Isogeny volcanoes

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A volcano

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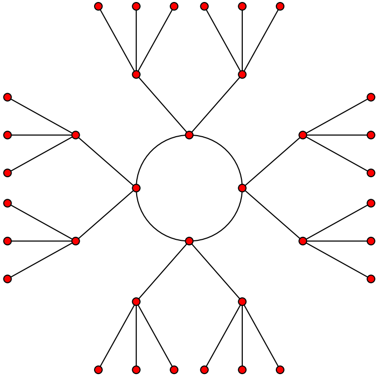


ℓ-volcanoes

For a prime ℓ, an ℓ-*volcano* is a connected undirected graph whose vertices are partitioned into levels $V_0, \ldots, V_d$.

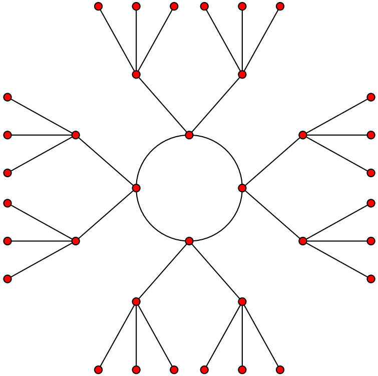
1. The subgraph on V_0 (the *surface*) is a connected regular graph of degree $0, 1$, or 2.

2. For $i > 0$, each $v \in V_i$ has exactly one neighbor in V_{i-1}. All edges not on the surface arise in this manner.

3. For $i < d$, each $v \in V_i$ has degree ℓ+1.

We allow self-loops and multi-edges in our graphs, but this can happen only on the surface of an ℓ-volcano.

A 3-volcano of depth 2



Elliptic curves

An elliptic curve E/k is a smooth projective curve of genus 1 with a distinguished k-rational point 0.

For any field extension k'/k, the set of k'-rational points $E(k')$ forms an abelian group with identity element 0.

When the characteristic of k is not 2 or 3 (which we assume for convenience) we may define E with an equation of the form

$$y^2 = x^3 + Ax + B,$$

where $A, B \in k$.

j-invariants

The $\bar{k}$-isomorphism classes of elliptic curves E/k are in bijection with the field k. For $E\colon\ y^2 = x^3 + Ax + B$, the j*-invariant* of E is

$$j(E) = j(A,B) = 1728\frac{4A^3}{4A^3 + 27B^2} \in k.$$

The j-invariants $j(0,B) = 0$ and $j(A,0) = 1728$ are special.
They correspond to elliptic curves with extra automorphisms.

For $j_0 \not\in \{0, 1728\}$, we have $j_0 = j(A,B)$, where

$$A = 3j_0(1728 - j_0) \qquad \text{and} \qquad B = 2j_0(1728 - j_0)^2.$$

Note that $j(E_1) = j(E_2)$ does not necessarily imply that E_1 and E_2 are isomorphic over k, but they must be isomorphic over $\bar{k}$.

ℓ-isogenies

An *isogeny* $\phi\colon E_1 \to E_2$ is a non-constant morphism of elliptic curves, a non-trivial rational map that fixes the point 0.

It induces a group homomorphism $\phi\colon E_1(\bar{k}) \to E_2(\bar{k})$ with finite kernel.
Conversely, every finite subgroup of $E_1(\bar{k})$ is the kernel of an isogeny.

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The *degree* of an isogeny is its degree as a rational map. For *separable* isogenies, we have $\deg\phi = |\ker\phi|$.

We are interested in isogenies of prime degree $\ell \neq \operatorname{char} k$, which are necessarily separable isogenies with cyclic kernels.

The *dual isogeny* $\hat{\phi}\colon E_2 \to E_1$ has the same degree ℓ as ϕ, and

$$\phi \circ \hat{\phi} = \hat{\phi} \circ \phi = [\ell]$$

is the *multiplication-by-ℓ* map.

The ℓ-torsion subgroup

For $\ell \neq \mathrm{char}(k)$, the *$\ell$-torsion subgroup*

$$E[\ell] = \{P \in E(\bar{k}) : \ell P = 0\}$$

is isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ and thus contains $\ell + 1$ cyclic subgroups of order ℓ, each of which is the kernel of an ℓ-isogeny.

These ℓ-isogenies are not necessarily defined over k.

An ℓ-isogeny is defined over k (and has image defined over k) if and only if its kernel is Galois-invariant.

The number of Galois-invariant order-ℓ subgroups of $E[\ell]$ is either $0, 1, 2$, or $\ell + 1$.

The modular equation

Let $j\colon \mathbb{H} \to \mathbb{C}$ be the classical modular function.

For any $\tau \in \mathbb{H}$, the values $j(\tau)$ and $j(\ell\tau)$ are the j-invariants of elliptic curves over $\mathbb{C}$ that are ℓ-isogenous.

The minimal polynomial $\Phi_\ell(Y)$ of the function $j(\ell z)$ over $\mathbb{C}(j)$ has coefficients that are actually integer polynomials of $j(z)$.

Replacing $j(z)$ with X yields the *modular polynomial* $\Phi_\ell \in \mathbb{Z}[X, Y]$ that parameterizes pairs of ℓ-isogenous elliptic curves $E/\mathbb{C}$:

$$\Phi_\ell\big(j(E_1), j(E_2)\big) = 0 \quad \Longleftrightarrow \quad j(E_1) \text{ and } j(E_2) \text{ are } \ell\text{-isogenous.}$$

This moduli interpretation remains valid over any field of characteristic not ℓ.

$\Phi_\ell(X, Y) = 0$ is a defining equation for the affine modular curve $Y_0(\ell) = \Gamma_0(\ell)\backslash\mathbb{H}$.

The graph of ℓ-isogenies

Definition
The *ℓ-isogeny graph* $G_\ell(k)$ has vertex set $\{j(E) : E/k\} = k$
and edges (j_1, j_2) for each root $j_2 \in k$ of $\Phi_\ell(j_1, Y)$ (with multiplicity).

Except for $j \in \{0, 1728\}$, the in-degree of each vertex of G_ℓ is equal to its out-degree. Thus G_ℓ is a bi-directed graph on $k\backslash\{0, 1728\}$, which we may regard as an undirected graph.

Note that we have an infinite family of graphs $G_\ell(k)$ with vertex set k, one for each prime $\ell \neq \mathrm{char}(k)$.

Ordinary and supersingular curves

For an elliptic curve E/k with $\mathrm{char}(k) = p$ we have

$$E[p] \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & (\textit{ordinary}), \\ \{0\} & (\textit{supersingular}). \end{cases}$$

For isogenous elliptic curves $E_1 \sim E_2$, either both are ordinary or both are supersingular. Thus the each isogeny graph G_ℓ decomposes into ordinary and supersingular components.

This has cryptographic applications; see [Charles-Lauter-Goren 2008], for example.

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Every supersingular curve is defined over $\mathbb{F}_{p^2}$. Thus the supersingular components of $G_\ell(\mathbb{F}_{p^2})$ are regular graphs of degree $\ell + 1$.

In fact, $G_\ell(\mathbb{F}_{p^2})$ has just one supersingular component, and it is a *Ramanujan graph* [Pizer 1990].

This has cryptographic applications; see [Charles-Lauter-Goren 2008], for example.

Endomorphism rings

Isogenies from an elliptic curve E to itself are *endomorphisms*.
They form a ring $\mathrm{End}(E)$ under composition and point addition.

We always have $\mathbb{Z} \subseteq \mathrm{End}(E)$, due to scalar multiplication maps.
If $\mathbb{Z} \subsetneq \mathrm{End}(E)$, then E has *complex multiplication* (CM).

For an elliptic curve E with complex multiplication:

$$\mathrm{End}(E) \simeq \begin{cases} \text{order in an imaginary quadratic field} & \text{(ordinary),} \\ \text{order in a quaternion algebra} & \text{(supersingular).} \end{cases}$$

Over a finite field, every elliptic curve has CM.

Horizontal and vertical isogenies

Let $\varphi\colon E_1 \to E_2$ by an ℓ-isogeny of ordinary elliptic curves with CM.
Let $\mathrm{End}(E_1) \simeq \mathcal{O}_1 = [1, \tau_1]$ and $\mathrm{End}(E_2) \simeq \mathcal{O}_2 = [1, \tau_2]$.

Then $\ell\tau_2 \in \mathcal{O}_1$ and $\ell\tau_1 \in \mathcal{O}_2$.

Thus one of the following holds:

- $\mathcal{O}_1 = \mathcal{O}_2$, in which case φ is *horizontal*;
- $[\mathcal{O}_1 : \mathcal{O}_2] = \ell$, in which case φ is *descending*;
- $[\mathcal{O}_2 : \mathcal{O}_1] = \ell$, in which case φ is *ascending*.

In the latter two cases we say that φ is a *vertical* isogeny.

The theory of complex multiplication

Let E/k have $\operatorname{End}(E) \simeq \mathcal{O} \subset K = \mathbb{Q}(\sqrt{D})$, with $D = \operatorname{disc} K$.

For each invertible $\mathcal{O}$-ideal $\mathfrak{a}$, the $\mathfrak{a}$-torsion subgroup

$$E[\mathfrak{a}] = \{P \in E(\bar{k}) : \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\}$$

is the kernel of an isogeny $\varphi_{\mathfrak{a}}\colon E \to E'$ of degree $N(\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$.
We necessarily have $\operatorname{End}(E) \simeq \operatorname{End}(E')$, so $\varphi_{\mathfrak{a}}$ is **horizontal**.

If $\mathfrak{a}$ is principal, then $E' \simeq E$. This induces a $\operatorname{cl}(\mathcal{O})$-action on the set.

$$\operatorname{Ell}_{\mathcal{O}}(k) = \{j(E) : E/k \text{ with } \operatorname{End}(E) \simeq \mathcal{O}\}.$$

This action is faithful and transitive; thus $\operatorname{Ell}_{\mathcal{O}}(k)$ is a principal homogeneous space, a *torsor*, for $\operatorname{cl}(\mathcal{O})$.

One can decompose horizontal isogenies of large prime degree into an equivalent sequence of isogenies of small prime degrees, which makes them **easy to compute**; see [Bröker-Charles-Lauter 2008, Jao-Souhkarev 2010].

Horizontal isogenies

Every horizontal ℓ-isogeny arises from the action of an invertible $\mathcal{O}$-ideal $\mathfrak{l}$ of norm ℓ.

If $\ell \mid [\mathcal{O}_K : \mathcal{O}]$, no such $\mathfrak{l}$ exists; if $\ell \nmid [\mathcal{O}_K : \mathcal{O}]$, then there are

$$1 + \left(\frac{D}{\ell}\right) = \begin{cases} 0 & \ell \text{ is inert in } K, \\ 1 & \ell \text{ is ramified in } K, \\ 2 & \ell \text{ splits in } K, \end{cases}$$

such ℓ-isogenies.

In the split case, $(\ell) = \mathfrak{l} \cdot \bar{\mathfrak{l}}$, and the $\mathfrak{l}$-orbits partition $\mathrm{Ell}_{\mathcal{O}}(k)$ into cycles corresponding to the cosets of $\langle [\mathfrak{l}] \rangle$ in $\mathrm{cl}(\mathcal{O})$.

Vertical isogenies

Let $\mathcal{O}$ be an imaginary quadratic order with discriminant $D_{\mathcal{O}} < -4$, and let $\mathcal{O}' = \mathbb{Z} + \ell\mathcal{O}$ be the order of index ℓ in $\mathcal{O}$.

The map that sends each invertible $\mathcal{O}'$-ideal $\mathfrak{a}$ to the (invertible) $\mathcal{O}$-ideal $\mathfrak{a}\mathcal{O}$ preserves norms and induces a surjective homomorphism

$$\phi\colon\ \mathrm{cl}(\mathcal{O}') \to \mathrm{cl}(\mathcal{O})$$

compatible with the class group actions on $\mathrm{Ell}_{\mathcal{O}}(k)$ and $\mathrm{Ell}_{\mathcal{O}'}(k)$.

It follows that each $j(E') \in \mathrm{Ell}_{\mathcal{O}'}(k)$ has a unique ℓ-isogenous "parent" $j(E)$ in $\mathrm{Ell}_{\mathcal{O}}(k)$, and every vertical isogeny must arise in this way.

The "children" of $j(E)$ correspond to a coset of the kernel of ϕ, which is a cyclic of order $\ell - \left(\frac{D_{\mathcal{O}}}{\ell}\right)$, generated by the class of an invertible $\mathcal{O}'$-ideal with norm ℓ^2.

Ordinary elliptic curves over finite fields

Let $E/\mathbb{F}_q$ be an ordinary elliptic curve with *trace of Frobenius*

$$t = \operatorname{tr}\pi_E = q + 1 - \#E(\mathbb{F}_q).$$

Then $\pi_E^2 - t\pi_E + q = 0$ and we have the *norm equation*

$$4q = t^2 - v^2 D,$$

where D is the (fundamental) discriminant of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{t^2 - 4q}) \simeq \operatorname{End}(E) \otimes \mathbb{Q}$ and $v = [\mathcal{O}_K : \mathbb{Z}[\pi_E]]$. We have

$$\mathbb{Z}[\pi_E] \subseteq \operatorname{End}(E) \subseteq \mathcal{O}_K.$$

Thus $[\mathcal{O}_K : \operatorname{End}(E)]$ divides v; this holds for any E with trace t.
If we define $\operatorname{Ell}_t(\mathbb{F}_q) = \{j(E) : E/\mathbb{F}_q \text{ with } \operatorname{tr}\pi_E = t\}$, then

$$\operatorname{Ell}_t(\mathbb{F}_q) = \bigcup_{\mathbb{Z}[\pi_E] \subseteq \mathcal{O} \subseteq \mathcal{O}_K} \operatorname{Ell}_{\mathcal{O}}(\mathbb{F}_q).$$

The main theorem

Theorem (Kohel)

Let V be an ordinary connected component of $G_\ell(\mathbb{F}_q)$ that does not contain $0, 1728$. Then V is an ℓ-volcano in which the following hold:

(i) *Vertices in level V_i all have the same endomorphism ring $\mathcal{O}_i$.*

(ii) $\ell \nmid [\mathcal{O}_K : \mathcal{O}_0]$*, and* $[\mathcal{O}_i : \mathcal{O}_{i+1}] = \ell$.

(iii) *The subgraph on V_0 has degree $1 + (\frac{D}{\ell})$, where $D = \operatorname{disc}(\mathcal{O}_0)$.*

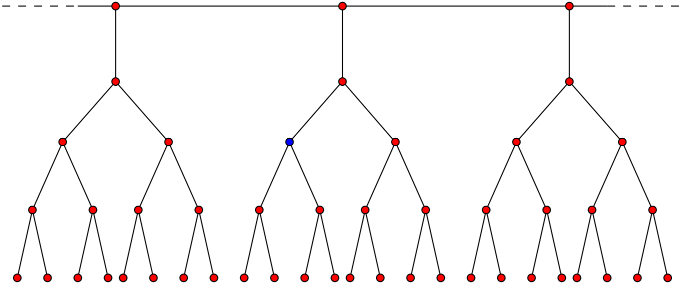
(iv) *If $(\frac{D}{\ell}) \geq 0$ then $|V_0|$ is the order of $[\mathfrak{l}]$ in $\operatorname{cl}(\mathcal{O}_0)$.*

(v) *The depth of V is $\operatorname{ord}_\ell(v)$, where $4q = t^2 - v^2 D$.*

The term *volcano* is due to Fouquet and Morain.

Applications

Finding the floor



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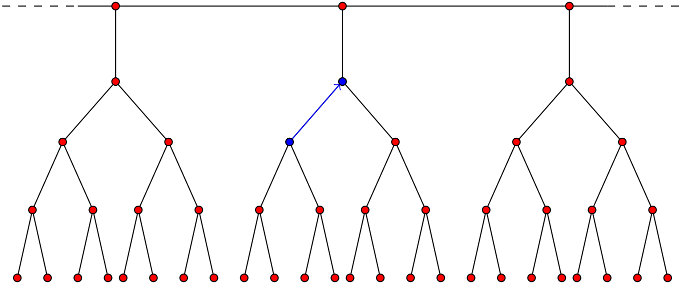

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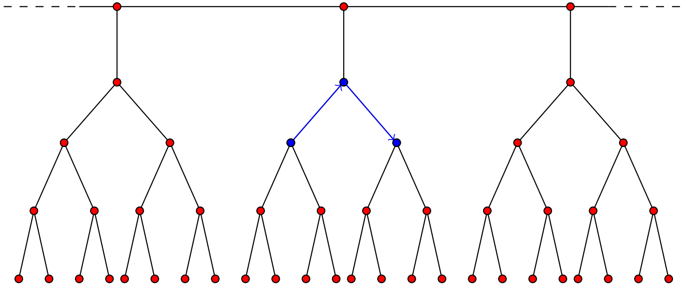

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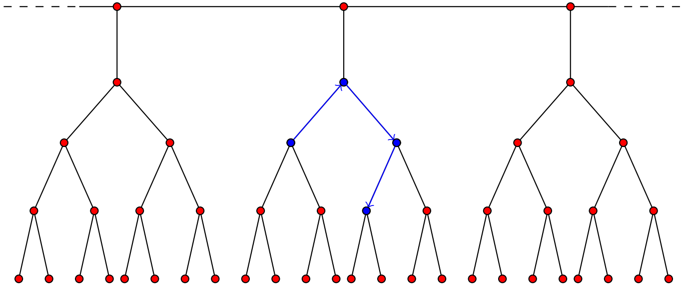

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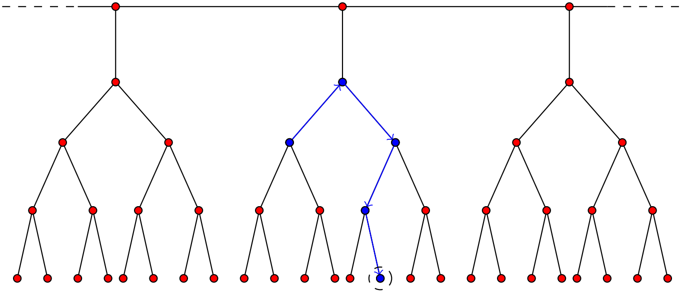
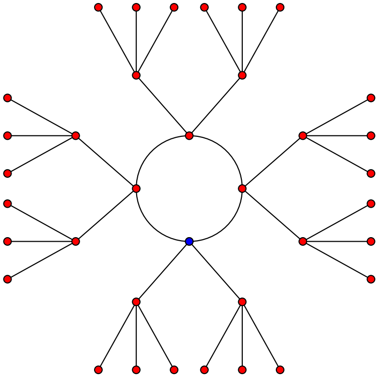


Curves on the floor necessarily have cyclic rational ℓ-torsion. This is useful, for example, when constructing Edwards curves with the CM method [Morain 2009].

Finding a shortest path to the floor



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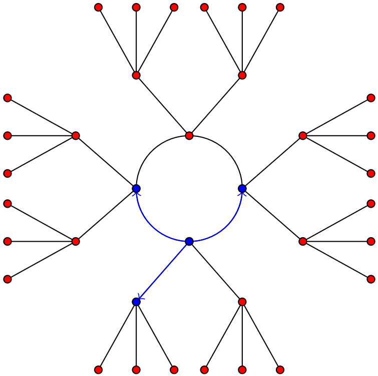

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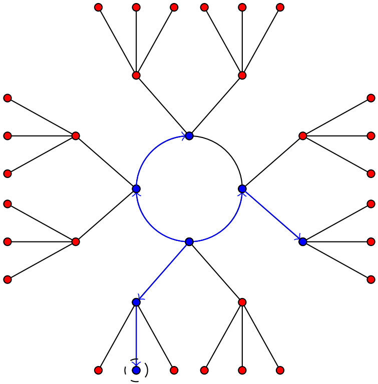

We now know that we are 2 levels above the floor.

Application: identifying supersingular curves

The equation $4q = t^2 - v^2 D$ implies that each ordinary component of $G_\ell(\mathbb{F}_q)$ is an ℓ-volcano of depth less than $\log_\ell \sqrt{4q}$.

Given $j(E) \in \mathbb{F}_{p^2}$, if we cannot find a shortest path to the floor in $G_2(\mathbb{F}_{p^2})$ within $\lceil \log_2 p \rceil$ steps, then E **must be supersingular**.

Conversely, if E is supersingular, our attempt to find the floor must fail, since every vertex in the supersingular component has degree $\ell + 1$.

See [S 2012] for details.

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This yields a (probabilistic) algorithm to determine supersingularity in $\tilde{O}(n^3)$ time, where $n = \log p$, improving the $\tilde{O}(n^4)$ complexity of the best previously known algorithms.

Moreover, the expected running time on a random elliptic curve is $\tilde{O}(n^2)$, matching the complexity of the best *Monte Carlo* algorithms, and faster in practice.

See [S 2012] for details.

Application: computing endomorphism rings

Given an ordinary elliptic curve $E/\mathbb{F}_q$, if we compute the Frobenius trace t and put $4q = t^2 - v^2 D$, we can determine $\mathcal{O} \simeq \mathrm{End}(E)$ by determining $u = [\mathcal{O}_K : \mathcal{O}]$, which must divide v.

It suffices to determine the level of $j(E)$ in its ℓ-volcano for $\ell | v$.

Problem: when ℓ is large it is not feasible to compute Φ_ℓ, nor is it feasible to directly compute a **vertical** ℓ-isogeny.

See [Bisson-S 2011] and [Bisson 2012] for more details.

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Solution: we may determine the primes $\ell | u$ by finding *smooth relations* that hold in $\mathrm{cl}((v/\ell)^2 D)$ but not in $\mathrm{cl}(\ell^2 D)$ and evaluating the corresponding **horizontal** isogenies (and similarly for ℓ^e)

This yields a probabilistic algorithm to compute $\mathrm{End}(E)$ with subexponential expected running time $L[1/2, \sqrt{3}/2]$, under GRH..

See [Bisson-S 2011] and [Bisson 2012] for more details.

Example

Let $q = 2^{320} + 261$ and suppose $\operatorname{tr} \pi_E = t$, where
$t = 2306414344576213633891236434392671392737040459558$.

Then $4q = t^2 - v^2 D$, where $D = -147759$ and $v = 2^2 p_1 p_2$ with

$$p_1 = 1644768905973582478403 9,$$
$$p_2 = 7100397697549005947257 1.$$

For $D_1 = 2^4 p_2^2 D$, and $D_1' = p_1^2 D$, the relation

$$\{\mathfrak{p}_5, \mathfrak{p}_{19}^2, \bar{\mathfrak{p}}_{23}^{210}, \mathfrak{p}_{29}, \mathfrak{p}_{31}, \bar{\mathfrak{p}}_{41}^{145}, \mathfrak{p}_{139}, \bar{\mathfrak{p}}_{149}, \mathfrak{p}_{167}, \bar{\mathfrak{p}}_{191}, \bar{\mathfrak{p}}_{251}^6, \mathfrak{p}_{269}, \bar{\mathfrak{p}}_{587}^7, \bar{\mathfrak{p}}_{643}\}$$

holds in $\operatorname{cl}(D_1)$ but not in $\operatorname{cl}(D_1')$ ($\mathfrak{p}_\ell$ is an ideal of norm ℓ).
For $D_2 = 2^4 p_1^2 D$, and $D_2' = p_2^2 D$, the relation

$$\{\mathfrak{p}_{11}, \bar{\mathfrak{p}}_{13}^{576}, \mathfrak{p}_{23}^2, \bar{\mathfrak{p}}_{41}, \bar{\mathfrak{p}}_{47}, \mathfrak{p}_{83}, \mathfrak{p}_{101}, \bar{\mathfrak{p}}_{197}^{28}, \bar{\mathfrak{p}}_{307}^3, \mathfrak{p}_{317}, \bar{\mathfrak{p}}_{419}, \mathfrak{p}_{911}\}$$

holds in $\operatorname{cl}(D_2)$ but not in $\operatorname{cl}(D_2')$.

Constructing elliptic curves with the CM method

Let $\mathcal{O}$ be an imaginary quadratic order with discriminant D.
The *Hilbert class polynomial* $H_D \in \mathbb{Z}[X]$ is defined by

$$H_D(X) = \prod_{j \in \mathrm{Ell}_{\mathcal{O}}(\mathbb{C})} (X - j).$$

Equivalently, it is the minimal polynomial of $j(\mathcal{O})$ over $K = \mathbb{Q}(\sqrt{D})$.
The field $K_{\mathcal{O}} = K(j(\mathcal{O}))$ is the *ring class field* for $\mathcal{O}$.

One can also construct supersingular curves with Hilbert class polynomials; see [Bröker 2008].

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If q splits completely in $K_{\mathcal{O}}$, then $H_D(X)$ splits completely in $\mathbb{F}_q[X]$, and every root of H_D is the j-invariant of an elliptic curve $E/\mathbb{F}_q$ with $N = q + 1 - t$ points, where $4q = t^2 - v^2 D$.

Every ordinary elliptic curve $E/\mathbb{F}_q$ can be constructed in this way, but computing H_D becomes quite difficult as $|D|$ grows.

The size of H_D is $O(|D| \log |D|)$ bits, exponential in $\log q$.

One can also construct supersingular curves with Hilbert class polynomials; see [Bröker 2008].

Application: computing Hilbert class polynomials

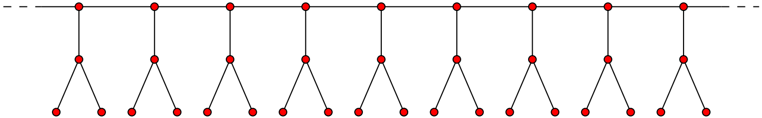
The CRT approach to computing $H_D(X)$, as described in [Belding-Bröker-Enge-Lauter 2008] and [S 2011].

1. Select a sufficiently large set of primes of the form $4p = t^2 - v^2 D$.
2. For each prime p, compute $H_D \bmod p$ as follows:
   - **a.** Generate random curves $E/\mathbb{F}_p$ until $\#E = p + 1 - t$.
   - **b.** Use volcano climbing to find $E' \sim E$ with $\mathrm{End}(E') \simeq \mathcal{O}$.
   - **c.** Enumerate $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p)$ by applying the $\mathrm{cl}(\mathcal{O})$-action to $j(E')$.
   - **d.** Compute $\prod_{j \in \mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p)} (X - j) = H_D(X) \bmod p$.
3. Use the CRT to recover H_D over $\mathbb{Z}$ (or mod q via the explicit CRT).

Under the GRH, the expected running time is $O(|D| \log^{5+\epsilon} |D|)$, quasi-linear in the size of H_D.

One can similarly compute other types of class polynomials [Enge-S 2010].

Running the rim



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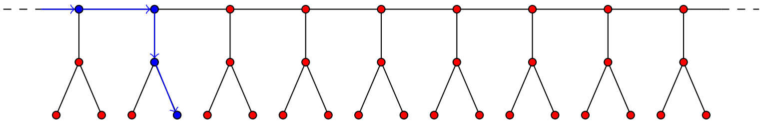

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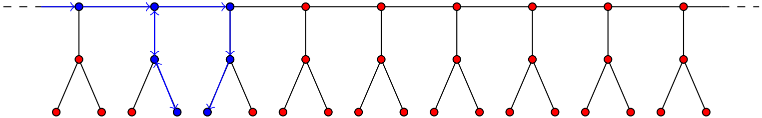

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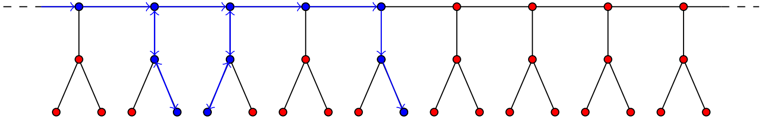

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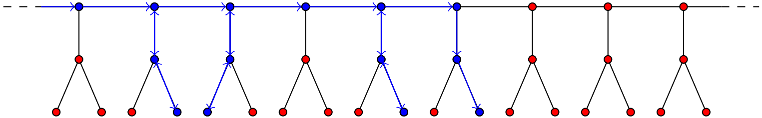

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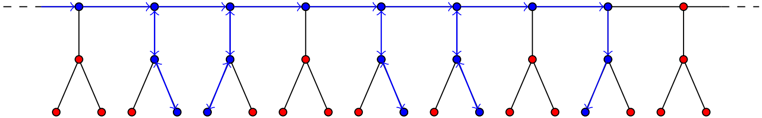
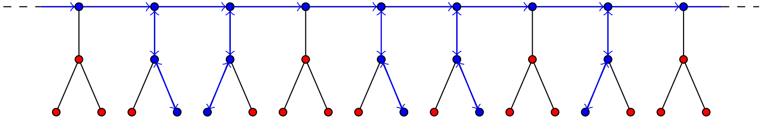


Running the rim



For particularly deep volcanoes, one may prefer to use a pairing-based approach; see [Ionica-Joux 2010].

Computational results

The CRT method has been used to compute $H_D(X)$ with $|D| > 10^{13}$, and using alternative class polynomials, with $|D| > 10^{15}$ (for comparison, the previous record was $|D| \approx 10^{10}$).

When $\mathrm{cl}(\mathcal{O})$ is composite (almost always the case), one can accelerate the CM method by decomposing the ring class field [Hanrot-Morain 2001, Enge-Morain 2003].

Combining this idea with the CRT approach has made CM constructions with $|D| > 10^{16}$ possible [S 2012].

Application: computing modular polynomials

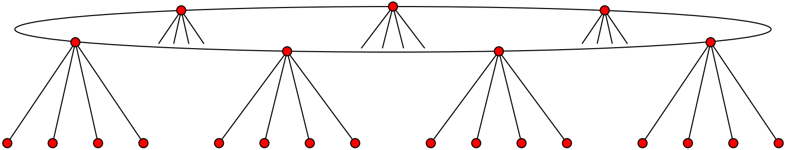
We can also use a CRT approach to compute $\Phi_\ell(X,Y)$ [Bröker-Lauter-S 2012].

1. Select a sufficiently large set of primes of the form $4p = t^2 - \ell^2 v^2 D$ with $\ell \nmid v$, $p \equiv 1 \bmod \ell$, and $h(D) > \ell + 1$.
2. For each prime p, compute $\Phi_\ell \bmod p$ as follows:
   a. Compute $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p)$ using $H_D \bmod p$.
   b. Map the ℓ-volcanoes intersecting $\mathrm{Ell}_{O}(\mathbb{F}_p)$ (without using Φ_ℓ).
   c. Interpolate $\Phi_\ell(X,Y) \bmod p$.
3. Use the CRT to recover Φ_ℓ over $\mathbb{Z}$ (or mod q via the explicit CRT).

Under the GRH, the expected running time is $O(\ell^3 \log^{3+\epsilon} \ell)$, quasi-linear in the size of Φ_ℓ.

We can similarly compute modular polynomials for other modular functions. See [Bruinier-Ono-S 2013] for an algorithm to compute Φ_N for composite N.

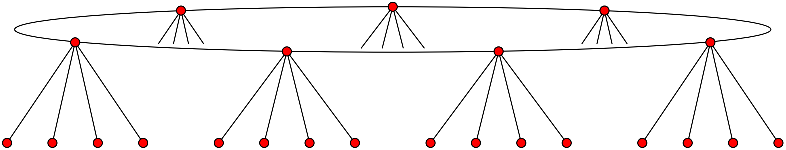
Mapping a volcano



Mapping a volcano

Example
$\ell = 5, \quad p = 4451, \quad D = -151$

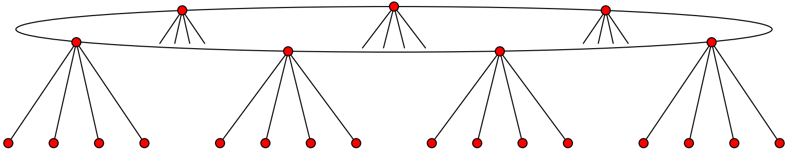
General requirements
$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$



Mapping a volcano

Example
$\ell = 5, \quad p = 4451, \quad D = -151$
$t = 52, \quad v = 2, \quad h(D) = 7$

General requirements
$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$
$\ell \nmid v, \quad \left(\frac{D}{\ell}\right) = 1, \quad h(D) \geq \ell + 2$



Mapping a volcano

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$t = 52, \quad v = 2, \quad h(D) = 7$

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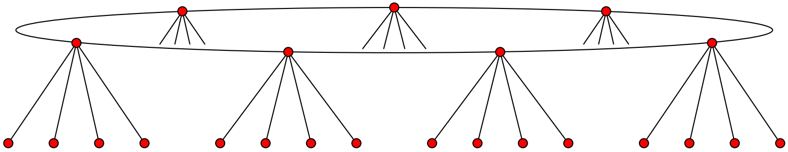

1. Find a root of $H_D(X)$

Mapping a volcano

Example
$\ell \ = 5, \quad p = 4451, \quad D = -151$
$t \ = 52, \quad v = 2, \quad h(D) = 7$

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$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$
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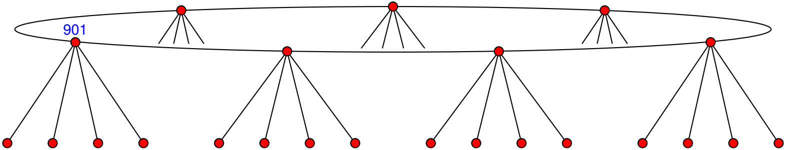

1. Find a root of $H_D(X)$: 901

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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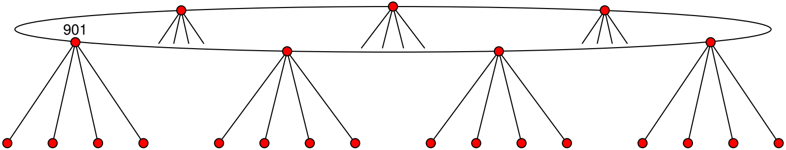

2. Enumerate surface using the action of α_{ℓ_0}

Mapping a volcano

Example

$\ell \ = 5, \quad p = 4451, \quad D = -151$

$t \ = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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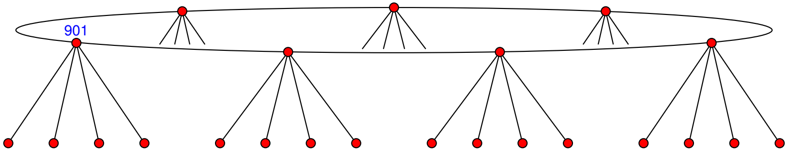
$\ell_0 \neq \ell,\ (\frac{D}{\ell_0}) = 1,\ \alpha_\ell = \alpha_{\ell_0}^k$



2. Enumerate surface using the action of α_{ℓ_0}

$901 \xrightarrow{2} 1582 \xrightarrow{2} 2501 \xrightarrow{2} 351 \xrightarrow{2} 701 \xrightarrow{2} 2872 \xrightarrow{2} 2215 \xrightarrow{2}$

Mapping a volcano

Example
$\ell = 5, \quad p = 4451, \quad D = -151$
$t = 52, \quad v = 2, \quad h(D) = 7$
$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements
$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$
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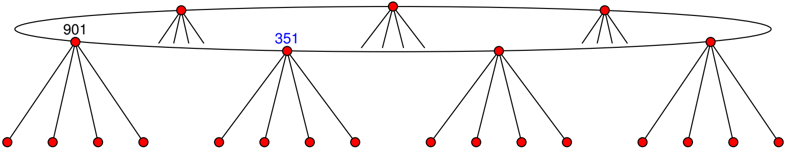

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Mapping a volcano

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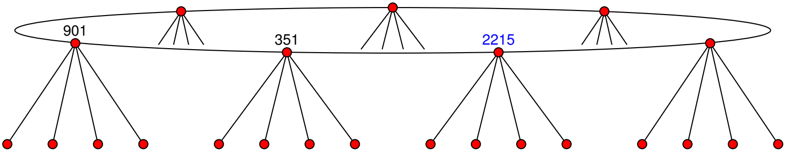

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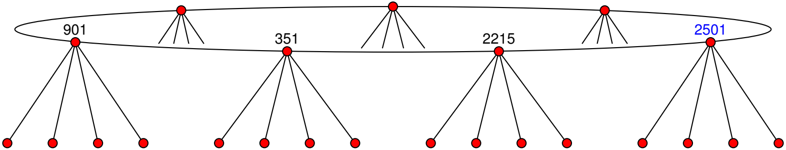

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Mapping a volcano

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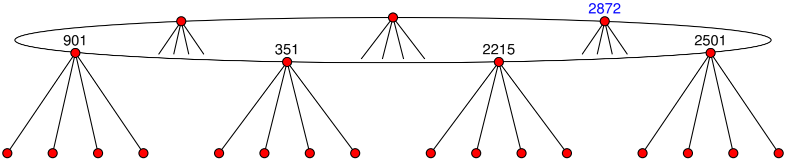

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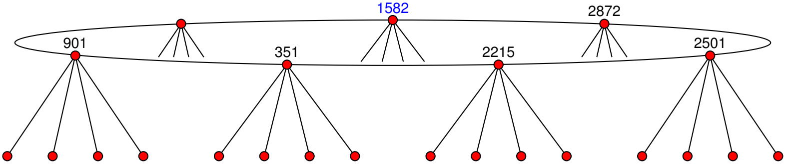

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Mapping a volcano

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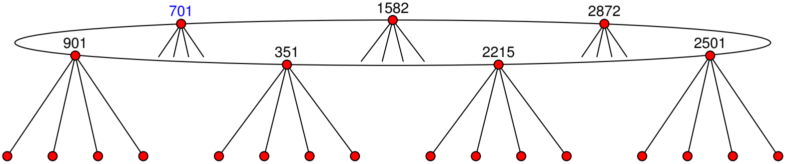

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Mapping a volcano

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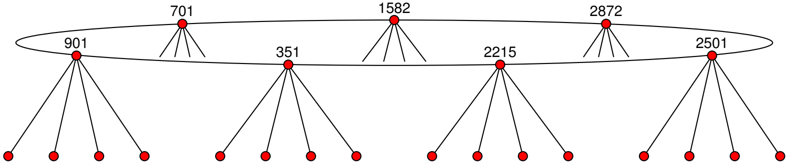

3. Descend to the floor using Vélu's formula

Mapping a volcano

Example

$\ell \;= 5, \quad p = 4451, \quad D = -151$

$t \;= 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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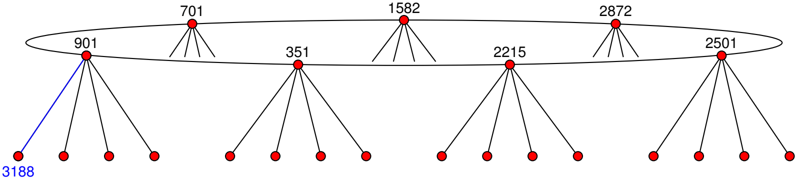

3. Descend to the floor using Vélu's formula: $901 \xrightarrow{5} 3188$

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

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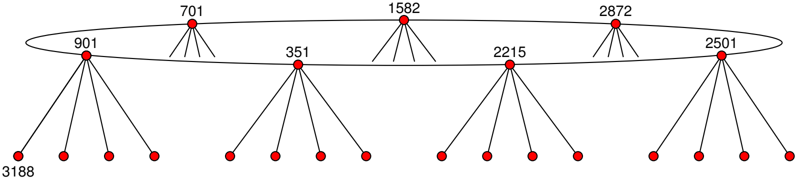

4. Enumerate floor using the action of β_{ℓ_0}

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3, \quad \beta_{25} = \beta_2^7$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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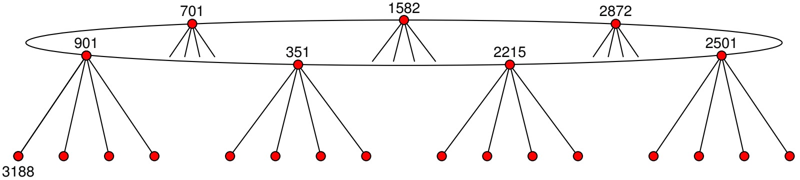

4. Enumerate floor using the action of β_{ℓ_0}

$$
\begin{array}{llllllll}
3188 \xrightarrow{2} 945 & \xrightarrow{2} 3144 & \xrightarrow{2} 3508 & \xrightarrow{2} 2843 & \xrightarrow{2} 1502 & \xrightarrow{2} 676 & \xrightarrow{2} \\
2970 \xrightarrow{2} 3497 & \xrightarrow{2} 1180 & \xrightarrow{2} 2464 & \xrightarrow{2} 4221 & \xrightarrow{2} 4228 & \xrightarrow{2} 2434 & \xrightarrow{2} \\
1478 \xrightarrow{2} 3244 & \xrightarrow{2} 2255 & \xrightarrow{2} 2976 & \xrightarrow{2} 3345 & \xrightarrow{2} 1064 & \xrightarrow{2} 1868 & \xrightarrow{2} \\
3328 \xrightarrow{2} 291 & \xrightarrow{2} 3147 & \xrightarrow{2} 2566 & \xrightarrow{2} 4397 & \xrightarrow{2} 2087 & \xrightarrow{2} 3341 & \xrightarrow{2}
\end{array}
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Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3, \quad \beta_{25} = \beta_2^7$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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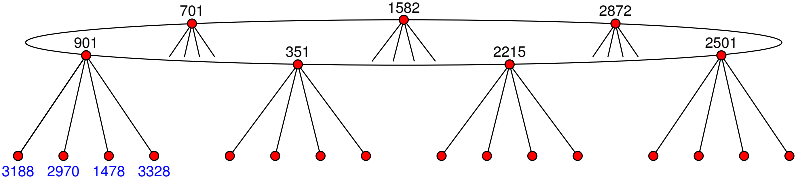

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Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

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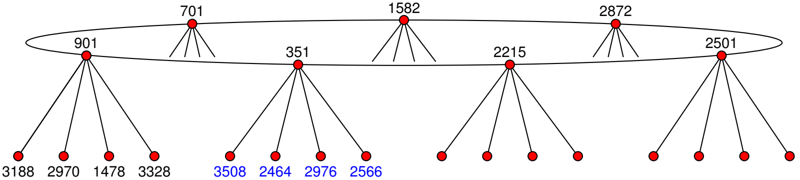

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Mapping a volcano

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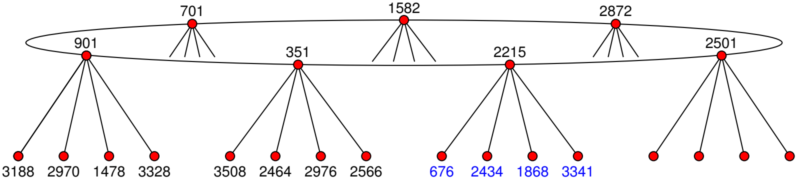

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Mapping a volcano

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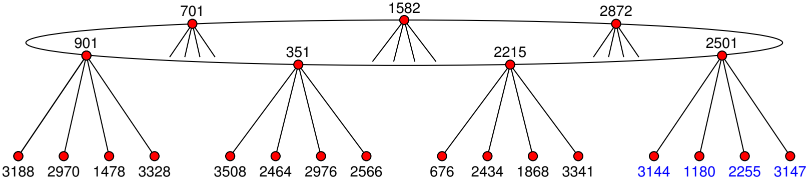

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Mapping a volcano

Example

$\ell \;= 5, \quad p = 4451, \quad D = -151$

$t \;= 52, \quad v = 2, \quad h(D) = 7$

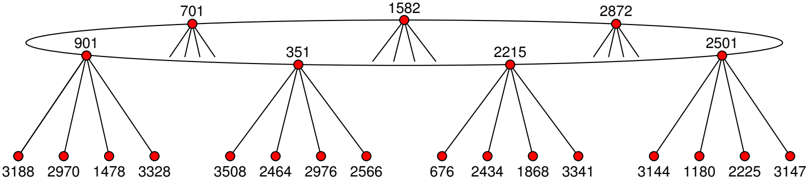
$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3, \quad \beta_{25} = \beta_2^7$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

$\ell \nmid v, \quad (\frac{D}{\ell}) = 1, \quad h(D) \geq \ell + 2$

$\ell_0 \neq \ell,\ (\frac{D}{\ell_0}) = 1,\ \alpha_\ell = \alpha_{\ell_0}^k,\ \beta_{\ell^2} = \beta_{\ell_0}^{k'}$



4. Enumerate floor using the action of β_{ℓ_0}

$$
\begin{array}{l}
3188 \xrightarrow{2} 945 \xrightarrow{2} 3144 \xrightarrow{2} 3508 \xrightarrow{2} 2843 \xrightarrow{2} {\color{blue}1502} \xrightarrow{2} 676 \xrightarrow{2} \\
2970 \xrightarrow{2} 3497 \xrightarrow{2} 1180 \xrightarrow{2} 2464 \xrightarrow{2} 4221 \xrightarrow{2} {\color{blue}4228} \xrightarrow{2} 2434 \xrightarrow{2} \\
1478 \xrightarrow{2} 3244 \xrightarrow{2} 2255 \xrightarrow{2} 2976 \xrightarrow{2} 3345 \xrightarrow{2} {\color{blue}1064} \xrightarrow{2} 1868 \xrightarrow{2} \\
3328 \xrightarrow{2} 291 \xrightarrow{2} 3147 \xrightarrow{2} 2566 \xrightarrow{2} 4397 \xrightarrow{2} {\color{blue}2087} \xrightarrow{2} 3341 \xrightarrow{2}
\end{array}
$$

Mapping a volcano

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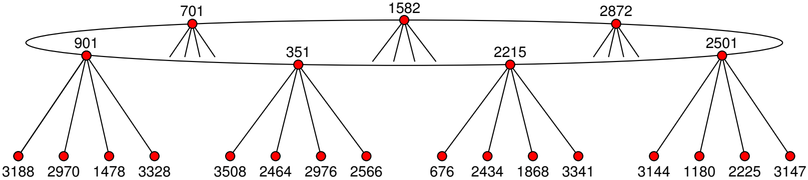

4. Enumerate floor using the action of β_{ℓ_0}

$3188 \xrightarrow{2} 945 \xrightarrow{2} 3144 \xrightarrow{2} 3508 \xrightarrow{2} 2843 \xrightarrow{2} 1502 \xrightarrow{2} 676 \xrightarrow{2}$

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Mapping a volcano

Example
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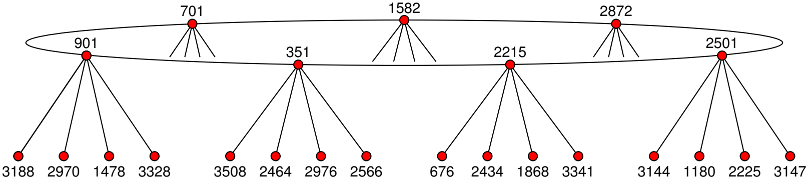
$\ell \nmid v, \quad \left(\frac{D}{\ell}\right) = 1, \quad h(D) \geq \ell + 2$
$\ell_0 \neq \ell, \ \left(\frac{D}{\ell_0}\right) = 1, \ \alpha_\ell = \alpha_{\ell_0}^k, \ \beta_{\ell^2} = \beta_{\ell_0}^{k'}$



4. Enumerate floor using the action of β_{ℓ_0}

$$
\begin{array}{l}
3188 \xrightarrow{2} 945 \xrightarrow{2} 3144 \xrightarrow{2} 3508 \xrightarrow{2} {\color{blue}2843} \xrightarrow{2} 1502 \xrightarrow{2} 676 \xrightarrow{2} \\
2970 \xrightarrow{2} 3497 \xrightarrow{2} 1180 \xrightarrow{2} 2464 \xrightarrow{2} {\color{blue}4221} \xrightarrow{2} 4228 \xrightarrow{2} 2434 \xrightarrow{2} \\
1478 \xrightarrow{2} 3244 \xrightarrow{2} 2255 \xrightarrow{2} 2976 \xrightarrow{2} {\color{blue}3345} \xrightarrow{2} 1064 \xrightarrow{2} 1868 \xrightarrow{2} \\
3328 \xrightarrow{2} 291 \xrightarrow{2} 3147 \xrightarrow{2} 2566 \xrightarrow{2} {\color{blue}4397} \xrightarrow{2} 2087 \xrightarrow{2} 3341 \xrightarrow{2}
\end{array}
$$

Mapping a volcano

Example

$\ell\ = 5,\quad p = 4451,\quad D = -151$

$t\ = 52,\quad v = 2,\quad h(D) = 7$

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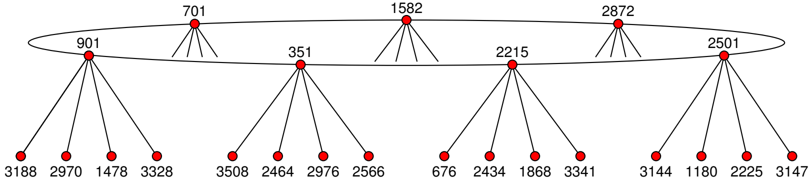
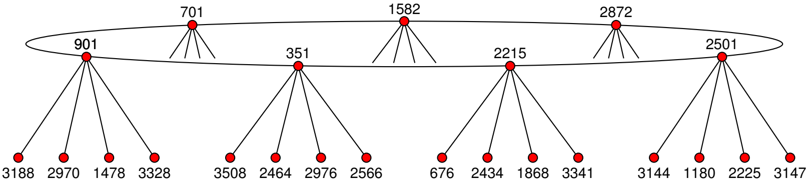
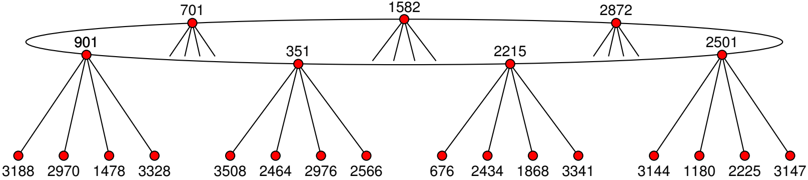


Interpolating $\Phi_\ell \bmod p$



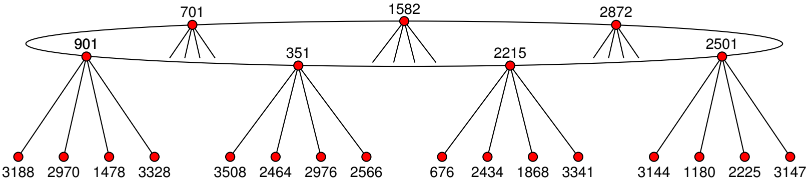
$$\begin{aligned}
\Phi_5(X, 901) &= (X-701)(X-351)(X-3188)(X-2970)(X-1478)(X-3328)\\
\Phi_5(X, 351) &= (X-901)(X-2215)(X-3508)(X-2464)(X-2976)(X-2566)\\
\Phi_5(X, 2215) &= (X-351)(X-2501)(X-3341)(X-1868)(X-2434)(X-676)\\
\Phi_5(X, 2501) &= (X-2215)(X-2872)(X-3147)(X-2255)(X-1180)(X-3144)\\
\Phi_5(X, 2872) &= (X-2501)(X-1582)(X-1502)(X-4228)(X-1064)(X-2087)\\
\Phi_5(X, 1582) &= (X-2872)(X-701)(X-945)(X-3497)(X-3244)(X-291)\\
\Phi_5(X, 701) &= (X-1582)(X-901)(X-2843)(X-4221)(X-3345)(X-4397)
\end{aligned}$$

Interpolating $\Phi_\ell \bmod p$



$$
\begin{aligned}
\Phi_5(X, 901) &= X^6 + 1337X^5 + 543X^4 + 497X^3 + 4391X^2 + 3144X + 3262\\
\Phi_5(X, 351) &= X^6 + 3174X^5 + 1789X^4 + 3373X^3 + 3972X^2 + 2932X + 4019\\
\Phi_5(X, 2215) &= X^6 + 2182X^5 + 512X^4 + 435X^3 + 2844X^2 + 2084X + 2709\\
\Phi_5(X, 2501) &= X^6 + 2991X^5 + 3075X^5 + 3918X^3 + 2241X^2 + 3755X + 1157\\
\Phi_5(X, 2872) &= X^6 + 389X^5 + 3292X^4 + 3909X^3 + 161X^2 + 1003X + 2091\\
\Phi_5(X, 1582) &= X^6 + 1803X^5 + 794X^4 + 3584X^3 + 225X^2 + 1530X + 1975\\
\Phi_5(X, 701) &= X^6 + 515X^5 + 1419X^4 + 941X^3 + 4145X^2 + 2722X + 2754
\end{aligned}
$$

Interpolating $\Phi_\ell \bmod p$



$$\begin{aligned}\Phi_5(X,Y) = X^6 &+ (4450Y^5 + 3720Y^4 + 2433Y^3 + 3499Y^2 + 70Y + 3927)X^5\\
&(3720Y^5 + 3683Y^4 + 2348Y^3 + 2808Y^2 + 3745Y + 233)X^4\\
&(2433Y^5 + 2348Y^4 + 2028Y^3 + 2025Y^2 + 4006Y + 2211)X^3\\
&(3499Y^5 + 2808Y^4 + 2025Y^3 + 4378Y^2 + 3886Y + 2050)X^2\\
&(\ \ 70Y^5 + 3745Y^4 + 4006Y^3 + 3886Y^2 + 905Y + 2091)X\\
&(Y^6 + 3927Y^5 + 233Y^4 + 2211Y^3 + 2050Y^2 + 2091Y + 2108)\end{aligned}$$

Computational results

Level records

1. **10009**: Φ_ℓ
2. **20011**: $\Phi_\ell \bmod q$
3. **60013**: $\Phi_\ell^{\mathfrak{f}}$

Speed records

1. **251**: Φ_ℓ in 28s $\Phi_\ell \bmod q$ in 4.8s (vs 688s)
2. **1009**: Φ_ℓ in 2830s $\Phi_\ell \bmod q$ in 265s (vs 107200s)
3. **1009**: $\Phi_\ell^{\mathfrak{f}}$ in 2.8s

Effective throughput when computing $\Phi_{1009} \bmod q$ is 100Mb/s.

Single core CPU times (AMD 3.0 GHz), using prime $q \approx 2^{256}$.
Polynomials $\Phi_\ell^{\mathfrak{f}}$ for $\ell < 5000$ available at `http://math.mit.edu/~drew`.

Application: point counting

Modular polynomials are the key ingredient to the Schoof-Elkies-Atkin (SEA) algorithm for computing $\#E(\mathbb{F}_q)$. Computing modular polynomials dominates the time and space complexity of SEA.

But the SEA algorithm does not actually require the full modular polynomial $\Phi_\ell(X, Y)$, it only needs the instantiated polynomials

$$\phi_\ell(Y) = \Phi_\ell(j(E), Y).$$

Using an isogeny volcano approach combined with the CRT, it is possible to directly compute ϕ_ℓ without computing Φ_ℓ [S 2012].

This dramatically reduces the space required by the SEA algorithm, and has led to several new point-counting records.

Elliptic curve point counting record

The number of points on the elliptic curve

$$y^2 = x^3 + 2718281828x + 3141592653$$

modulo the 5011 digit prime $16219299585 \cdot 2^{16612} - 1$ is

8323769891444946600619018491391378260069836370604500159309667928183741136740938227669912830997846627009617004020582940190774831705166648378125548174433
6222360544000538839492022451911485986733819166009550859216525385267852842524240978796544500427958734245859103650693623260065854955676905842760404211102
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4700901798590441199208750379215971112344019653309999668029194772178482699210001668960742884085944350942098735441112464897682811881029409157742761498481
8236133983076302692999418138548552140105778012525989072405641889553339872433242793570967700290860169473820597303300518069505065832587533308670748048008
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2635494364468486039656664278370935750099790919223024134537160958876614320893731637296530257682556027127545666105422232328156220481118882835904832158925
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7046203327761442798102604298830732855895993246333047147994546492842426742530314565704272126471147496267335652133743455000287920232413723922825839150351
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3284624677436722012981685194580777814009291336645358525962424649443734012222395524 8