_3)$	$x^6 + 10x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$J(D_4)$	$x^5 + 3x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2}, \sqrt[4]{3})$
$J(D_6)$	$x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6$	$\mathbb{Q}$	see entry for D_6
$J(T)$	$x^6 + 6x^5 - 20x^4 + 20x^3 - 20x^2 - 8x + 8$	$\mathbb{Q}$	see entry for T
$J(O)$	$x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$	$\mathbb{Q}$	see entry for O
$C_{2,1}$	$x^6 + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$
$C_{4,1}$	$x^5 + 2x$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt[4]{2})$
$C_{6,1}$	$x^6 + 6x^5 - 30x^4 + 20x^3 + 15x^2 - 12x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, a); a^3 - 3a + 1 = 0$
$D_{2,1}$	$x^5 + x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$D_{4,1}$	$x^5 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$D_{6,1}$	$x^6 + 6x^5 - 30x^4 - 40x^3 + 60x^2 + 24x - 8$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, a); a^3 - 9a + 6 = 0$
$D_{3,2}$	$x^6 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$D_{4,2}$	$x^6 + x^5 + 10x^3 + 5x^2 + x - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a); a^4 - 14a^2 + 28a - 14 = 0$
$D_{6,2}$	$x^6 + 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{2})$
O_1	$x^6 + 7x^5 + 10x^4 + 10x^3 + 15x^2 + 17x + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2}, a, b);$
			$a^3 + 5a + 10 = b^4 + 4b^2 + 8b + 2 = 0$

Genus 2 curves realizing Sato-Tate groups with $G^0 \neq \mathrm{U}(1)$

Group	Curve $y^2 = f(x)$	k	K
F	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i, \sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_a	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(i)$	$\mathbb{Q}(i, \sqrt{2})$
F_{ab}	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$
F_{ac}	$x^5 + 1$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 + 5a^2 + 5 = 0$
$F_{a,b}$	$x^6 + 3x^4 + x^2 - 1$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
E_1	$x^6 + x^4 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
E_2	$x^6 + x^5 + 3x^4 + 3x^2 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$
E_3	$x^5 + x^4 - 3x^3 - 4x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^3 - 3a + 1 = 0$
E_4	$x^5 + x^4 + x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(a); a^4 - 5a^2 + 5 = 0$
E_6	$x^5 + 2x^4 - x^3 - 3x^2 - x$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{7}, a); a^3 - 7a - 7 = 0$
$J(E_1)$	$x^5 + x^3 + x$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$J(E_2)$	$x^5 + x^3 - x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt{2})$
$J(E_3)$	$x^6 + x^3 + 4$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$
$J(E_4)$	$x^5 + x^3 + 2x$	$\mathbb{Q}$	$\mathbb{Q}(i, \sqrt[4]{2})$
$J(E_6)$	$x^6 + x^3 - 2$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3}, \sqrt[6]{-2})$
$G_{1,3}$	$x^6 + 3x^4 - 2$	$\mathbb{Q}(i)$	$\mathbb{Q}(i)$
$N(G_{1,3})$	$x^6 + 3x^4 - 2$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$G_{3,3}$	$x^6 + x^2 + 1$	$\mathbb{Q}$	$\mathbb{Q}$
$N(G_{3,3})$	$x^6 + x^5 + x - 1$	$\mathbb{Q}$	$\mathbb{Q}(i)$
$\mathrm{USp}(4)$	$x^5 - x + 1$	$\mathbb{Q}$	$\mathbb{Q}$

Searching for curves

We surveyed the $\bar{L}$-polynomial distributions of genus 2 curves

$$y^2 = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

$$y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

with integer coefficients $|c_i| \leq 128$, over 2^{48} curves.

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We found over 10 million non-isogenous curves with exceptional distributions, including at least 3 apparent matches for all of our target Sato-Tate groups.

Representative examples were computed to high precision $N = 2^{30}$.

For each example, the field K was then determined, allowing the Galois type, and hence the Sato-Tate group, to be **provably** identified.

Computational methods

There are four standard ways to compute $L_p(T)$ for a genus 2 curve:

1. point counting: $O(p^2 \log^{1+\epsilon} p)$.
2. group computation: $O(p^{3/4} \log^{1+\epsilon} p)$.
3. p-adic methods: $O(p^{1/2} \log^{2+\epsilon} p)$.
4. CRT approach: $O(\log^{8+\epsilon} p)$.

For the feasible range of $p \leq N$, we found (2) to be the best [KS08].
We can accelerate the computation with partial use of (1) and (4).

The `smalljac` software package provides an open source implementation of this approach.

A recent breakthrough

All of the methods above perform separate computations for each p. But we want to compute $L_p(T)$ for all good $p \leq N$ using reductions of *the same curve* in each case.

Is their a way to take advantage of this?

A recent breakthrough

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Is their a way to take advantage of this?

Theorem (Harvey, 2012)

Let $y^2 = f(x)$ be a hyperelliptic curve of genus g with $\log \|f\| = O(\log N)$. One can compute $L_p(T)$ for all odd $p \le N$ with $p \nmid \operatorname{disc}(f)$ in time

$$O\big(g^{8+\epsilon} N \log^{3+\epsilon} N\big).$$

This yields an average time of $O\big(g^{8+\epsilon} \log^{4+\epsilon} N\big)$ per prime.

This is the first algorithm to achieve an average running time that is polynomial in both g and $\log p$.

Some preliminary implementation results

With suitable optimizations, this algorithm can be made quite practical.

In genus 2 we are able to surpass the performance of `smalljac` for $N \geq 2^{18}$, with more than a $10\times$ improvement for $N \geq 2^{25}$.

When combined with group computations in genus 3, we expect to obtain a dramatic improvement over all existing methods.

We are also looking at adapting the algorithm to handle certain families of non-hyperelliptic curves, including Picard curves.

[Harvey-S, Achter-S work in progress]

Harvey's algorithm in genus 1

The Hasse invariant h_p of an elliptic curve $y^2 = f(x) = x^3 + ax + b$ over $\mathbb{F}_p$ is the coefficient of x^{p-1} in the polynomial $f(x)^{(p-1)/2}$.

We have $h_p \equiv t_p \bmod p$, which uniquely determines t_p for $p > 13$.

Naïve approach: iteratively compute $f, f^2, f^3, \ldots, f^{(N-1)/2}$ in $\mathbb{Z}[x]$ and reduce the x^{p-1} coefficient of $f(x)^{(p-1)/2}$ mod p for each prime $p \leq N$.

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But the polynomials f^n are huge, each has $\Omega(n^2)$ bits.
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So this is a terrible idea...

But we don't need all the coefficients of f^n, we only need one;
and we only need to know its value modulo $p = 2n + 1$.

A better approach

Let $f(x) = x^3 + ax + b$, and let f_k^n denote the coefficient of x^k in $f(x)^n$. Using $f^n = ff^{n-1}$ and $(f^n)' = nf'f^{n-1}$, one obtains the relations

$$
\begin{aligned}
(n+2)f_{2n-2}^n &= n\left(2af_{2n-3}^{n-1} + 3bf_{2n-2}^{n-1}\right),\\
(2n-1)f_{2n-1}^n &= n\left(3f_{2n-4}^{n-1} + af_{2n-2}^{n-1}\right),\\
2(2n-1)bf_{2n}^n &= (n+1)af_{2n-4}^{n-1} + 3(2n-1)bf_{2n-3}^{n-1} - (n-1)a^2f_{2n-2}^{n-1}.
\end{aligned}
$$

These allow us to compute the vector $w_n = [f_{2n-2}^n, f_{2n-1}^n, f_{2n}^n]$ from the vector $w_{n-1} = [f_{2n-4}^{n-1}, f_{2n-3}^{n-1}, f_{2n-2}^{n-1}]$ via multiplication by a 3×3 matrix M_n with entries in $\mathbb{Q}$. We have

$$w_n = w_0 M_1 M_2 \cdots M_n.$$

For $n = (p-1)/2$, the Hasse invariant of the elliptic curve $y^2 = f(x)$ over $\mathbb{F}_p$ is obtained by reducing the third entry f_n^{2n} of w_n modulo p.

Computing $t_p \bmod p$

To compute $t_p \bmod p$ for all odd primes $p \le N$ it suffices to compute

$$\begin{aligned} M_1 &\bmod 3 \\ M_1M_2 &\bmod 5 \\ M_1M_2M_3 &\bmod 7 \\ M_1M_2M_3M_4 &\bmod 9 \\ &\vdots \\ M_1M_2M_3\cdots M_{(N-1)/2} &\bmod N \end{aligned}$$

Doing this naïvely would take $O\big(N^{2+\epsilon}\big)$ time.
But it can be done in $O\big(N^{1+\epsilon}\big)$ time using a *remainder tree*.

Remainder trees

Given matrices $M_1, M_2, \ldots, M_N$ and moduli $m_1, m_2, \ldots, m_N$, we wish to compute remainders $R_1, R_2, \ldots, R_N$, where $R_n = \prod_{i=1}^{n-1} M_i \bmod m_n$.

Algorithm for $N = 2^k$:

1. Compute a binary *product tree* with leaf values $M_1, \ldots, M_N$ and internal nodes whose values that are the product of their children, and do the same for the moduli $m_1, \ldots, m_N$.
2. Working from the top down, compute each node's *remainder* as the product of its parent's remainder and its left sibling's value, reduced modulo the node's modulus.

Each node's remainder is the product of the values in the leaves to its left, reduced modulo the node's modulus.

The leaf remainders are precisely $R_1, \ldots, R_N$.
Using FFT-based arithmetic, this algorithm runs in quasi-linear time.

Hyperelliptic curves of genus $g > 1$.

The general algorithm uses Monsky-Washnitzer cohomology (as in Kedlaya's algorithm), but for $g \leq 3$ it is enough to just compute the Hasse-Witt matrix. This is the $g \times g$ matrix $W = [w_{ij}]$ with entries

$$w_{ij} = f_{pi-j}^{(p-1)/2} \bmod p.$$

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$$w_{ij} = f_{pi-j}^{(p-1)/2} \bmod p.$$

The w_{ij} can each be computed using recurrence relations between the coefficients of f^n and those of f^{n-1}, as in genus 1.

The characteristic polynomial of W determines the $L_p(T) \bmod p$.

Using group computations in the Jacobian of the curve, one can determine $L_p(T)$ exactly. This takes $\tilde{O}(1)$ time in genus 2, and $\tilde{O}(p^{1/4})$ time in genus 3, which turns out to be negligible within the feasible range of computation.

Sato-Tate in dimension 3

For $g = 3$ there are 15 possibilities for the connected part of ST_A. There are at least 400 groups that satisfy the Sato-Tate axioms.

In order to realize cases with large component groups, one needs abelian threefolds with many endomorphisms. An obvious place to start is with Jacobians of curves with large automorphism groups (and their twists). Some notable cases enumerated by Wolfart:

$$y^2 = x^8 - x, \quad y^2 = x^7 - x, \quad y^2 = x^8 - 1$$
$$y^2 = x^8 - 14x^4 + 1, \quad y^3 = x^4 - x, \quad y^3 = x^4 - 1$$
$$x^4 + y^4 = 1, \quad x^3y + y^3z + z^3x = 0.$$

However, Jacobians may not be enough!